Kerr-Schild-Kundt metrics in generic Einstein-Maxwell theories

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We study the Kerr-Schild-Kundt class of metrics in generic gravity theories with Maxwell's field. We prove that these metrics linearize and simplify the field equations of generic gravity theories with Maxwell's field.

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I. INTRODUCTION

In the last decade, in a series of papers [1–6], we showed that the Kerr-Schild-Kundt (KSK) types of metrics are universal. This means that the KSK metrics reduce the field equations of any generic gravity theory to a linear equation for the metric function $V(x)$ (see below). By using this result, we have studied some special cases, such as quadratic gravity, $F(\text{Riemann})$ gravity, cubic gravity theories, and found AdS-plane and pp-wave solutions of these theories. For the universality and almost universality of the KSK metrics, see also Refs. [7–12].

The KSK metrics are defined by the spacetime metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V l_\mu l_\nu, \tag{1}$$

which is in the "generalized" Kerr-Schild form [13,14]. Here, $\bar{g}_{\mu\nu}$ represents the background spacetime, $V(x)$ is a scalar field called the profile function, and l^μ is a null vector field. The background metric $\bar{g}_{\mu\nu}$ is assumed to be maximally symmetric, and as such, its Riemann tensor satisfies the following property:

$$\bar{R}_{\mu\alpha\nu\beta} = K(\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} - \bar{g}_{\nu\alpha}\bar{g}_{\mu\beta}), \quad K = \frac{\bar{R}}{D(D-1)} = \text{const.}, \tag{2}$$

where K is the curvature constant, and it is related to the background Ricci scalar $\bar{R}$ and the spacetime dimension D, as seen. Therefore, depending on the value of K, the background might be either the Minkowski, de Sitter (dS), or anti–de Sitter (AdS) spacetime, for which $K = 0$, $K > 0$, or $K < 0$, respectively. The profile function $V(x)$ and the vector field l^μ in Eq. (1) together satisfy the relations

$$l_\mu l^\mu = 0, \qquad \nabla_\mu l_\nu = \frac{1}{2}(l_\mu \xi_\nu + l_\nu \xi_\mu), \tag{3}$$

$$l_\mu \xi^\mu = 0, \qquad l^\mu \partial_\mu V = 0, \tag{4}$$

with ξ^μ being an arbitrary vector field which becomes explicit for a specific background metric. With these properties, one can show that the inverse metric $g^{\mu\nu}$, the Einstein tensor $G_{\mu\nu}$, and the trace-free Ricci tensor $S_{\mu\nu}$ are (see, e.g., Ref. [2])

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - 2V l^\mu l^\nu, \tag{5}$$

$$G_{\mu\nu} = -\frac{(D-1)(D-2)}{2} K g_{\mu\nu} + S_{\mu\nu}, \quad S_{\mu\nu} = -\rho l_\mu l_\nu, \tag{6}$$

where

$$\rho = \left[\bar{\Box} + 2\xi^\alpha \partial_\alpha + \frac{1}{2}\xi_\alpha \xi^\alpha + 2(D-2)K\right] V \equiv -\mathcal{O}V, \tag{7}$$

with $\bar{\Box} \equiv \bar{\nabla}_\mu \bar{\nabla}^\mu$ and $\bar{\nabla}_\mu$ being the covariant derivative with respect to the background metric $\bar{g}_{\mu\nu}$.

In this work, we consider the most general gravity theory coupled with an electromagnetic field. The Lagrange function of the whole theory depends on the curvature tensor, the electromagnetic field, and their covariant derivatives at any order. We call such a theory a "generic Einstein-Maxwell theory." We then assume that the spacetime metric is of the KSK form defined above. With this assumption, we prove a theorem stating that the KSK metrics simplify the field equations of any generic Einstein-Maxwell theory. To prove this theorem, we use the technique that has been used in Ref. [6]. As an explicit example, we examine the Horndeski vector-tensor theory [15], which generalizes the Einstein-Maxwell theory by adding some special curvature-electromagnetic couplings, and we write its field equations in the KSK spacetimes.

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Our paper is structured as follows: In Sec. II, we review the universality of the KSK metrics for a generic gravity theory. In Sec. III, we give the generalization of the universality property given in Sec. II by considering a generic gravity theory with Maxwell's field. In Sec. IV, we present the Horndeski vector-tensor theory as an explicit example for our formulation, and we conclude in Sec. V.

II. UNIVERSALITY OF KERR-SCHILD-KUNDT METRICS

In a recent paper [6], it has been proved that the KSK metrics given by the form in Eq. (1) satisfying Eqs. (3) and (4) simplify the field equations of any generic gravity theory constructed from the Riemann tensor and its covariant derivatives at any order. Here we shall now give a brief review of this property—the *universality* of the KSK metrics.

A vacuum generic gravity theory can be described by the action

$$I = \int d^D x \sqrt{-g} f(g, R, \nabla R, \ldots), \tag{8}$$

where f is a smooth function of the metric tensor g, the Riemann tensor R, the covariant derivative of the Riemann tensor ∇R, and the higher-order covariant derivatives of R, respectively. For the KSK metrics, it can be shown that the field equations of the theory in Eq. (8), obtained by variation with respect to the metric $g_{\mu\nu}$, take the form (see, e.g., Ref. [6])

$$E_{\mu\nu} \equiv e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0, \tag{9}$$

where $S_{\mu\nu}$ is the traceless Ricci tensor and $\Box$ is the d'Alembertian with respect to $g_{\mu\nu}$. The derivative order of the theory becomes $2N+2$, such that $N=0$ represents the Einstein gravity and $N=1$ represents the quadratic curvature gravity, or more generally $F(\text{Riemann})$ theories. Taking the trace of Eq. (9) produces the scalar equation

$$e = 0, \tag{10}$$

which determines the effective cosmological constant in terms of the parameters of the theory. Inserting Eq. (10) into Eq. (9) produces the traceless part

$$\sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0, \tag{11}$$

which must be satisfied independently. This is a nontrivial nonlinear differential equation which cannot be solved in general, except for some trivial cases. However, it has been shown in Ref. [4] that Eq. (11) can also be written as the linear equation

$$l_\mu l_\nu \sum_{n=0}^{N} a_n (-1)^n (\mathcal{O} - 2K)^n \mathcal{O} V = 0, \tag{12}$$

since $S_{\mu\nu} = -\rho l_\mu l_\nu$ and

$$\Box^n S_{\mu\nu} = (-1)^n l_\mu l_\nu (\mathcal{O} - 2K)^n \mathcal{O} V \tag{13}$$

for the KSK metrics. Here, $\mathcal{O}$ is the operator defined in Eq. (7). This result is true for any ξ_μ satisfying $l_\mu \xi^\mu = 0$, the first condition in Eq. (4). For $N \geq 1$, it is further possible to factorize Eq. (12) as

$$\prod_{n=0}^{N} (\mathcal{O} + b_n) \mathcal{O} V = 0, \tag{14}$$

where b_n's are related to a_n's, and so to the parameters of the theory. Now, if all b_n's are distinct and nonzero, the most general solution of Eq. (14) can be given in the form

$$V = V_E + V_1 + V_2 + \cdots + V_N, \tag{15}$$

where V_E is the solution of the Einstein gravity equation

$$\mathcal{O} V_E = 0, \tag{16}$$

and each V_n, for $n = 1, 2, \ldots, N$, is the solution of the quadratic curvature gravity equation

$$(\mathcal{O} + b_n) V_n = 0. \tag{17}$$

At this point, it is worth mentioning that there are some special cases in which some or all of the b_n's coincide or vanish. In these cases, fourth- or higher-power operators, such as $(\mathcal{O} + b_n)^2$, appear, and log-type solutions, which exist in the so-called critical theories, arise in the solution spectrum of the generic gravity theory. Equations (16) and (17) can easily be solved for V_E and V_n by using such techniques as the method of separation of variables and the method of Green's function.

In the proof of the universality theorems for the KSK metrics [6], we use some properties of the null vector l_μ. First, note that the contractions of l^μ with l_μ, ξ_μ, and $\partial_\mu V$ yield zero. Second, the contractions of l^μ with the first-order derivatives of ξ_μ and $\partial_\mu V$ yield

$$l^\nu \nabla_\mu \xi_\nu = -\frac{1}{2} l_\mu \xi^\nu \xi_\nu, \tag{18}$$

$$\nabla_\mu \xi^\mu = -\frac{1}{4} \xi^\mu \xi_\mu + \frac{2D-3}{D(D-1)} R, \tag{19}$$

$$l^\mu \nabla_\mu \xi_\alpha = -l_\alpha \left(\frac{1}{4} \xi^\mu \xi_\mu - \frac{1}{D(D-1)} R \right), \tag{20}$$

$$l^\mu \nabla_\mu \partial_\nu V = l^\mu \nabla_\nu \partial_\mu V = -\frac{1}{2} l_\nu \xi^\mu \partial_\mu V. \tag{21}$$

So, here are the important points to observe (see Ref. [6] for more details):

(1) The number of l vectors is preserved, since a free-index l always appears in the results.
(2) The contraction with the l vector removes the first-order derivatives acting on ξ_μ and $\partial_\mu V$.
(3) The contraction of the l vector with the higher-order derivatives of ξ_μ and $\partial_\mu V$ produce free-indexed l vectors.

We define the l-degree of a tensor as the number of free-indexed l vectors contained. For example, the l-degree of the Weyl tensor is 2 [6]. According to this definition, from the above discussions, we can say that the contraction of the l^μ vector with the covariant derivatives of the vectors ξ_μ and $\partial_\mu V$ preserves the l-degree of the relevant tensor. Our definition of l-degree of a tensor is equivalent to the boost weight of a tensor defined by Coley *et al.* [7] (and see the references therein).

III. GENERIC GRAVITY THEORIES WITH MAXWELL'S FIELD

Now, we wish to extend the theorem given in Sec. II on the universality of the KSK metrics [6] to generic gravity theories with the Maxwell field. The Lagrange function of such a theory should contain the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, the Riemann tensor $R_{\alpha\beta\mu\nu}$, the Maxwell field tensor $F_{\mu\nu}$, and the covariant derivatives of these tensors of all orders. That is, in D dimensions, the most general action for the Einstein-Maxwell theory is

$$I = \int d^D x \sqrt{-g} L(g, R, \nabla\nabla \ldots \nabla R, F, \nabla\nabla \ldots \nabla F). \tag{22}$$

Let the electromagnetic vector potential be given by $A_\mu = \phi l_\mu$, where ϕ is a function satisfying the condition $l^\mu \phi_{,\mu} = 0$. Then, the Maxwell field tensor takes the form

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \phi_{,\mu} l_\nu - \phi_{,\nu} l_\mu, \tag{23}$$

which satisfies the following conditions:

$$F_{\mu\nu} F^{\mu\nu} = 0, \tag{24}$$

$$l_\mu F^{\mu\nu} = 0, \tag{25}$$

$$F_{\mu\alpha} F_\nu{}^\alpha = \psi l_\mu l_\nu, \tag{26}$$

where $\psi = g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}$. For the extension of the universality theorem to generic gravity theories with an antisymmetric tensor $F_{\alpha\beta}$, we use the following notation:

(1) $\nabla^n F$ denotes n-number of covariant derivatives of the F tensor—i.e., $\nabla_{\alpha_1} \nabla_{\alpha_2} \cdots \nabla_{\alpha_n} F_{\mu\nu}$.
(2) $[(\nabla^n F)(\nabla^m F)]_{\mu\nu}$ denotes a second-rank symmetric tensor obtained from the product tensors $(\nabla^n F)(\nabla^m F)$ of rank $(4 + m + n)$.

With all these, we now have the following theorem:

Theorem 1: Let the spacetime metric be given by the Kerr-Schild-Kundt (KSK) type

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V l_\mu l_\nu,$$

with the properties

$$l^\mu l_\mu = 0, \quad \nabla_\mu l_\nu = \xi_{(\mu} l_{\nu)}, \quad \xi_\mu l^\mu = 0, \quad l^\mu \partial_\mu V = 0,$$

and let the electromagnetic vector potential $A_\mu = \phi l_\mu$, or the Maxwell field tensor

$$F_{\mu\nu} = \phi_{,\mu} l_\nu - \phi_{,\nu} l_\mu,$$

with the property $l^\mu \phi_{,\mu} = 0$, where $\bar{g}_{\mu\nu}$ is the metric of a space of constant curvature (A)dS. Then, any second-rank symmetric tensor constructed from the Riemann tensor, Maxwell's field tensor, and their covariant derivatives can be written as a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, $F_\mu{}^\alpha F_{\nu\alpha}$, and their higher derivatives in the forms $\Box^n S_{\mu\nu}$ and $[(\nabla^n F)(\nabla^m F)]_{\mu\nu}$ for all m and n, where $\Box$ represents the d'Alembertian with respect to $g_{\mu\nu}$; that is,

$$E_{\mu\nu} \equiv e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} + \sum_{m=0,n=0}^{M} b_{mn} [(\nabla^n F)(\nabla^m F)]_{\mu\nu}, \tag{27}$$

and

$$E^\mu \equiv \nabla_\alpha \left[\sum_{n=0}^{M} c_n \Box^n F^{\alpha\mu} \right], \tag{28}$$

where a_n, b_{mn}, and c_n are constants coming from the parameters of the theory and N and M are numbers related to the derivative orders in the theory. Then the associated field equations of the generic Einstein-Maxwell theory are $E_{\mu\nu} = 0$ and $E_\mu = 0$.

Sketch of the Proof: The most general Lagrange function for the generic Einstein-Maxwell theory given in Eq. (22) can be written as follows:

$$\begin{aligned} L = {} & L_1(g, R, \nabla R, \nabla\nabla R, \ldots) \\ & + L_2(g, R, \nabla R, \nabla\nabla R, \ldots, F, \nabla\nabla F, \ldots) \\ & + L_3(g, F, \nabla F, \nabla\nabla F, \ldots), \end{aligned} \tag{29}$$

where L_1 is a function of the curvature tensor and its covariant derivatives of any order, L_2 is a function representing the coupling of the electromagnetic tensor F and the curvature tensor R at any order, and L_3 is a function depending solely on F and its covariant derivatives of any order. Then the field equations associated with the above Lagrange function can be written as

$$E^1_{\mu\nu} + E^2_{\mu\nu} + E^3_{\mu\nu} = 0, \tag{30}$$

$$E^2_{\mu} + E^3_{\mu} = 0, \tag{31}$$

where $E^1_{\mu\nu}, E^2_{\mu\nu}, E^3_{\mu\nu}$ are the tensors obtained from the variation of the action in Eq. (22) with respect to the metric tensor, and E^2_μ and E^3_μ are the vectors obtained from the variation of the action with respect to the electromagnetic vector potential vector A_μ. All of these two-rank symmetric tensors and the vectors have the following forms in general:

$$E^1_{\mu\nu} = e g_{\mu\nu} + \sum_{n_0,n_1,\ldots,n_k} C^1_{n_0,n_1,\ldots n_k}[R^{n_0}\nabla^{n_1}R\nabla^{n_2}R\cdots\nabla^{n_k}R]_{\mu\nu}, \tag{32}$$

$$E^2_{\mu\nu} = \sum_{n_0,n_1,\ldots,n_k,s_0,s_1,\ldots s_k} C^2_{n_0,n_1,\ldots n_k,s_0,s_1,\ldots s_k} \times [R^{n_0}\nabla^{n_1}R\nabla^{n_2}R\cdots\nabla^{n_k}RF^{s_0}\nabla^{s_1}F\cdots]_{\mu\nu}, \tag{33}$$

$$E^3_{\mu\nu} = \sum_{t_0,t_1,\ldots t_k} C^3_{t_0,t_1,\ldots t_k}[F^{t_0}\nabla^{t_1}F\cdots\nabla^{t_k}F]_{\mu\nu}, \tag{34}$$

where the coefficients C^1, C^2, and C^3 are all constants; e is a function of scalars obtained from the Riemann tensor, the electromagnetic field tensor, and their covariant derivatives; and

$$E^2_{\mu} = \sum_{n_0,n_1,\ldots,n_k,s_0,s_1\ldots s_k} C^4_{n_0,n_1,\ldots n_k,s_0,s_1\ldots s_k} \times [R^{n_0}\nabla^{n_1}R\nabla^{n_2}R\cdots\nabla^{n_k}RF^{s_0}\nabla^{s_1}F\cdots]_{\mu}, \tag{35}$$

$$E^3_{\mu} = \sum_{t_0,t_1,\ldots t_k} C^5_{t_0,t_1,\ldots t_k}[F^{t_0}\nabla^{t_1}F\cdots\nabla^{t_k}F]_{\mu}, \tag{36}$$

where C^4 and C^5 are constants. To proceed further, we now consider typical monomials in each of $E^1_{\mu\nu}, E^2_{\mu\nu}, E^3_{\mu\nu}$, and in E^2_μ and E^3_μ.

After inserting the KSK metric tensor into Eq. (32) and using $R_{\mu\nu\alpha\beta} = K(g_{\mu\alpha}g_{\nu\beta} - g_{\nu\alpha}g_{\mu\beta}) + r_{\mu\nu\alpha\beta}$, where $r_{\mu\nu\alpha\beta}$ is a tensor depending on the vectors ξ_μ, $\partial_\nu V$ and their covariant derivative at any order (see Ref. [6] for the explicit expression), one can reduce $E^1_{\mu\nu}$ to

$$E^1_{\mu\nu} = e_0 g_{\mu\nu} + \sum_{n_0,n_1,\ldots,n_k} \bar{C}^1_{n_0,n_1,\ldots n_k}[r^{n_0}\nabla^{n_1}r\nabla^{n_2}r\cdots\nabla^{n_k}r]_{\mu\nu}, \tag{37}$$

where e_0 is a constant and $\bar{C}^1$ are constants. A typical monomial in $E^1_{\mu\nu}$ is, therefore,

$$[r^{n_0}\nabla^{n_1}r\nabla^{n_2}r\cdots\nabla^{n_k}r]_{\mu\nu}. \tag{38}$$

Since the l-degree of $r_{\mu\nu\alpha\beta}$ is 2, the number of free l vectors in such a monomial is $2n_0 + 2k$. Since the contraction of the l vector with ξ_μ and $\partial_\mu V$ yields zero, and with their covariant derivatives of any order, this keeps the number of free l-vectors unchanged, in order to have a nonzero term in the monomial [Eq. (38)] at the end of the contractions, it must be that $2n_0 + 2k = 2$, which can only be satisfied either when $n_0 = 1$ or when $k = 1$. This means that [6]

$$E^1_{\mu\nu} = e_0 g_{\mu\nu} + \rho_1 S_{\mu\nu} + \rho_2[\nabla\nabla\cdots\nabla r]_{\mu\nu}, \tag{39}$$

where ρ_1 and ρ_2 are some scalars containing V, ϕ, and their partial derivatives. This result is equivalent to (by the use of Bianchi identities) [6]

$$E^1_{\mu\nu} = e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu}. \tag{40}$$

A typical monomial of $E^2_{\mu\nu}$ can be written from Eq. (33) as

$$[R^{n_0}\nabla^{n_1}R\nabla^{n_2}R\cdots\nabla^{n_k}RF^{s_0}\nabla^{s_1}F\cdots\nabla^{s_k}F]_{\mu\nu}. \tag{41}$$

When we insert $R_{\mu\nu\alpha\beta} = K(g_{\mu\alpha}g_{\nu\beta} - g_{\nu\alpha}g_{\mu\beta}) + r_{\mu\nu\alpha\beta}$ into the above monomial, the terms coming from the K part of the curvature tensor reduce to monomials with fewer r tensors, and also to the monomials containing only Maxwell fields—i.e., they join with $E^3_{\mu\nu}$. There will be no contributions to the e_0 part of the field equations $E^1_{\mu\nu}$ in Eq. (39) from such monomials. The remaining part of the monomial will therefore be exactly of the above form, but instead of R's we now have r's:

$$[r^{n_0}\nabla^{n_1}r\nabla^{n_2}r\cdots\nabla^{n_k}rF^{s_0}\nabla^{s_1}F\cdots\nabla^{s_k}F]_{\mu\nu}. \tag{42}$$

For the KSK ansatz, we let $A_\mu = \phi l_\mu$, where ϕ is a function satisfying $l^\mu\partial_\mu\phi = 0$, and $F_{\mu\nu} = \phi_{,\mu}l_\nu - \phi_{,\nu}l_\mu$. Then, the l-degrees of $r_{\mu\nu\alpha\beta}$ and $F_{\mu\nu}$ are 2 and 1, respectively. The number of free l vectors in the bracket is $2n_0 + 2k + s_0 + k$, which must be equal to 2 for having nonvanishing terms. Since these monomials must contain both r and F tensors, it is easy to see that $2n_0 + 2k + s_0 + k > 2$; for this reason, all such coupling terms must vanish. This means that, for KSK metrics and for $F_{\mu\nu} = \phi_{,\mu}l_\nu - \phi_{,\nu}l_\mu$, there will be no coupling of the tensors r and F; such terms vanish identically. A typical monomial of $E^3_{\mu\nu}$ in Eq. (34) can be given as

$$[F^{t_0}\nabla^{t_1}F\cdots\nabla^{t_k}F]_{\mu\nu}. \tag{43}$$

The number of free l vectors in this expression is t_0+k. After contractions, this number will be preserved, and hence, for nonvanishing terms, we must have $t_0+k=2$. This means that either $t_0=2$ (F^2 term), or $k=2$ ($\nabla\nabla\cdots\nabla F\nabla\nabla\cdots\nabla F$ terms), or $t_0=1, k=1$ (symmetrized $F\nabla\nabla\cdots\nabla F$ terms). Combining these, we get

$$E^3_{\mu\nu}=\sum_{m=0,n=0}^{M}b_{mn}[(\nabla^n F)(\nabla^m F)]_{\mu\nu}, \tag{44}$$

where b_{mn}'s are constants. This completes the proof of the first part of Theorem 1.

To prove the second part of the theorem, we use the same approach. A typical monomial of E^2_μ in Eq. (35) can be written as

$$[R^{n_0}\nabla^{n_1}R\nabla^{n_2}R\cdots\nabla^{n_k}RF^{s_0}\nabla^{s_1}F\cdots\nabla^{s_k}F]_\mu. \tag{45}$$

After inserting $R_{\mu\nu\alpha\beta}=K(g_{\mu\alpha}g_{\nu\beta}-g_{\nu\alpha}g_{\mu\beta})+r_{\mu\nu\alpha\beta}$, the terms related to the K part of the curvature tensors in the above monomials reduce either to the same type of monomials with fewer r's or to monomials of E^3_μ in Eq. (36). Hence, we can study the above monomials, only with r's instead of R's. In such a case, the number of free l vectors is $2n_0+2k+s_0+k$. To have nonzero terms in the monomial, we must have $2n_0+2k+s_0+k=1$, but this is not possible, because such monomials represent couplings between r and F tensors, and so $2n_0+2k+s_0+k\neq 1$ for all cases. That is to say, $E^2_\mu=0$ identically. Finally, a typical monomial of E^3_μ in Eq. (36) can be given by

$$[F^{t_0}\nabla^{t_1}F\cdots\nabla^{t_k}F]_\mu. \tag{46}$$

The number of free l vectors in the above expression is t_0+k. For nonzero terms, this number must be equal to 1; therefore, we must have either $t_0=1$ (not possible) or $k=1$ ($\nabla\nabla\cdots\nabla F$ terms). Thus, we obtain, by the use of Bianchi identities,

$$E^3_\mu=\nabla_\alpha\left[\sum_{n=0}^{M}c_n\Box^n F^\alpha{}_\mu\right]=0. \tag{47}$$

This completes the proof of Theorem 1.

Remark 1: In the case of the KSK metrics, it is straightforward to show that the Maxwell equations in Eqs. (28) and (47) can also be written as

$$E^\mu\equiv\sum_{n=0}^{M}\bar{c}_n\Box^n(\nabla_\alpha F^{\alpha\mu})=0. \tag{48}$$

For the KSK metrics, the trace of Eq. (27) reduces to $e=0$, which gives a relation between the parameters of the theory and the cosmological constant, and the remaining part of Eq. (27) gives

$$\sum_{n=0}^{N}a_n\mathcal{O}^{n+1}V+\rho_e=0, \tag{49}$$

where ρ_e is the source term for the equation of V, and the operator $\mathcal{O}$ is defined in Eq. (7)—namely, it is given by

$$\mathcal{O}V=-\left[\bar{\Box}+2\xi^\alpha\partial_\alpha+\frac{1}{2}\xi_\alpha\xi^\alpha+2(D-2)K\right]V. \tag{50}$$

On the other hand, Eq. (28) reduces to

$$\sum_{n=0}^{M}c_n\mathcal{R}^n\eta=0, \tag{51}$$

where

$$\eta=\bar{\Box}\phi+\xi^\alpha\phi_{,\alpha}, \tag{52}$$

and the operator $\mathcal{R}$ is defined by

$$\mathcal{R}\eta=[\bar{\Box}+\xi^\alpha\partial_\alpha+(D-1)K]\eta. \tag{53}$$

To derive the above operators, we used the following identities:

$$\Box l_\mu=(D-1)Kl_\mu, \tag{54}$$

$$l^\alpha\bar{\nabla}_\alpha\xi_\mu=\left(K-\frac{1}{4}\xi^\alpha\xi_\alpha\right)l_\mu, \tag{55}$$

$$\bar{\nabla}^\alpha\xi_\alpha+\frac{1}{4}\xi^\alpha\xi_\alpha-(2D-3)K=0, \tag{56}$$

and

$$\bar{\nabla}_\nu\xi_\beta=\frac{1}{2}\xi_\nu\xi_\beta+2K\bar{g}_{\nu\beta}+n_\nu l_\beta+2n_\beta l_\nu-\mu l_\nu l_\beta, \tag{57}$$

where μ is a function and n_μ is a vector satisfying

$$l^\alpha n_\alpha=-\frac{1}{4}\xi^2-K,$$

where $\xi^2=\xi^\alpha\xi_\alpha$. We also have

$$\nabla_\nu\xi^2=(\xi^2+4K)\xi_\nu+4(\xi^\alpha n_\alpha)l_\nu.$$

Since $p_\mu l^\mu = 0$, where $p_\mu \equiv \partial_\mu \phi$, it is now easy to calculate

$$(\bar{\nabla}_\mu \xi_\nu) p^\mu p^\nu = \frac{1}{2}(\xi_\mu p^\mu)^2 + 2K p^\mu p_\mu. \tag{58}$$

As a final remark, by using the three steps below, we can express $[(\nabla^n F)(\nabla^m F)]_{\mu\nu}$ in Eq. (27) for any n and m proportional to $l_\mu l_\nu$:

(1) The number of l vectors is preserved, since a free-index l always appears in the results.
(2) The contraction with the l vector removes the first-order derivatives acting on ξ_μ, $\partial_\mu V$, and $\partial_\mu \phi$.
(3) The contraction of the l vector with the higher-order derivatives of ξ_μ, $\partial_\mu V$, and $\partial_\mu \phi$ produces free-indexed l vectors.

For illustration, we give the special cases where (i) the Lagrange function depends only on F, and (ii) the Lagrange function depends on F up to the first-order covariant derivatives as the following corollaries.

Corollary 1: If the Lagrange function contains only F's (no derivatives), then the reduced field equations are

$$E_{\mu\nu} = e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} + b\tau_{\mu\nu} = 0 \tag{59}$$

and

$$E^\mu = \nabla_\alpha F^{\alpha\mu} = 0, \tag{60}$$

where $\tau_{\mu\nu} = F_\mu{}^\alpha F_{\nu\alpha} = \psi l_\mu l_\nu$.

Corollary 2: If the Lagrange function contains F's and first derivatives of F's, then the reduced field equations are

$$\begin{aligned} E_{\mu\nu} = {} & e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} + b_1 \tau_{\mu\nu} + b_2 \Box \tau_{\mu\nu} \\ & + b_3 \nabla_\gamma F_{\mu\alpha} \nabla^\gamma F_\nu{}^\alpha + b_4 \nabla_\mu F_{\beta\alpha} \nabla_\nu F^{\beta\,\alpha} \\ & + b_5 \nabla_\alpha F_{\beta\mu} \nabla^\beta F^\alpha{}_\nu = 0 \end{aligned} \tag{61}$$

and

$$E^\mu = c_1 \nabla_\alpha F^{\alpha\mu} + c_2 \nabla_\alpha \Box F^{\alpha\mu} = 0. \tag{62}$$

Remark 2: In Corollary 2, there are five different symmetric tensors obtained by the first derivatives of F's:

$$1)\quad (\Box F_{\mu\alpha}) F_\nu{}^\alpha + F_{\mu\alpha} \Box F_\nu{}^\alpha, \tag{63}$$

$$2)\quad \Box (F_{\mu\alpha} F_\nu{}^\alpha), \tag{64}$$

$$3)\quad \nabla_\gamma F_{\mu\alpha} \nabla^\gamma F_\nu{}^\alpha, \tag{65}$$

$$4)\quad \nabla_\mu F_{\beta\alpha} \nabla_\nu F^{\beta\,\alpha}, \tag{66}$$

$$5)\quad \nabla_\alpha F_{\beta\mu} \nabla^\beta F^\alpha{}_\nu. \tag{67}$$

But the first and second terms are not independent; they can be expressed in terms of the others:

$$\Box (F_{\mu\alpha} F_\nu{}^\alpha) = 2\nabla_\gamma F_{\mu\alpha} \nabla^\gamma F_\nu{}^\alpha + (\Box F_{\mu\alpha}) F_\nu{}^\alpha + F_{\mu\alpha} \Box F_\nu{}^\alpha.$$

Using the Bianchi identity for F's, we get

$$\nabla_\mu F_{\beta\alpha} \nabla_\nu F^{\beta\,\alpha} = 2\nabla_\gamma F_{\mu\alpha} \nabla^\gamma F_\nu{}^\alpha - 2\nabla_\alpha F_{\beta\mu} \nabla^\beta F^\alpha{}_\nu.$$

On the other hand, for the KSK metric and $M = 2$, we find

$$\nabla_\gamma F_{\mu\alpha} \nabla^\gamma F_\nu{}^\alpha = \rho_1 l_\mu l_\nu, \tag{68}$$

$$\nabla_\mu F_{\beta\alpha} \nabla_\nu F^{\beta\,\alpha} = \rho_2 l_\mu l_\nu, \tag{69}$$

$$\nabla_\alpha F_{\beta\mu} \nabla^\beta F^\alpha{}_\nu = \rho_3 l_\mu l_\nu, \tag{70}$$

where

$$\rho_1 = -\frac{1}{2}(p_\alpha \xi^\alpha)^2 + \left(\nabla_\alpha p_\beta + \frac{1}{2} p_\alpha \xi_\beta\right)\left(\nabla^\alpha p^\beta + \frac{1}{2} p^\alpha \xi^\beta\right), \tag{71}$$

$$\rho_2 = -(p_\alpha \xi^\alpha)^2 + \frac{1}{2}(p_\alpha p^\alpha)(\xi^\beta \xi_\beta), \tag{72}$$

$$\rho_3 = \nabla_\alpha p_\beta \nabla^\alpha p^\beta + p^\alpha \xi^\beta \nabla_\alpha p_\beta, \tag{73}$$

so that $\rho_2 = 2(\rho_1 - \rho_3)$, and hence we can set $b_5 = 0$. Here we have defined $p_\mu \equiv \partial_\mu \phi$.

IV. HORNDESKI'S VECTOR-TENSOR THEORY: AN EXPLICIT EXAMPLE

As an explicit example, we shall consider Horndeski's vector-tensor theory, which is a generalization of Einstein-Maxwell theory that leads to second-order equations of motion and satisfying charge conservation. This theory is described, in D dimensions, by the action [15]

$$I = \int d^D x \sqrt{-g} \left[\frac{R - 2\Lambda}{2\kappa^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sigma \mathcal{R}^{\mu\nu}_{\alpha\beta} F_{\mu\nu} F^{\alpha\beta}\right], \tag{74}$$

where the parameters κ^2, Λ, and σ are the gravitational constant, the cosmological constant, and the Horndeski coupling constant, respectively, and

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{75}$$

$$\mathcal{R}^{\mu\nu}{}_{\alpha\beta} \equiv -\frac{1}{4} \delta^{\mu\nu\lambda\sigma}_{\alpha\beta\rho\tau} R^{\rho\tau}{}_{\lambda\sigma}. \tag{76}$$

Making explicit use of the generalized Kronecker delta defined by

$$\delta^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_k} = k!\delta^{[\alpha_1}_{\beta_1}\ldots\delta^{\alpha_k]}_{\beta_k} = k!\delta^{\alpha_1}_{[\beta_1}\ldots\delta^{\alpha_k}_{\beta_k]}, \tag{77}$$

one can show that the Horndeski interaction term in Eq. (74) can also be written as

$$\mathcal{R}^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}F^{\alpha\beta} = -RF^2 + 4R_\mu{}^\nu F_{\nu\alpha}F^{\mu\alpha} - R^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}F^{\alpha\beta}, \tag{78}$$

where $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$, and $R_\mu{}^\nu$ is the Ricci tensor. The field equations derived from the action (74) are

$$G_\mu{}^\nu + \Lambda\delta^\nu_\mu = \kappa^2(T_\mu{}^\nu + \sigma\tau_\mu{}^\nu), \tag{79}$$

$$\nabla_\nu\mathcal{F}^{\mu\nu} = 0, \tag{80}$$

where

$$T_\mu{}^\nu \equiv F_{\mu\alpha}F^{\nu\alpha} - \frac{1}{4}\delta^\nu_\mu F^2, \tag{81}$$

$$\tau_\mu{}^\nu \equiv \delta^{\nu\alpha\beta\gamma}_{\mu\rho\sigma\tau}\nabla_\alpha F^{\sigma\tau}\nabla^\rho F_{\beta\gamma} - 4\mathcal{R}^{\nu\rho}{}_{\mu\alpha}F_{\rho\kappa}F^{\alpha\kappa}, \tag{82}$$

$$\mathcal{F}^{\mu\nu} \equiv F^{\mu\nu} - 4\sigma\mathcal{R}^{\mu\nu}{}_{\alpha\beta}F^{\alpha\beta}. \tag{83}$$

Now, using the KSK ansatz [Eq. (1)] having the properties in Eqs. (2)–(6) together with the electromagnetic vector potential of the form

$$A_\mu = \phi(x)l_\mu, \tag{84}$$

where l_μ and $p_\mu \equiv \partial_\mu\phi$ satisfy $l_\mu p^\mu = 0$, one can show that

$$F_{\mu\nu} = p_\mu l_\nu - p_\nu l_\mu, \qquad T_\mu{}^\nu = \psi l_\mu l^\nu, \tag{85}$$

$$\tau_\mu{}^\nu = -4\Big\{\bar{\nabla}_\alpha p_\beta\bar{\nabla}^\alpha p^\beta + \frac{1}{2}\xi^\alpha\partial_\alpha\psi - (\xi_\alpha p^\alpha)^2 + \frac{1}{2}[\xi^2 + (D-2)(D-3)K]\psi\Big\}l_\mu l^\nu, \tag{86}$$

$$\mathcal{F}^{\mu\nu} \equiv [1 + 4\sigma(D-2)(D-3)K]F^{\mu\nu}, \tag{87}$$

where $\psi \equiv p_\mu p^\mu$. Then, Eqs. (79) and (80) become

$$\Big[\Lambda - \frac{(D-1)(D-2)}{2}K\Big]\delta^\nu_\mu - \rho l_\mu l^\nu = \kappa^2\Big\{\psi - 4\sigma\Big[\bar{\nabla}_\alpha p_\beta\bar{\nabla}^\alpha p^\beta + \frac{1}{2}\xi^\alpha\partial_\alpha\psi - (\xi_\alpha p^\alpha)^2 + \frac{1}{2}[\xi^2 + (D-2)(D-3)K]\psi\Big]\Big\}l_\mu l^\nu, \tag{88}$$

$$-[1 + 4\sigma(D-2)(D-3)K][\bar{\Box}\phi + \xi^\nu p_\nu]l^\mu = 0. \tag{89}$$

From these, we find that

$$\Lambda = \frac{(D-1)(D-2)}{2}K, \tag{90}$$

$$\bar{\Box}V + 2\xi^\alpha\partial_\alpha V + \Big[\frac{1}{2}\xi_\alpha\xi^\alpha + 2(D-2)K\Big]V = -\kappa^2\Big\{\psi - 4\sigma\Big[\bar{\nabla}_\alpha p_\beta\bar{\nabla}^\alpha p^\beta + \frac{1}{2}\xi^\alpha\partial_\alpha\psi - (\xi_\alpha p^\alpha)^2 + \frac{1}{2}[\xi^2 + (D-2)(D-3)K]\psi\Big]\Big\}, \tag{91}$$

$$\bar{\Box}\phi + \xi^\nu p_\nu = 0. \tag{92}$$

Observe that, in writing the last equation, we assume the coefficient in Eq. (89) is nonzero—i.e.,

$$1 + 4\sigma(D-2)(D-3)K \neq 0. \tag{93}$$

Using the relation

$$\bar{\nabla}_\alpha p_\beta\bar{\nabla}^\alpha p^\beta = \frac{1}{2}\bar{\Box}\psi + \frac{1}{2}\xi^\alpha\partial_\alpha\psi - (D-1)K\psi + p^\alpha p^\beta\bar{\nabla}_\alpha\xi_\beta - p^\beta\bar{\nabla}_\beta(\bar{\Box}\phi + \xi^\nu\partial_\nu\phi), \tag{94}$$

together with Eqs. (58) and (92), we can equivalently write Eq. (91) as

$$\bar{\Box}V + 2\xi^\alpha\partial_\alpha V + \Big[\frac{1}{2}\xi_\alpha\xi^\alpha + 2(D-2)K\Big]V = -\kappa^2\Big\{\psi - 4\sigma\Big[\frac{1}{2}\bar{\Box}\psi + \xi^\alpha\partial_\alpha\psi - \frac{1}{2}(\xi_\alpha p^\alpha)^2 + \frac{1}{2}[\xi^2 + (D-3)(D-4)K]\psi\Big]\Big\}. \tag{95}$$

Note that when $\xi_\mu = 0$ and $K = 0$, all these expressions recover the flat background (pp wave) case in Horndeski theory [16]. In a recent paper [17], we studied a modified version of this theory by adding extra couplings to Eq. (74) of the form $R^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}F^{\alpha\beta}$ and obtained exact plane wave solutions to its field equations.

Remark 3: Equations (91) and (92) are special cases of the general field equations (49) and (51) for $n = 0$. Furthermore, the Horndeski theory is a special case of Corollary 2 with no derivatives of $F_{\mu\nu}$.

V. CONCLUSION

In this work, we considered the most general Einstein-Maxwell theory in which the pure gravity and Maxwell parts and their couplings are thought to be arbitrary. The Lagrange function associated with such a theory is any function of the curvature tensor, the electromagnetic field, and their covariant derivatives of any order. When the metric of the spacetime is assumed to be the Kerr-Schild-Kundt type of metrics, we proved a theorem stating that the most general Einstein-Maxwell field equations reduce to two coupled simple equations for the functions V and ϕ representing the gravitational and electromagnetic potentials, respectively. As an explicit application of the theorem, we presented the field equations of the Horndeski vector-tensor theory in the KSK spacetimes.

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