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A serial inventory system with lead-time-dependent backordering: A reduced-state approximation

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ABSTRACT
We study a serial inventory system where the external customers may have a maximum time that they would be willing to wait for delivery in cases of stock-out and the demand would be lost if the remaining delivery lead time of the next available item is longer. This lead-time-dependent backordering behavior subsumes the models of partial backordering regardless of the wait that a customer would experience. In the inventory literature, this behavior has only been analyzed in single-location settings. We study this behavior in a multi-stage setting. We consider continuous review $(S-1, S)$ policies at all stages facing external Poisson demands. Using the method of supplementary variables, we define the stochastic process representing the inventory system and obtain the expressions for the operating characteristics of the inventory system. Based on the solution structures for the special cases, we propose an approximate solution which rests on replacing the state-dependent purchasing decision of the customer with an averaged-out purchase probability computed using only the age of the oldest item. An extensive numerical study indicates that the proposed approximation performs very well. Our numerical study provides additional insights about the sensitivity and allocation of stock levels across stages.

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Inventory theory and control; multi-echelon inventory; lead-time-dependent backordering; method of supplementary variables

1. Introduction

This article considers a multi-stage serial inventory system with external customers whose purchase decision is sensitive to the quoted lead time in cases of stock-out. Specifically, we envision that external customers may have a maximum time that they would be willing to wait for delivery and the demand would be lost if the remaining delivery lead time of the next available item is longer. The purchase behavior of customers assumed in our model is quite common in practice.

In retail settings, customers exhibit five basic responses to stock-outs: waiting for the stocked-out item, substitution by another size, product or outlet, and leaving the supply chain outlet (The National Association of Food Chains and Nielsen, 1968; Dadzie and Winston, 2007). Willingness to wait for the stocked-out item can be as high as 29.8% (Schary and Christopher, 1979) and as low as 2.5% (Walter and Grabner, 1975). Gruen *et al.* (2002) found that stock-outs are very common with a frequency of 7–8%, and that customers facing a stock-out behaves differently: 15% choose to wait until the product arrives, 9% choose not to purchase and the rest choose to substitute. Verhoef and Sloot (2006) report similar rates and indicate that 23% of customers are willing to wait during the quoted time to delivery. Marketing research also indicates that customer behavior during stock-outs may be explained partially through a customer's tolerance to wait for delivery, which reveals itself as the immediacy of need or sense of urgency (Emmelhainz *et al.*, 1991; Zinn and Liu, 2001; Corsten and Gruen, 2003; Sloot *et al.*, 2005), and the perceived cost of the time needed to acquire the item (Campo *et al.*, 2000). This type of customer whose purchase decision is sensitive to the quoted lead time is considered in this work. Henceforth, we will refer to the maximum time a customer is willing to wait for delivery as the customer's *waiting time tolerance*, the time between an order placement and its delivery as the *lead time* at all stages, the remaining time for delivery at the retailer quoted to the customer as the *remaining lead time*.

The vast majority of the literature on inventory solely considers the extreme stock-out responses of full backordering or fully lost sales. (See Hadley and Whitin (1963) and Zipkin (2000) for the general modeling approaches.) There are only a few works on settings where not all customers exhibit such extreme shortage behavior. With constant demand rates, Montgomery *et al.* (1973) are the first to introduce the notion of partial backordering. Incorporation of positive (finite) waiting time tolerances is due to Abad (1996). For a review of subsequent works with time-dependent backordering in deterministic demand environments, we refer the reader to Lodree (2007). With random demands, full backordering has also been the most commonly assumed shortage behavior. Schmidt and Nahmias (1985) provide the

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earliest study of lost sales for unit Poisson demands. Their model has been later extended to the (Q, r) policy by Berk and Gurler (2008). Moinzadeh (1989) models a Bernoulli-type partial backordering setting where a customer either backorders or is lost with a probability. The only study that considers lead-time-dependent backordering with random demands is Perry and Posner (1998). Olsson (2019) provides a unifying framework and extends the partial backordering models. All of the above works assume single location settings with an outside ample supplier.

For multi-echelon systems, lead-time-dependent backordering has not been studied in its generality. For arborescent (or divergent) systems only the special case of full lost sales has been considered. All of the existing works develop decomposition approximations for a two-echelon system with unit Poisson demands. Andersson and Melchiors (2001) and Hill *et al.* (2007) employ a heuristic approach where the lower stage is modeled as a single location with the delay at the upper stage modeled as an $M/D/S$ queue. Seifbarghy and Akbari-Jokar (2006) extend the heuristic to the (Q, r) environment. Thangam and Uthayakumar (2008) introduce a self-imposed limit on maximum backorder level and consider emergency shipments beyond this threshold under the same heuristic decomposition.

For serial systems, the seminal work is by Clark and Scarf (1960) who show the optimality of echelon-stock policies for a periodic review, finite horizon problem with negligible fixed replenishment costs and full backordering at all stages. Federguren and Zipkin (1984) extend their results to infinite horizon, and Rosling (1989) shows that assembly systems can be modeled as a serial inventory system. Chen and Zheng (1994) establish the optimality of the echelon-stock policy class under continuous review. Svoronos and Zipkin (1991) incorporate random lead times to the $(S-1, S)$ setting. Muharremoglu and Tsitsiklis (2008) establish that the echelon-stock policy class is also optimal for random lead times (so that orders do not cross). A number of re-formulations and tight bounds have also been proposed. (See Boyaci and Gallego (2001), Boyaci *et al.* (2003), Gallego and Ozer (2005) and Shang and Song (2006).) For surveys of studies on serial inventory systems, see Axsäter (2003) and Van-Houtum *et al.* (1996). All of the existing works on serial systems assume full backordering except a study by Huh and Janakiramany (2010) who consider full lost sales in a periodic review setting with echelon-stock policies. For a complete survey of models with general shortage behavior, we refer the reader to Bijvank and Vis (2011). For general reviews of multi-stage inventory systems, see de Kok *et al.* (2018) and Zijm (2019).

Our work differs from the practices where lead times are communicated to customers such as *commitment lead time* (see Ahmadi *et al.* (2019a), Ahmadi *et al.* (2019b)) or *guaranteed service* (see Kimball (1988), Graves and Willems (2000), Graves and Willems (2003), Huang *et al.* (2011), Eruguz *et al.* (2016)). The former practice is a pre-order strategy as a lead time reduction technique by which a retailer announces a constant time window for customers to place their orders before they actually need an item, thus enabling the retailer to effectively reduce its replenishment lead time. Customers are not sensitive to waits and, in cases of a stock-out, all unmet demand is backordered. In our model, an arriving customer receives the item if it is immediately available on hand and decides to backorder or not depending on the remaining time to delivery in case of a stock-out. The main difference of our model is that, in GSM, each stage has inbound service times set to a constant value decoupling it from the upperstream stages except its immediate suppliers and backordering is prohibited throughout the inventory system. In our model, all stages are allowed to have backorders (so long as the external customer is willing to wait.) In the classification of the literature into the GSM and SSM (stochastic service models), our model falls into the SSM category (Graves and Willems, 2000).

This article considers a multi-stage serial inventory system where the external customers may have a maximum time that they would be willing to wait for delivery in cases of stock-out and would walk away without purchase if the remaining delivery lead time of the next available item is longer. This lead-time-dependent backordering behavior subsumes the models of partial backordering regardless of the wait that a customer would experience. In the inventory literature, it has only been analyzed in single-location settings. We study this behavior in a multi-stage setting. As such, we attempt to fill an important gap in the literature. Specifically, we model a serial N-stage inventory system with constant delivery (transit) times and unit Poisson demands. The optimal policy class for the problem at hand is not known. Therefore, we use a local (as opposed to echelon) base-stock policy; that is, we employ one-for-one replenishment policies at all stages.

Only the demand at the lowest stage ($i = 1$) is backordered during stock-outs; the backordering behavior is of a general structure that subsumes the special cases of full backordering and lost sales among others. The main modeling difficulty in the setting is that the information on the time that the oldest unassigned item has spent in the system is not sufficient to estimate the time when the item will be available to the customer, but rather it is a complicated function of the times of all other subsequently ordered items.

We express the purchase decision of an arriving customer as a function of his/her waiting time tolerance and a $\sigma_{1,N}$-dimensional array representing the ordered sequence of the times elapsed since the most recent $\sigma_{1,N}$ orders have been placed. That is, the purchase (sales) process is, in general, a multi-dimensional process. Inspired by the exact solutions for the special cases of (i) a single-location system with general lead-time-dependent backordering, and (ii) the serial system with full backordering, we propose a reduced-state approximation.

An extensive numerical study demonstrates that the proposed approximation performs very well. Additionally, we provide insights and observations about the structural properties of the stocking policies in the presence of lead-time-dependent backordering.

The serial inventory model was motivated by the authors' encounter with a particular practice at a national household appliance manufacturer, which operates its own dealership/

distribution network. In that practice, each local dealership holds its own stock at its outlet (retailer in our model) and keeps "earmarked or retailer-dedicated" stocks at upper stages (city warehouses, regional depots and manufacturer's warehouse.) The managers explain the rationale of the practice as that "they can plan better for customer deliveries and accordingly communicate exact delivery times to their customers while taking advantage of economies of scale in warehousing for their individual stocks". This 'earmarked' inventory practice effectively renders the supply network, which may be of an arborescent nature, into a purely serial system for each retailer in operation. Although against the spirit of "risk-pooling", such dedicated stock policies have been shown to be beneficial in Poisson demand environments (Yang and Schrage, 2009). Since we assume Poisson demands as well, we believe that the studied serial system has a real-life application, as well as being of theoretical interest and making a contribution to the serial inventory literature.

The rest of this article is organized as follows. In Section 2, we present the model assumptions, develop the model and the expressions for the operating characteristics of the inventory system. In Section 3, we provide a reduced-state approximation of the model In Section 4, we provide our numerical study. We present our conclusions in Section 5.

2. The model

In this section we introduce the basic assumptions of the model. We define the stochastic process representing the system, and develop the expressions for the operating characteristics based on the stationary probability distribution of the defined stochastic process. To this end, we use the method of supplementary variables. (See Cox (1955) and Schmidt and Nahmias (1985).) We then develop the system of integro-differential equations that describe the stationary behavior of the process and provide exact solutions for the special cases, which will be used later to develop a reduced-state approximation.

2.1. The basic assumptions

We consider a single item, multi-stage serial inventory system with N levels with the lowest stage $(i = 1)$ being closest to the demand. Demands arrive at the retailer according to a unit Poisson process with rate λ. An $(S_i - 1, S_i)$ control policy is employed at all stages, where S_i denotes the maximum stocking level at stage i for $i = 1, 2, \ldots, N$. Hence, a satisfied demand causes an order to be placed by each stage at the next higher stage. Each stage corresponds to a physical location of stock. In a general supply chain setting the lowest stage may be referred to as retailer and upper stages may be referred to as distribution centers, warehouses, manufacturers, and suppliers. We assume that the system is supplied externally by an ample supplier which may be viewed to constitute the $(N+1)$th stage. (Similarly, customers may be viewed as constituting stage 0.) There are non-negative transit times between stages; L_i denotes the transit time from stage $i+1$ to stage i for all stages and is constant. (Transit time from stage 1 to customer is assumed to be negligible.) All stages use the *FIFO* issuance policy. (As discussed later in Section 2.3.1., any serial system with some stages having zero stock levels can be equivalently represented by one where $S_i > 0$ for all i. Therefore, we introduce and develop our model with the assumption of all stock levels being positive.)

Upon a customer arrival, if the retailer has stock on hand, the demand is immediately satisfied. However, in the case of a stock-out at the retailer, we assume that a customer exhibits a particular purchase behavior which takes into account how long she/he is willing to wait for delivery (i.e., customer's waiting time tolerance). If the time until an item becomes available does not exceed the waiting time tolerance of the customer, the demand is backordered; otherwise, it is lost. Because all transit times are deterministic in the system, the exact time when an outstanding order will be received by a stage is fully known. This enables the retailer to communicate the exact remaining lead time. Let the customer waiting time tolerance be denoted by τ (≥ 0), which may be distributed across the customer population with $g(\tau)$. Then, if the retailer communicates a remaining lead time r, the customer backorders when $\tau \leq r$, and goes away, otherwise. We model the purchase decision of a customer arriving at a particular state of the system in detail later in this article.

The following costs are charged to the inventory system: Holding cost rate h_i per unit of stock held at stage i, backordering cost b per unit of demand backordered per unit time and π per unit of demand lost. (Note that the shortage costs are charged at the lowest stage only.) The inventory system is managed centrally and the objective is to minimize the expected total cost rate for the entire system.

2.2. Preliminaries

Customers arrive at the inventory system and demand an item. A purchase is said to occur whenever a demand is satisfied. At each purchase instance, a unit of inventory allocated to Stage 1 is issued to the customer to satisfy the demand. Under the employed one-for-one policy, this purchase triggers (i) an order placement to the next higher stage and (ii) simultaneously an item issuance to the next lower stage along the entire system. Issuing an item by a stage may be in the form of either initiating an actual shipment of the item if it is on hand or earmarking an outstanding order for delivery upon its arrival. As each order corresponds to a single item, the age of an item is the time elapsed since the order associated with that item was placed at the ample supplier. The employed inventory control policy allocates S_i (> 0) units of inventory to each stage i $(1 \leq i \leq N)$ with a total system stock of $\sigma_{1,N} (= \sum_{i=1}^{N} S_i)$ units. For notational convenience, we number the items at any point in time according to their ages where the item with number $\sigma_{1,N}$ is the youngest. Items 1 through S_1 are allocated to Stage 1, items $(S_1 + 1)$ through $(S_1 + S_2)$ to Stage 2 and so on with items $(\sigma_{1,N} - S_N + 1)$ through $\sigma_{1,N}$ being allocated to Stage N.

The *FIFO* issuance policy dictates that a stage always issues the oldest unit that is allocated to that stage. Therefore, at each purchase instance, the oldest item that is currently allocated to Stage 1 (item 1) is issued to the customer. If item 1 has not yet reached Stage 1 when a customer arrives, the inventory system experiences a stock-out. In the lead-time-dependent backordering environment considered herein, the purchase decision of the customer depends on the remaining lead time for item 1 to reach Stage 1. To calculate this remaining lead time, one needs to know how long the item has stayed on the shelf at the upper stages as well as its current age.

Our modeling rests on obtaining the operating characteristics (on-hand inventories, number of backorders, etc.) of the inventory system by means of a multi-dimensional stochastic process. (See Schmidt and Nahmias (1985) for a similar approach for single-location systems.) We obtain the steady state behavior of this process and the corresponding probability distribution function of finding it in a particular state via its temporal evolution. Using this, we obtain the expected values of the operating characteristics and develop the expression for the objective function of our model. However, to describe the evolution of the stochastic process, we first need the purchasing and ordering dynamics within the inventory system. To this end, we obtain the expressions for remaining lead times and purchase decision of an arriving customer for a given state of the inventory system. Then, we obtain the expressions that describe the evolution of the inventory system over time.

Due to the *FIFO* policy the lengths of stay at upper stages themselves can be computed using the ages of certain subsequent items. Therefore, the multi-dimensional stochastic process employed herein is a stochastic process based on the times elapsed since the most recent $\sigma_{1,N}$ orders were placed. Next, we introduce this process formally.

Stochastic Process, $\xi(t)$

Define the $\sigma_{1,N}$-dimensional stochastic process:

$$\xi(t) = \{\xi_1(t), \xi_2(t), ..., \xi_{S_1}(t), \xi_{S_1+1}(t), ..., \xi_{S_1+\cdots+S_N}(t)\}$$

where $\xi_i(t)$, $(i = 1, 2, ..., \sigma_{1,N})$, denotes the time elapsed at time t since the $(\sigma_{1,N} - i + 1)$th most recent order placement at the ample supplier (i.e., occurrence of satisfied demand). Clearly, $\xi_1(t) \geq \xi_2(t) \geq \cdots \geq \xi_{S_1+\cdots+S_N}(t) \geq 0$. The elapsed times of orders also correspond to the ages of the items in the system (in stock or in transit) at time t; $\xi_i(t)$ is the age of the ith oldest item in the system at time t, where $\xi_1(t)$ is the age of the oldest item, $\xi_2(t)$ is the age of the second oldest item, and so forth.

Remaining Lead Times

The remaining lead time for item 1 to reach Stage 1 is crucial in the model herein. We illustrate below how it can be computed when one finds the stochastic process $\xi(t)$ in state $\mathbf{x} = (x_1, x_2, ..., x_{S_1+\cdots+S_N})$ at some time t; $\{\xi_1(t) = x_1, ..., \xi_{S_1+\cdots+S_N}(t) = x_{S_1+\cdots+S_N}\}$.

For the case of a single-stage inventory system ($N = 1$), knowing only the age of item 1 is sufficient to quote the remaining lead time for this item in the case of a stock-out: $r(\mathbf{x}) = \max(0, [L_1 - x_1])$. For serial systems with $N(\geq 2)$ stages, the remaining lead time depends on the age of item 1 *and* how long it has stayed on the shelf at the upper stages. Let $z_1^k(\mathbf{x})$ denote how long item 1 has stayed on the shelf at stage k when the stochastic process is currently in state $\mathbf{x}$

$$z_1^k(\mathbf{x}) = \max\left(0, (x_1 - x_{1+\sigma_{k,N}}) - \left(\sum_{m=k}^{N} L_m + \sum_{m=k+1}^{N} z_1^m(\mathbf{x})\right)\right)$$

where $\sigma_{k,N} = \sum_{i=k}^{N} S_i$ for $1 \leq k \leq N$. Then, the remaining lead time for item 1 to reach Stage 1, $r(\mathbf{x})$, is given by

$$r(\mathbf{x}) = \max\left(0, \left\{\left[\sum_{i=1}^{N} L_i + \sum_{i=2}^{N} z_1^i(\mathbf{x})\right] - x_1\right\}\right)$$

for an N-stage system. (For a detailed discussion of the construction of $r(\mathbf{x})$ with specific illustrations for $N = 1$, 2 and 3, see the Supplemental Online Materials.)

Purchase Decision of an Arriving Customer

Next, we formally characterize the purchase decision of a customer based on the remaining lead time obtained above. If an arriving customer finds the system in state $\mathbf{x}$ such that $r(\mathbf{x}) = 0$, the demand is immediately satisfied; otherwise $(r(\mathbf{x}) > 0)$, the demand is either backordered or lost depending on the waiting time tolerance of the customer, τ. If $r(\mathbf{x}) \leq \tau$, the sale is realized as a backorder; otherwise, it is lost. Hence, the arrival of a customer who finds the system in state $\mathbf{x}$ results in a purchase with probability

$$\alpha(\mathbf{x}) = \int_\tau g(\tau)\delta(\tau, \mathbf{x})d\tau$$

where $\delta(\tau, \mathbf{x})$ is a binary variable that attains the value of one if (i) $r(\mathbf{x}) = 0$ or (ii) $0 < r(\mathbf{x}) \leq \tau$, and zero, otherwise.

Evolution of $\xi(t)$ over Time

So far, we have discussed how the state of the process is used to obtain the key characteristics of the inventory system at hand at any time t. Next, we illustrate how the stochastic process evolves over time. Suppose that at some initial time $t = t_0$, we have $\xi(t)$ in state $(x_1^0, x_2^0, ..., x_{S_1+\cdots+S_N}^0)$; and we observe a sequence of purchase instances at times $t_i, i = 1, 2, ...$ Until the first purchase instance, ages of the items in the system grow at equal rates. Immediately before the purchase instance at $t = t_1^-$, the system is in state $(x_1^0 + (t_1^- - t_0), x_2^0 + (t_1^- - t_0), x_3^0 + (t_1^- - t_0), ..., x_{S_1+\cdots+S_N}^0 + (t_1^- - t_0))$. At the purchase instance t_1, the oldest item in the system (item 1) is withdrawn to satisfy the demand. Hence, immediately after the purchase instance (at $t = t_1^+$), $\xi(t)$ moves to the state $(x_2^0 + (t_1^+ - t_0), x_3^0 + (t_1^+ - t_0), ..., x_{S_1+\cdots+S_N}^0 + (t_1^+ - t_0), 0^+)$. That is, immediately after the first purchase instance, we have $\xi_{S_1+\cdots+S_N}(t_1^+) = 0^+$ and $\xi_i(t_1^+) = x_{i+1}^0 + (t_1^+ - t_0)$ for $i = 1, ..., S_1 + \cdots + S_N - 1$. Similarly, the ages of the items grow at equal rates between consecutive purchase instances; and, hence, immediately after some purchase instance t_j, we have $\xi_{S_1+\cdots+S_N}(t_j^+) = 0^+$ $\forall j$, $\xi_i(t_j^+) = x_{i+j}^0 + (t_j^+ - t_0)$ for $i = 1, ..., S_1 + \cdots + S_N - j$ and $j < S_1 + \cdots + S_N$, and $\xi_{S_1+\cdots+S_N-i}(t_j^+) = t_j^+ - t_{j-i}^+$ for $i = 1, ..., S_1 + \cdots + S_N - j$ and $j \geq S_1 + \cdots + S_N$. In general, given the initial state and the sequence of purchase instances t_1, t_2 and so on, the state of the stochastic process at time t ($t_j \le t < t_{j+1}\ \forall j$) can be found to be: $\xi_{S_1+\cdots+S_N}(t) = t - t_j\ \forall j$; $\xi_i(t) = x^0_{i+j} + t$ for $i = 1, \ldots, S_1 + \cdots + S_N - j$ and $j < S_1 + \cdots + S_N$; and $\xi_{S_1+\cdots+S_N-i}(t) = t - t_{j-i}$ for $i = 1, \ldots, S_1 + \cdots + S_N - 1$ and $j \ge S_1 + \cdots + S_N$.

As demand arrivals and, thereby, purchase decisions are random, finding the process $\xi(t)$ in a particular state at any time is also random. Let $p(t, \mathbf{x})$ denote the probability density of $\xi(t)$ being in state $\mathbf{x}$ at time t. Then, the system of equations that this density must satisfy are given in the following.

Proposition 2.1. *For $t > 0$, and $x_{S_1+\cdots+S_N} > 0$,*

$$\frac{\partial}{\partial t}p(t, x_1, \ldots, x_{S_1+\cdots+S_N}) + \sum_{i=1}^{\sigma_{1,N}} \frac{\partial}{\partial x_i}p(t, x_1, \ldots, x_{S_1+\cdots+S_N}) = -\lambda\alpha(x_1, \ldots, x_{S_1+\cdots+S_N})p(t, x_1, \ldots, x_{S_1+\cdots+S_N}) \tag{1}$$

and, for $t > 0$, and $x_{S_1+\cdots+S_N} = 0$,

$$p(t, x_1, x_2, \ldots, x_{S_1+\cdots+S_N-1}, 0) = \int_{\eta=x_1}^{\infty} \lambda\alpha(\eta, x_1, \ldots, x_{S_1+\cdots+S_N-1})p(t, \eta, x_1, x_2, \ldots, x_{S_1+\cdots+S_N-1})d\eta. \tag{2}$$

Proof. Provided in the Supplemental Online Materials.

Note that (1) describes the evolution of the process over time after one purchase instance until the next, and (2) provides the boundary condition at the purchase instances at which the process experiences a jump.

2.3. Steady state analysis

Our modeling of the inventory system rests on the stationary (steady state) behavior of the above defined stochastic process and the corresponding stationary distribution function. Let $p(\mathbf{x})$ denote the stationary probability density of $\xi(t)$ being in state $\mathbf{x}$ at steady state (as $t \to \infty$). A key property in this approach is the existence of a stationary distribution. In the following result, we establish that the stationary distribution exists for this stochastic process, and thereby, for the corresponding inventory system.

Theorem 2.2. *The stationary probability density function (p.d.f.) of $\xi(t)$ denoted by $p(\mathbf{x})$ exists and it satisfies the following system of equations. For $x_{S_1+\cdots+S_N} > 0$,*

$$\sum_{i=1}^{\sigma_{1,N}} \frac{\partial}{\partial x_i}p(x_1, \ldots, x_{S_1+\cdots+S_N}) = -\lambda\alpha(x_1, \ldots, x_{S_1+\cdots+S_N})p(x_1, \ldots, x_{S_1+\cdots+S_N}) \tag{3}$$

and $x_{S_1+\cdots+S_N} = 0$,

$$p(x_1, x_2, \ldots, x_{S_1+\cdots+S_N-1}, 0) = \int_{\eta=x_1}^{\infty} \lambda\alpha(\eta, x_1, \ldots, x_{S_1+S_2+\cdots+S_N-1}) p(\eta, x_1, x_2, \ldots, x_{S_1+\cdots+S_N-1})d\eta. \tag{4}$$

Proof. Provided in the Supplemental Online Materials.

Next, we obtain the operating characteristics of the inventory system. Specifically, we derive the expressions for on-hand inventories, number of demands that are backordered per unit time, the number of demands that are lost per unit time, the time an arriving customer would have to wait for delivery, and their expected values.

2.3.1. Operating characteristics of the inventory system

The on-hand inventory at each stage when the process is in a particular state $\mathbf{x}$ can also be obtained as follows. Let $n(j)$ denote the stage to which item j ($1 \le j \le \sigma_{1,N}$) remains allocated and is given as $n(j)$ is the smallest integer such that $j \le S_1 + \cdots + S_{n(j)}$. Also let $z^k_j(\mathbf{x})$ denote the time that item j has spent on the shelf at stage k ($n(j) \le k \le N$) if the process is in state $\mathbf{x} = (x_1, x_2, \ldots, x_{S_1+\cdots+S_N})$ at time t. Using the same argumentation for the lengths of stay for item 1 above, we have, for item j,

$$z^k_j(\mathbf{x}) = \max\left(0, (x_j - x_{j+\sigma_{k,N}}) - \left(\sum_{m=k}^{N} L_m + \sum_{m=k+1}^{N} z^m_j(\mathbf{x})\right)\right)$$

for $n(j) + 1 \le k \le N$. For notational brevity, we use $a_{i,k}(\mathbf{x})$ to denote the total time that the $(S_1 + \cdots + S_{i-1} + k)$th oldest item in the inventory system has spent in transit and on the shelf at higher stages prior to joining stage i computed when the process is in state $\mathbf{x}$. By definition,

$$a_{i,k}(\mathbf{x}) = \sum_{m=i}^{N} L_m + \sum_{m=i+1}^{N} z^m_{\{S_1+\cdots+S_{i-1}+k\}}(\mathbf{x})$$

for $k = 0, 1, \ldots, S_i$ and $i = 1, \ldots, N$. Then, the on-hand inventory at stage i when the process is in state $\mathbf{x}$, $OH_i(\mathbf{x})$ is given by

$$OH_1(\mathbf{x}) = \begin{cases} 0 & \text{if } a_{1,1}(\mathbf{x}) > x_1 \ge x_2 \ge \cdots \ge x_{S_1} \ge \cdots \ge x_{\{S_1+\cdots+S_N\}} \ge 0 \\ k & \text{if } \begin{array}{c} x_1 \ge \cdots \ge x_k \ge a_{1,k}(\mathbf{x}); \\ a_{1,k+1}(\mathbf{x}) > x_{k+1} \ge \cdots \ge x_{S_1} \ge \cdots \ge x_{\{S_1+\cdots+S_N\}} \ge 0 \end{array} \\ S_1 & \text{if } \begin{array}{c} x_1 \ge \cdots \ge x_{S_1} \ge a_{1,S_1}(\mathbf{x}); \\ x_{S_1} \ge x_{S_1+1} \ge \cdots \ge x_{\{S_1+\cdots+S_N\}} \ge 0 \end{array} \end{cases}$$

and for $2 \le i \le N$,

$$OH_i(\mathbf{x}) = \begin{cases} 0 & \text{if } \begin{array}{c} x_1 \ge \cdots \ge x_{\{S_1+\cdots+S_{i-1}\}} \ge x_{\{S_1+\cdots+S_{i-1}+1\}}; \\ a_{i,1}(\mathbf{x}) > x_{\{S_1+\cdots+S_{i-1}+1\}} \ge x_{\{S_1+\cdots+S_{i-1}+2\}} \ge \cdots \\ \ge x_{\{S_1+\cdots+S_N\}} \ge 0 \end{array} \\ k & \text{if } \begin{array}{c} x_1 \ge \cdots \ge x_{\{S_1+\cdots+S_{i-1}+k\}} \ge a_{i,S_1+\cdots+S_{i-1}+k}(\mathbf{x}); \\ a_{i,S_1+\cdots+S_{i-1}+k+1}(\mathbf{x}) > x_{\{S_1+\cdots+S_{i-1}+k+1\}}; \\ x_{\{S_1+\cdots+S_{i-1}+k\}} \ge x_{\{S_1+\cdots+S_{i-1}+k+1\}} \ge \cdots \\ \ge x_{\{S_1+\cdots+S_N\}} \ge 0 \end{array} \\ S_i & \text{if } \begin{array}{c} x_1 \ge \cdots \ge x_{\{S_1+\cdots+S_i\}} \ge a_{i,S_1+\cdots+S_i}(\mathbf{x}); \\ x_{\{S_1+\cdots+S_i\}} \ge x_{\{S_1+\cdots+S_i+1\}} \ge \cdots \ge x_{\{S_1+\cdots+S_N\}} \ge 0 \end{array} \end{cases}.$$

Let P^i_k denote the probability that there are k items on hand at stage i at steady state. Then, $P^i_k = \int_{\mathbf{x}} I[OH_i(\mathbf{x}) = k]p(\mathbf{x})d\mathbf{x}$ where $p(\mathbf{x})$ is the stationary distribution, $I[\cdot]$ denotes the indicator function and integration operation over $\mathbf{x}$ is performed over the entire domain $\{x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_{S_1+\cdots+S_N} \geq 0\}$. Thus, the probability that the retailer experiences a stock-out at steady state is $P_{out} = P_0^1$.

The expected on-hand inventory at stage i at steady state, $E[OH_i] = \sum_{k=1}^{S_i} kP_k^i$, the expected number of demands that are backordered per unit time at steady state, $E[NBO] = \lambda \int_{\mathbf{x}} \alpha(\mathbf{x}) I[OH_1(\mathbf{x}) = 0] p(\mathbf{x}) d\mathbf{x}$ where $\int_{\mathbf{x}} \alpha(\mathbf{x}) I[OH_1(\mathbf{x}) = 0] p(\mathbf{x}) d\mathbf{x}$ is the probability that *a* customer arrival results in a purchase during a stock-out. The expected number of demands that are lost per unit time at steady state, $E[LS] = \lambda \int_{\mathbf{x}} [1-\alpha(\mathbf{x})] I[OH_1(\mathbf{x}) = 0] p(\mathbf{x}) d\mathbf{x} = \lambda P_{out} - E[NBO]$.

We can use Little's Law (Little, 1961) to find the expected number of backorders at steady state. Letting $E[W]$ denote the expected time that an arriving customer would wait for his/her demand to be satisfied,

$$E[W] = \int_{\mathbf{x}} r(\mathbf{x}) \alpha(\mathbf{x}) p(\mathbf{x}) d\mathbf{x}$$

where $r(\mathbf{x})$ denotes the remaining lead time (if any) for item 1 to reach Stage 1. Note that the customer faces no wait if there is positive stock on arrival or the remaining lead time of the oldest item to the retailer is longer than the customer's tolerance. Then, using Little's Law, we find the expected backorders at steady state, $E[BO] = \lambda\, E[W]$.

The expected total cost rate $E[TC]$ used in our analysis is written as follows:

$$E[TC] = \sum_{i=1}^{N} h_i E[OH_i] + bE[BO] + \pi E[LS]$$

Remarks. (1) In the above construction, we exclude the pipeline inventory costs (that is, costs accruing to items in transit). If this cost is to be included, the expected pipeline inventory cost rate, $E[PIC]$ is simply added to $E[TC]$. Each item sold has to spend the transit time L_i from stage $i+1$ to the lower stage, by definition. Assuming that the stage to which an item is issued incurs the holding cost, we have $E[PIC] = \sum_{i=1}^{N} h_i(\lambda - E[LS])L_i$. (2) There is an equivalence between an N-stage serial system with zero stocking levels at some stages and an $(N-1)$-stage serial system with positive stock at all stages. Suppose that the N-stage serial system, denoted by $^{(1)}$ has lead times $L_i^{(1)} (>0)$ and stock levels $S_i^{(1)}$ for $1 \leq i \leq N$.

(i) If $S_j^{(1)} = 0$ for $j = N$, then system $^{(1)}$ is equivalent to an $(N-1)$-stage serial system, denoted by $^{(2)}$, where lead times and stocking levels are as follows. $L_i^{(2)} = L_i^{(1)}$ for $1 \leq i < N-1$ and $L_{N-1}^{(2)} = L_{N-1}^{(1)} + L_N^{(1)}$; $S_i^{(2)} = S_i^{(1)}$ for $1 \leq i \leq N-1$.

(ii) If $S_j^{(1)} = 0$ for $1 < j < N$, then system $^{(1)}$ is equivalent to an $(N-1)$-stage serial system, denoted by $^{(2)}$, where lead times and stocking levels are as follows. $L_i^{(2)} = L_i^{(1)}$ for $1 \leq i < j-1$, $L_{j-1}^{(2)} = L_{j-1}^{(1)} + L_j^{(1)}$, $L_i^{(2)} = L_{i+1}^{(1)}$ for $j \leq i \leq N-1$. $S_i^{(2)} = S_i^{(1)}$ for $1 \leq i < j-1$, $S_i^{(2)} = S_{i+1}^{(1)}$ for $j \leq i \leq N-1$.

(iii) If $S_1^{(1)} = 0$, then system $^{(1)}$ is equivalent to a $(N-1)$-stage system, denoted by $^{(2)}$, having lead times $L_i^{(2)} = L_{i+1}^{(1)}$ and stocking levels $S_i^{(2)} = S_{i+1}^{(1)}$ for $1 \leq i \leq N-1$ and a modified waiting tolerance $\tau - L_1^{(1)}$ ensuring positive purchase probability for at least some system states. Otherwise, it corresponds to a system where all demands are lost at all times.

2.3.2. Computation of the operating characteristics

The computation of the operating characteristics and the expected cost rate at steady state is possible if the functional form of $p(\mathbf{x})$ is available. The explicit, closed form solution for the system of Volterra-type equations (3) and (4) is available for the following special cases:

1. For an N-stage serial inventory system with a customer population where $g(\tau) > 0$ iff $\tau \geq \sum_{i=1}^{N} L_i$, $\alpha(x_1, \ldots, x_{S_1+\cdots+S_N}) = 1\ \forall \mathbf{x}$. The resulting inventory system is a serial system with full backordering.
2. For a single-location inventory system $(N=1)$ with lead-time-dependent backordering as considered herein, $\alpha(x_1, \ldots, x_{S_1+\cdots+S_N}) = \int_{\tau} g(\tau)\delta'(\tau, x_1) d\tau$ where $\delta'(\tau, x_1)$ is a binary variable that attains the value of one iff $x_1 \geq L_1$. This is a generalization (to non-perishables) of the (constant shelf life) model in Perry and Posner (1998).

In both cases, the stationary solution $p(\mathbf{x})$ is given as follows.

Proposition 2.3. *Let* $\alpha(x_1, \ldots, x_{S_1+\cdots+S_N}) = f(x_1)$ *where* $f(x_1)$ *is a univariate function. Then, for* $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_{S_1+\cdots+S_N} \geq 0$,

$$p(\mathbf{x}) = C_0 e^{-\left\{\lambda \int^{x_1} f(\eta) d\eta\right\}}$$

with

$$C_0 = \left[\int_{x_1=0}^{\infty} e^{-\left\{\lambda \int^{x_1} f(\eta) d\eta\right\}} x_1^{(\sigma_{1,N}-1)} dx_1\right]^{-1} (\sigma_{1,N}-1)!$$

However, for the general system described by (3) and (4), there does not appear to be a simple explicit, closed form solution. We may conjecture that, for $N > 1$ and $\alpha(\mathbf{x})$ of a general form, the solution would be a complicated function of (at least) the entire state vector (if not also a function of a number of prior order times) due to the purchase decision being a function of the age information on all of the unassigned items in the system. Such integro-differential equation systems can be solved numerically as suggested in Perry and Posner (1998), but this would also involve considerable tedium as N and $\sigma_{1,N}$ get large. In the absence of the solution for (3) and (4) to describe the stationary probability distribution $p(\mathbf{x})$, it would be beneficial to solve the system of equations approximately. We propose an approximation that allows for an analytical expression for $p(\mathbf{x})$, which, in turn, enables one to compute analytically the operating characteristics of the inventory system. Instead we propose a reduced-state approximation for the purchase probability $\alpha(\mathbf{x})$ which renders the sales process a uni-dimensional process; thereby, an explicit approximate solution is possible as discussed above. Next, we discuss this approach and the solution.

3. A reduced-state approximation

The approximation of the purchase probability rests on the observations that (i) the explicit solution for $p(\mathbf{x})$ obtained above for a single-location inventory system holds for a serial system with $N > 1$ *if* the purchase behavior were to depend only on the age of the oldest item in the system and (ii) the conditional probability that the system is in state $\mathbf{x}$ given that the oldest item's age is x_1 has the same structure as in Proposition 2.3. Suppose we approximate $\alpha(\mathbf{x})$ with $\widetilde{\alpha}(x_1)$ which denotes the probability of purchase for an arriving customer based only on the information provided in the age of the oldest item in the system. That is, suppose the purchase process is approximated by a reduced-state-dependent process. Let $\widetilde{p}(\mathbf{x})$ denote the solution to the system of (3) and (4) that describe such a process. Then, directly from Proposition 2, we have

$$\widetilde{p}(\mathbf{x}) = C_0 e^{-\left\{\lambda\int^{x_1}\widetilde{\alpha}(\eta)d\eta\right\}} \text{ for } x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_{S_1+\cdots+S_N} \geq 0$$

with

$$C_0 = (\sigma_{1,N}-1)!\left[\int_{x_1=0}^{\infty} e^{-\left\{\lambda\int^{x_1}\widetilde{\alpha}(\eta)d\eta\right\}} x_1^{(\sigma_{1,N}-1)}\right]^{-1}$$

To develop the approximation for the serial system, the question of how to choose $\widetilde{\alpha}(x_1)$ remains. We proceed in two steps. First, we approximate $\alpha(\mathbf{x})$ by $\alpha_1(x_1)$ which is a function that only depends on the age of the oldest item:

$$\alpha_1(x_1) = \int_{\{x_2,\ldots,x_{S_1+\cdots+S_N}\}\in\Omega(x_1)} \times \left[\int_\tau g(\tau)\delta(\tau,\mathbf{x})d\tau\right]\Phi(\mathbf{x}|x_1)\ dx_2\ldots dx_{S_1+\cdots+S_N}$$

where $\delta(\tau,\mathbf{x})$ is as defined before, $\Omega(x_1)$ is a $(\sigma_{1,N}-1)$-dimensional space defined as $\{x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_{S_1+\cdots+S_N} \geq 0 | x_1\}$ and $\Phi(\mathbf{x}|x_1)$ denotes the conditional probability that the system is in state $\mathbf{x}$ given that the oldest item's age is x_1. In this step, we lose information as we reduce the purchase probability to a univariate function. Next, since we do not have $\Phi(\mathbf{x}\,|\,x_1)$ which would be computed with the exact solution $p(\mathbf{x})$, we approximate $\Phi(\mathbf{x}\,|\,x_1)$ by its counterpart $\widetilde{\Phi}(\mathbf{x}\,|\,x_1)$ based on $\widetilde{p}(\mathbf{x})$ which assumes an approximate purchase probability $\widetilde{\alpha}(x_1)$:

$$\widetilde{\Phi}(\mathbf{x}|x_1) = \frac{\widetilde{p}(\mathbf{x})}{\widetilde{p}(x_1)} = \frac{\widetilde{p}(\mathbf{x})}{\int_{x_2=0}^{x_1}\cdots\int_{x_{S_1+\cdots+S_N}=0}^{x_{S_1+\cdots+S_N-1}}\widetilde{p}(\mathbf{x})dx_{\sigma_{1,N}}\ldots dx_2}$$
$$= \frac{C_0\exp\{-\int^{x_1}\widetilde{\alpha}(y)dy\}}{C_0\exp\{-\int^{x_1}\widetilde{\alpha}(y)dy\}x_1^{(\sigma_{1,N}-1)}/(\sigma_{1,N}-1)!}$$
$$= \frac{(\sigma_{1,N}-1)!}{x_1^{(\sigma_{1,N}-1)}}$$

Hence,

$$\widetilde{\alpha}(x_1) = \int_{\{x_2,\ldots,x_{S_1+\cdots+S_N}\}\in\Omega(x_1)}\left[\int_\tau g(\tau)\delta(\tau,\mathbf{x})d\tau\right] \times \frac{(\sigma_{1,N}-1)!}{x_1^{(\sigma_{1,N}-1)}}\ dx_2\ldots dx_{S_1+\cdots+S_N}$$

The approximate counterparts of the operating characteristics (denoted by the notation "~") can then be obtained by substituting $\widetilde{\alpha}(x_1)$ and $\widetilde{p}(\mathbf{x})$ in lieu of $\alpha(\mathbf{x})$ and $p(\mathbf{x})$, resp.

The rationale of the approximation is as follows. In reality, the arriving customer is communicated the true remaining lead time for the next available item, which uses the exact state information $\mathbf{x}$. However, in the approximation, we replace the state-dependent purchasing decision with an averaged-out purchase probability. Furthermore, $\widetilde{\Phi}(\mathbf{x}|x_1)$ implies that the placement times of the subsequent orders are uniformly distributed over $[0, x_1]$ *as if* purchases follow a Poisson process as do customer arrivals. With this approximation, we are able to compute the operating characteristics of the inventory system.

4. Numerical study

Our numerical study has three objectives: (i) to investigate the quality of the proposed approximation in capturing the true behavior of the inventory system at given stock levels, (ii) to investigate the goodness of the solutions obtained with the approximation *vis a vis* those obtained with simulation, and (iii) to examine the sensitivity of the optimal policy parameters with respect to various system and cost parameters as well as some structural observations on multi-stage systems in the presence of lead-time-dependent backordering.

4.1. Parameter set

We considered four experiment groups in our numerical study. The first group consists of a two-stage system with system parameters constitute an extended set of Andersson and Melchiors (2001); the remaining groups consider $N > 2$ and have been generated on the basis of the studies provided by Gallego and Zipkin (1999) and Shang and Song (2003). In all experiments in our numerical study, we retain the following overall setting. The demand process follows a unit Poisson distribution with mean λ. All transit times between stages are constant. Customers are identical in their waiting time tolerances. (We retain τ to denote the particular value of waiting time tolerance, which is the same for all customers.) We assume that the system does not incur any cost for a customer whose demand is satisfied within its waiting time tolerance ($b = 0$); customer dissatisfaction is subsumed in the lost sales cost. We discuss the particulars of each group in the following.

In Group 1, we set $\lambda = 1, 2, 4, 8$. The transit times are selected as $L_1/L_2 = 0.5, 1,$ and 2 with $L_2 = 1$. The holding cost at the retailer, h_1, is defined in multiples of the holding cost at the warehouse such that $h_1/h_2 = 1, 1.25, 1.5, 2$ with $h_2 = 1$. Unit lost sales cost, $\pi = \widetilde{\pi} - \sum_{i=1}^{N} h_i L_i$ with $\widetilde{\pi} =$

5, 25, 125. The customer waiting time tolerance is $\tau = 0, L_1/4, L_1/2, L_1, (L_1 + L_2)/2$. Note that, instances in which the customer tolerance exceeds the retailer transit time ($\tau > L_1$) are excluded from the experimental set. Overall, in this experiment group, we considered 624 instances. (For the special case of $N = 2$, we provide the analytical expressions in the Supplemental Online Materials.)

Group 2 considers a four-stage system ($N=4$) with equal transit times, $L_i = 0.25$ for $\forall i$ and Poisson demand with $\lambda = 16$. The customer waiting tolerance is τ $(= 0, L_1/4, L_1/2)$. The unit lost sales cost π is selected as $\pi = 5, 25, 125$. We assume the unit holding cost rates h_j are the highest at the retailer ($j=1$) and decreasing away from it such that $\Delta h_j = h_j - h_{j+1}$ with $h'_{N+1} = 0$ and $\Delta h_j \in \{0.25, 2.5\}$ $\forall j$. This selection of holding cost rates gives 16 possible cases for each τ and π pair. However, some combinations resulted in the pipeline inventory holding cost exceeding unit lost sales cost. Discarding such cases, in this group, we tested 135 distinct instances.

In Group 3, we considered $N=4$, 8, 16 stages. Following Gallego and Zipkin (1999) and Shang and Song (2003), we chose the holding cost structure in linear, affine, kink and jump forms. For linear holding cost structure, $\Delta h_j = 1/N$. (We retain the notation above herein.) For affine holding costs, $\Delta h_N = \alpha + (1 - \alpha)/N$ and $\Delta h_j = (1 - \alpha)/N$, $j = 1, 2, \ldots, N - 1$. We used $\alpha = 0.25$ and 0.75. For the kink form, we assumed that the change occurs in the middle, $\Delta h_j = (1 - k)/N$, $j \le N/2$ and $\Delta h_j = (1 + k)/N$, $j > N/2$ with $k=0.25$ and 0.75. For the jump cost form, the jump occurs at stage $(N/2) + 1$, $\Delta h_j = u + (1 - u)/N$, $j = N/2$ and $\Delta h_j = (1 - u)/N$ otherwise with $u=0.75$. All transit times are equal across the chain and the total system transit time is set to one. For all holding cost forms, we set the retailer holding cost rate, $h_1 = 1$. We considered $\tau = 0, L_1/4, L_1/2$. The rest of the system parameters are $\lambda = 16$ and $\pi = 5, 25, 125$. In this phase, we test $N \times \tau \times \mathbf{h} \times \pi$ $(162 = 3 \times 3 \times 6 \times 3)$ instances.

Finally, in Group 4, we considered a four-stage ($N=4$) benchmark system where $L_j = 0.25, 1.5$, the holding cost structure is linear with $\Delta h_j = 0.25$ and $h_{N+1} = 0$ $\forall j$, $\lambda = 4$ and $\pi = 25$. We, then, changed the transit time from the benchmark to 0.5 for stages 1, 2, 3, or 4 one at a time. We used $\tau = 0, L_1/4, L_1/2$. For each benchmark cases, we tested $\tau \times \mathbf{L}$ $(15 = 3 \times 5)$ instances. Overall, we have 327 $(= 135 + 162 + 30)$ instances in this group.

4.2. Optimization

In our numerical study, we consider the operating characteristics and the corresponding expected cost rate of (i) the system as modeled by the proposed approximation and (ii) the true system at hand. The true system's characteristics are obtained via simulation as the exact analytical results are not available. Optimization is done in both cases employing the search algorithms discussed below. We use "$\sim$" to refer to the entities obtained by the proposed approximation. That is, $\widetilde{S_i^*}$ denotes the best stocking level for stage i obtained using the approximation, $E[\widetilde{TC^*}]$ denotes the corresponding expected total cost rate as computed using the expressions provided in Section 2. The entities obtained in optimization via simulation of the true system are denoted by the notations $S^*_{i\,sim}$ and $E[TC^*]_{sim}$.

For $2 \le N \le 4$, the search is conducted in a nested manner considering j-stage subsets of the system ($j = 2, \ldots, N$). It starts with a two-stage subset of the system ($j=2$) consisting of the retailer and the immediate upstream stage fed directly by the ample supplier and proceeds by augmenting the subset with the next upstream stage until $j = N$. For every j-stage subset, a local directional improvement algorithm is called with the initial points $\mathbf{S}^0(j) = [S_1^0, \ldots, S_j^0]$ selected as $\{\mathbf{S}^*(j-1) \cup \{S_j^0\}\}$ such that the optimal stocking level vector for a $(j-1)$-stage system denoted by $\mathbf{S}^*(j-1)$ is merged with $\{S_j^0\}$ denoting the mean demand during transit time for stage j, λL_j. The initial point $\mathbf{S}^0(1)$ is taken as the optimal stocking level for the full lost sales single-location model with transit time set to L_1 (retailer's transit time). The improvement algorithm proceeds by comparing all immediate neighbors of a given solution until no further improvement occurs. Immediate neighbors for a solution $\mathbf{S}(j)$ are defined as $([S_1, \ldots, S_j] : S_i \in \{S_i - 1, S_i, S_i + 1\}$ for all i, $1 \ge i \ge j)$; for a j-stage subset, there are $3^j - 1$ immediate nonnegative neighbors of a solution.

For larger systems we considered alternative search routine options that are computationally less burdensome. For systems with $N > 4$, we developed three variants of a greedy single pass algorithm inspired by the Majorization Heuristic proposed in Boyaci and Gallego (2001). The algorithm builds on the neighborhood search results for a four-stage subset of the system as obtained by the search procedure described above. Two variants take the initial four-stage subset as consisting of the stages $(1, 2, 3, 4)$ for any N-stage system, whereas the third variant takes it as consisting of stages $(1, 3, 6, 8)$ for $N=8$ and of stages $(1, 6, 12, 16)$ for $N=16$. Then, the initial solution for the N-stage system is selected such that the stocking levels of the corresponding stages are set to their optimal values obtained for the considered four-stage subsystem and those of the rest of the stages are set to zero. The search algorithm proceeds by re-adjusting the stocking levels and/or increasing the total system stock. We observed that there is no single dominant variant for larger systems; each variant ended up obtaining the best solution in roughly equal proportions. The best performer among the solutions obtained thus is reported as the optimal solution.

Simulation optimization has been carried out using the same search algorithms as for the proposed approximation with 500,000 arrivals and 100,000 time units as stable bounds revealed by preliminary tests.

Admittedly, our search routines do not guarantee optimality. It is reasonable to assume that the objective function is relatively flat around its global optimal. Arguably, our search procedures result in solutions in the neighborhood of the global optimum if not truly the global optimum although being trapped in local optima cannot be wholly excluded.

4.3. Quality of the approximation

We begin our discussion with the quality of the approximation; that is, how well one can compute the operating

Table 1. Summary statistics for quality of the approximation, $\Delta_q\% = 100 \times \frac{(E[\widetilde{TC(\mathbf{S})}] - E[TC(\mathbf{S})]_{sim})}{E[TC(\mathbf{S})]_{sim}}$.

		(mean; median; max); count
Group 1	Underestimation	(−0.05; −0.03; −0.25); 114
	Overestimation	(0.48; 0.27; 2.96); 510
	Overall	(0.39; 0.16; 2.96); 624
Group 2	Underestimation	(−0.025; −0.015; −0.09); 12
	Overestimation	(1.58; 1.25; 3.31); 123
	Overall	(1.17; 1.13; 3.31); 135
Group 3	Underestimation	(−0.21; −0.13; −0.78); 20
	Overestimation	(0.98; 0.83; 3.44); 142
	Overall	(0.83; 0.74; 3.44); 162
Group 4	Underestimation	–
	Overestimation	(1.19; 1.14; 2.33); 30
	Overall	(1.19; 1.14; 2.33); 30

characteristics and the resulting expected cost rate using the approximation provided herein as compared to the values obtained via simulation for a given $\mathbf{S}$ vector. To assess this quality, we considered the solutions (base-stock levels) obtained as best using the approximation. Formally, the quality is defined as

$$\Delta_q\% = 100 \times (E\left[\widetilde{TC(\mathbf{S})}\right] - E[TC(\mathbf{S})]_{sim})/E[TC(\mathbf{S})]_{sim}$$

evaluated at $\mathbf{S} = \widetilde{\mathbf{S}}^*$ where $E[TC(\mathbf{S})]_{sim}$ denotes the expected cost rate obtained by simulating the system at $\mathbf{S}$.

In Table 1, we provide for each experiment group, the average, median and maximum values of $\Delta_q\%$ for all of the instances and for those in which the approximation results in under- or overestimations. Overall, we conclude that the quality of the proposed approximation is very good and robust over system size and cost structure.

4.4. Goodness of the approximate solution

Next, we consider the goodness of the approximate solution $(\widetilde{S_1^*}, \dots, \widetilde{S_N^*})$. This can be assessed by (i) fraction of instances for which the approximation results in the optimal total system stock and the policy parameter values found by simulation ($\widetilde{\sigma_{1,N}^*} = \sigma_{1,N\,sim}^*$, and $\widetilde{S_i^*} = S_{i\,sim}^*\ \forall i$) and, (ii) how much the resulting cost rates deviate from the optimal.

(i) When we consider the total system stock, we find $\widetilde{\sigma_{1,N}^*} = \sigma_{1,N\,sim}^*$ in 829 out of 951 total instances. For the individual experimental groups, we have 619/624, 94/135, 112/162 and 19/30, respectively. For the special case of full lost sales ($\tau = 0$) the corresponding breakdown is 139/144, 33/45, 40/54, and 9/10, respectively. When we consider the optimal solution $(S_{1\,sim}^*, \dots, S_{N\,sim}^*)$, we find that the approximation provides this in 373 out of 951 total instances. In each experimental group, we have 368/624, 2/135, 0/162 and 3/30, respectively. For the special case of full lost sales ($\tau = 0$) the corresponding breakdown is 63/144, 1/45, 0/54 and 0/10, respectively.

The ability of the approximation to find the optimal total system stock value is important in that the solution thus obtained may then serve as a basis for a more refined (but more time-intensive) search if desired. As discussed below, the cost deviations in the approximate solutions *vis a vis* the

Table 2. The overall goodness summary with respect to groups, $\Delta_g\% = 100 \times \frac{(E[TC(\widetilde{\mathbf{S}^*})]_{sim} - E[TC^*]_{sim})}{E[TC^*]_{sim}}$.

		Average	*Median*	*Max*
Group 1	overall	0.14	0	2.41
	$\tau = 0$	0.24	0.06	1.61
Group 2	overall	0.77	0.72	2.56
	$\tau = 0$	0.88	0.85	2.42
Group 3	overall	1.13	1.04	3.26
	$\tau = 0$	1.11	0.91	3.22
Group 4	overall	0.88	1	1.64
	$\tau = 0$	0.92	1.1	1.43
Overall	overall	0.42	0.11	3.26

simulation optimal are reasonably small; this indicates that the impact of allocation of stock across stages on costs is important, but not as significant as that of the total system stock.

(ii) Next, we investigate the goodness of the proposed approximation in terms of the percentage deviations observed in the expected cost rates *vis a vis* the (simulation) optima, defined as

$$\Delta_g\% = 100 \times (E\left[TC(\widetilde{\mathbf{S}^*})\right]_{sim} - E[TC^*]_{sim})/E[TC^*]_{sim}.$$

Across all 951 instances, the proposed approximation results in an average percentage deviation of 0.42%, a median of 0.11% with a maximum of 3.26%. (See Table 2.)

For Group 1 the percentage deviation is larger than 1% in only 13 out of 624 instances, without a discernible pattern in parameters. For Group 2 it is larger than 1% in 48 out of 135 instances and 2% in 2 out of 135 instances and it is larger than 1% in 15 out of 30 instances for Group 4. We observe that the cost structure along the serial system has a significant impact on how well the approximation performs in this measure. We study this in Group 3 whose results are presented in Tables 3 and 4.

Our extensive numerical study demonstrates that the proposed approximation has very high efficacy in obtaining results close to the optima found via simulation.

4.5. Structural observations

The behavior of optimal stocking levels with respect to system and cost parameters exhibit expected patterns: The optimal system stock is non-decreasing in the unit lost sales cost π, arrival rate λ and total transit time across the system.

Numerical evidence also supports the following observations about stock allocation.

OBSERVATION 1. *When $\tau = L_1$, it may be optimal for an N-stage lead-time-dependent serial system to hold stock at the retailer.*

For illustration, consider the stock allocation in the optimal solutions for $N = 2$. There are three possible resulting structures: a two-stage system in which stock is kept at both levels $(S_1 > 0, S_2 > 0)$, a single-stage system without the warehouse $(S_1 > 0, S_2 = 0)$, and, a single-stage system without the retailer $(S_1 = 0, S_2 > 0)$. As reported in Table 5, we observe the last structure in only 6 out of 624 instances, all for $\tau = L_1$. When $\tau = L_1$, all customers are willing to wait

Table 3. Effect of holding cost structure on $\Delta_g\% = 100 \times \frac{(E[TC(\widetilde{S^*})]_{sim} - E[TC^*]_{sim})}{E[TC^*]_{sim}}$ for Group 3.

	Affine 0.25	Affine 0.75	Jump	Kink 0.25	Kink 0.75	Linear	Overall
Average	1.11	0.30	1.38	1.56	1.05	1.40	1.13
Median	1.09	0.27	1.40	1.46	0.89	1.42	1.04
Max	1.68	1.31	2.62	3.26	3.15	2.90	3.26
Count	27	27	27	27	27	27	162

Table 4. Effect of holding cost structure on $\Delta_g\% = 100 \times \frac{(E[TC(\widetilde{S^*})]_{sim} - E[TC^*]_{sim})}{E[TC^*]_{sim}}$ for Group 3 when full lost sales ($\tau = 0$).

	Affine 0.25	Affine 0.75	Jump	Kink 0.25	Kink 0.75	Linear	Overall
Average	1.14	0.23	1.36	1.66	0.75	1.55	1.11
Median	1.09	0.21	1.83	1.46	0.57	1.96	0.91
Max	1.68	1.58	2.62	3.23	1.90	2.35	3.22
Count	9	9	9	9	9	9	54

Table 5. Breakdown of the stocking structures (overall). The number of instances in which a particular structure has been found optimal for the system as modeled by (the proposed approximation; the true system at hand).

τ/L_1	0	0.25	0.5	1
single-stage system without warehouse	(30; 60)	(27; 53)	(27; 48)	(20; 38)
single-stage system without retailer	(0; 0)	(0; 0)	(0; 0)	(6; 6)
two-stage system	(114; 84)	(117; 91)	(117; 96)	(118; 100)

as long as the transit time between the retailer and the next upstream stage. In this case, one might think that holding stock at the retailer may not be needed at all since $b = 0$. However, in 138 such instances of our numerical study we observe that stock is kept at the retailer ($S_1 > 0$). See also Table 6. We explain this finding as follows.

Consider an arbitrary solution such that $S_1 = 0$ and $S_2 = \sigma_{1,2}(> 0)$. By design, the retailer is always out of stock and its expected on-hand inventory is zero. An arriving demand is satisfied if and only if the warehouse has positive on-hand inventory. Now, consider perturbing the stock levels while keeping the total system stock the same, so that $S_1 = 1$ and $S_2 = \sigma_{1,2} - 1(> 0)$. Having *positive* maximum stocking level at the retailer ($S_1 > 0$) now creates an inventory system which may have some items in transit between the two stages. The magnitude of the remaining lead time of the item in transit has no impact on the purchasing decision of an arriving customer that finds the system in this state since $\tau = L_1$. Hence, the demand (sales) process is the same as before. However, the expected on-hand inventory and, thereby, the holding cost at stage 2 are lower, but those of stage 1 are now non-zero. Recalling that holding costs for in-transit items are sunk in the model, if the saving at stage 2 is higher than the cost increase at stage 1, the solution of the perturbed system where the retailer holds stock dominates the previous one. The 138 cases observed in Tables 5 and 6 correspond to such settings. Similar marginal cost improvement argumentation holds for $N > 2$.

Shang and Song (2006) have certain propositions about full backordering serial inventory systems. Similar observations can be made for the setting analyzed herein.

OBSERVATION 2. In an N-stage supply chain with equal transit time and echelon holding cost increments for all stages (that is, the difference between the unit holding cost rates at two neighboring stages), the optimal system stock is non-decreasing in N as total lead time across the chain is kept

Table 6. Breakdown of the stocking structures ($\tau = L_1$). The number of instances in which a particular structure has been found optimal for the system as modeled by (the proposed approximation; the true system at hand).

h_1/h_2	1	1.25	1.5	2
single-stage system without warehouse	(16; 35)	(3; 2)	(0; 0)	(1; 1)
single-stage system without retailer	(0; 0)	(0; 0)	(0; 0)	(6; 6)
two-stage system	(20; 1)	(33; 34)	(36; 36)	(29; 29)

constant (at one), and that as N changes the unit holding cost at the lowest echelon (h_1) stays the same.

This is similar in spirit to Proposition 3 in Shang and Song (2006) for full backordering, which has also been numerically observed by Gallego and Zipkin (1999). To support this observation, we can use the results of Group 3 with the linear structure of holding costs with equal transit times across the chain with a total transit time of one. In creating our test bed, customer tolerances are expressed in terms of the retailer transit times, which change as N changes. Hence, we consider the nominal values of $\tau = 0$ and $\tau = 1/16, 1/32$ for the respective system size pairs of ($N = 4$, 8) and ($N = 8$, 16). For example with $\tau = 0$, we have $\sigma^*_{1,N} = 23, 23, 23$ for $\pi = 5$ and $\sigma^*_{1,N} = 30, 30, 31$ for $\pi = 125$ for $N = 4, 8, 16$.

OBSERVATION 3. There might be instances where, in an N-stage system with fixed total transit time and equal echelon holding cost increment for all stages, given the same amount of transit-time increment, the required total system stock needed remains the same or is smaller if that transient time increment happens at a downstream stage.

This is similar in spirit to Proposition 4 in Shang and Song (2006) for full backordering. To support this observation, we can use the results obtained for experimental Group 4 where we change the transit times one at a time while the holding costs are linear across the system. For the case with base transit time equal to 0.25, our results support the observation. However, for the case with the base transit time of 1.5 we observed some anomalies that cannot be explained via single-location insights. Related to this phenomenon, we conjecture that as τ increases, this is equivalent to decreasing the retailers transit time, which implies that the total transit time demand decreases so that total system stock decreases.

OBSERVATION 4. In an N-stage supply chain with equal transit time for all stages, system stock increases if a substantially value-adding step moves to a downstream stage.

This is similar in spirit to Proposition 5 in Shang and Song (2006) for full backordering. For this observation we can use the results obtained for experimental Group 2 where we change echelon holding costs one at a time.

OBSERVATION 5. In an N-stage supply chain with equal holding cost increments and transit times for all stages, the optimal base-stock level at a stage does not increase as we move upstream.

This is similar in spirit to Proposition 6 in Shang and Song (2006) for full backordering. This observation is supported by our results in Group 1 and 3 with linear holding cost structure for $N = 4$ taking into account the flatness of the cost surface around the optima.

OBSERVATION 6. For any stage $j = 1, \ldots, N$, more stock is allocated to a stage with longer transit time and/or lower holding cost.

This is similar in spirit to Proposition 7 in Shang and Song (2006) for full backordering. The observation is supported by our results for Group 4 with $L_j = 0.25$ for transit times and Group 2 for holding costs.

5. Conclusion

In this article, we consider a serial inventory system under continuous review $(S-1, S)$ policies at all stages facing Poisson demand when customers have waiting time tolerances. We develop the system equations that define the stationary probability distribution of a stochastic process which mimics the inventory system at hand. Based on the solution structures for the special cases, we propose an approximate solution for the general serial system. The approximation rests on re-computing the purchase probability of a customer on the basis of the age of the oldest item at the lowest stage *as if* purchase instances follow a Poisson process. An extensive numerical study indicates that the proposed approximation performs very well in comparison with a simulation optimization benchmark. Our approximation allows for a generalization of the demand process to consider purchase decisions that may depend on the actual age or resulting utility of an item. This would be the case when customers have age preferences (corresponding utilities) for the items. When customers are not allowed to choose the item that they purchase or when the inventory manager is unaware of their utility functions (that is, the *FIFO* issuance policy is employed), our model is directly applicable. Another direct use of our model is that it provides an approximation for the full lost sales model in a serial setting. Furthermore, we assumed constant transit times between stages. In the case of exogenous random transit times (where order crossing cannot occur), our model is also directly applicable with the slight modification that $\alpha(.)$ needs to be computed by considering possible values of the transit times between stages. Thus, the model herein provides a building block for richer operational problems. As future work, we envision incorporating some form of rationing policies and/or novel issuing policies which may be beneficial in lead-time-dependent backordering settings.

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