ONE LEVEL DENSITY OF HECKE L-FUNCTIONS ASSOCIATED WITH CUBIC CHARACTERS AT PRIME ARGUMENTS

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

MATHEMATICS

By

Cazibe Kavalcı

January 2023

One level density of Hecke L-functions associated with cubic characters at prime arguments

By Cazibe Kavalcı

January 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Ahmet Muhtar Güloğlu(Advisor)

Hamza Yeşilyurt

Yıldırım Akbal

Approved for the Graduate School of Engineering and Science:

Orhan Arıkan
Director of the Graduate School

ABSTRACT

ONE LEVEL DENSITY OF HECKE L-FUNCTIONS ASSOCIATED WITH CUBIC CHARACTERS AT PRIME ARGUMENTS

Cazibe Kavalcı
M.S. in Mathematics
Advisor: Ahmet Muhtar Güloğlu
January 2023

We study the one-level density of low-lying zeros of a family of L-functions associated with cubic Hecke characters defined over the Eisenstein field. We show that this family of L-functions satisfies the Katz-Sarnak conjecture for all test functions whose Fourier transforms are supported in $(-1, 1)$, under the Generalized Riemann Hypothesis.

Keywords: One level density, cubic Hecke L-functions, Katz-Sarnak Conjecture.

ÖZET

KÜBİK HECKE L-FONKSİYONLARININ 1/2 NOKTASINA YAKIN SIFIRLARININ DAĞILIMI

Cazibe Kavalcı
Matematik, Yüksek Lisans
Tez Danışmanı: Ahmet Muhtar Güloğlu
Ocak 2023

Bu tezde Eisenstein cismi üstünde, asal argümanlarda tanımlı üçüncü derece ilkel Dirichlet karakterlerine karşılık gelen Hecke L-fonksiyonlarının $s = 1/2$ noktasına yakın sıfırlarının dağılımı üstünde çalışılmıştır. Genel Riemann Hipotezi varsayımı altında bu L-fonksiyonlarının Katz-Sarnak sanısını desteklediği gösterilmiştir.

Anahtar sözcükler: Katz-Sarnak sanısı, Kübik Hecke L-fonksiyonları.

Acknowledgement

I would like to express my gratitude to my advisor Ahmet Muhtar Güloğlu for everything he has done for me and everything I have learned from him. He inspired me to be a mathematician with his teaching methods in my undergraduate years and it would be impossible to conclude this thesis without the generous support of him.

I would also like to thank my family and my friends, for their continuous support.

I would like to thank my committee members Hamza Yeşilyurt and Yıldırım Akbal for their time, comments and suggestions.

I would like to thank Bilkent University for providing me the facilities to work in and TÜBİTAK for the financial support provided by BİDEB 2210.

Contents

Chapter 1

Introduction

In this thesis, we study the statistics of low lying zeros, zeros near the central point $s = 1/2$, of a family of Hecke L-functions associated with cubic characters. The reason to study these statistics is to provide evidence to the Katz-Sarnak Conjecture, which suggests a relation between the distribution of low lying zeros of a family of L-functions and that of the eigenvalues near 1 of a corresponding group of matrices from some classical compact group. The family of L-functions is defined in (2.0.10). We denote the set of cubic characters by $\mathscr{F}$ and for each character $\chi \in \mathscr{F}$ we have an associated L-function $L(s, \chi)$. In order to define one level density for the family $\mathscr{F}$, we first fix an even Schwartz test function $\phi(x)$ whose Fourier transform is supported in $(-v, v)$. We assume the Generalized Riemann Hypothesis (GRH) for the L-functions in this thesis; that is, we assume that all nontrivial zeros of all the L-functions lie on the critical line $s = 1/2$. For each $\chi \in \mathscr{F}$, we consider the corresponding sum

$$\sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ L(\rho,\chi)=0}} \phi\Big(\frac{\gamma \log X}{2\pi}\Big)$$

where the sum runs over the zeros ρ of $L(s, \chi)$ with multiplicity. This sum counts zeros that are within $O(1/\log X)$ of the central point $s = 1/2$.

It is not possible to asymptotically evaluate this sum by taking a single L-function, because such a sum captures essentially a bounded number of zeros. Therefore, we take an average over the family $\mathscr{F}$ by considering

$$D(X;\phi,\mathscr{F}) = \frac{1}{A_{\mathscr{F}}(X)} \sum_{\chi\in\mathscr{F}} w(n_\chi/X) \sum_{\substack{\rho=1/2+i\gamma \\ L(\rho,\chi)=0}} \phi\Big(\frac{\gamma\log X}{2\pi}\Big),$$

where $A_{\mathscr{F}}(X) = \sum_{\chi\in\mathscr{F}} w(n_\chi/X)$, $w:\mathbb{R}\to(0,\infty)$ is a Schwartz function, and n_χ is the norm of the conductor of χ. The one level density for $\mathscr{F}$ then is

$$\lim_{X\to\infty} D(X;\phi,\mathscr{F}).$$

Now we state Katz-Sarnak conjecture:

Conjecture 1.0.1 (Katz-Sarnak[1],[2])**.** *Statistics of low-lying zeros of a family $\mathscr{F}$ of L-functions correspond to those of eigenvalues near 1 of an underlying group $G(\mathscr{F})$ of matrices determined by the family $\mathscr{F}$; that is, for any test function ϕ,*

$$\lim_{X\to\infty} D(X;\phi,\mathscr{F}) = \int_{-\infty}^{\infty} \phi(x) W_{G(\mathscr{F})}(x)dx,$$

where $W_{G(\mathscr{F})}(x)$ is the one-level scaling density of eigenvalues near 1 in the group of matrices with respect to the symmetry type of the family $\mathscr{F}$.

Note that in this conjecture, there is no restriction on the test function $\phi(x)$. All known results in the literature, however, assume that the Fourier transform $\hat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x)e^{-2\pi ixy}dx$ of ϕ has compact support in $(-v,v)$ for some v. In particular, for a family with a unitary symmetry type, going beyond the range $(-1,1)$ is crucial to show that a positive proportion of L-functions in the family does not vanish at $s=1/2$. Therefore, it is important to obtain a result where the support is as large as possible. For instance, Chantal and Güloğlu [3] used Poisson summation to go beyond this range and obtained a positive proportion of L-functions in a family associated with cubic reside symbols with square-free conductors. This family is also a thin subfamily of all Dirichlet characters and it contains our family as a subset. In our case, however, since we cannot use Poisson summation and some of the other analytic tools that they are able to use, we cannot obtain a non-vanishing result using the one-level density approach. In this thesis, the family $\mathscr{F}$ leads to an incomplete sum which unables one to use such tools. Therefore, we cannot establish a non-vanishing proportion of L-functions by passing the range $(-1, 1)$.

Chowla claims in [4] that quadratic Dirichlet L-functions $L(s, \chi)$ do not vanish for $s \in (0, 1)$. In particular, $L(1/2, \chi) \neq 0$. It is believed that the same holds for all Dirichlet characters. This is also predicted by Katz-Sarnak conjecture since assuming this conjecture one can show that almost all L-functions in a family do not vanish at the central point $s = 1/2$. There are other results in the literature that support Chowla's conjecture which will be mentioned below. On the other hand, Li [5] showed that there are infinitely many L-functions over the rational function field which vanish at the central point $s = 1/2$. She remarks that although Chowla's conjecture is not strictly true, it may hold for almost all quadratic characters.

Some of the results which support Chowla's conjecture are as follows: Özlük and Snyder [6] showed that at least $15/16$ of the L-functions attached to quadratic characters do not vanish at the central point by computing one level density under GRH. Soundararajan [7] proved without assuming GRH that at least 87.5% of quadratic Dirichlet L-functions do not vanish at the central point by computing mollified moments of L-functions. Gao-Zhao [8] proved that at least 75% of the members of a family of quadratic Hecke L-functions of prime moduli do not vanish at the central point under GRH.

We study one level density for Hecke L-functions associated with primitive cubic Dirichlet characters of prime moduli. The above mentioned work done by Chantal-Güloğlu [3] is the first one extending the support $(-1, 1)$ of the Fourier transforms of the test functions ϕ for a family of cubic L-functions. In their paper, Poisson summation leads to averages of Gauss sums which is more difficult to work with compared to the quadratic case. For the cubic case, there is much less work in the literature compared to the quadratic case for this reason. Chantal-David-Lalin [9] recently showed that a positive proportion of cubic Hecke L-functions do not vanish at the central point over the rational function field by computing mollified moments. Much other work has been done towards showing non-vanishing of various families of L-functions and studying one-level density, some of which are [10, 11, 12, 13, 14].

Theorem 1.0.2. *Let $\mathscr{F}$ be the family of cubic Hecke characters that is defined by Definition 2.0.10. Let ϕ be an even Schwarz function with $\hat{\phi}(y)$ supported in (-1,1). Then assuming GRH,*

$$\lim_{X\to\infty} D(X;\phi,\mathscr{F}) = \int_{-\infty}^{\infty} \phi(x)W_U(x)dx$$

where $W_U = 1$.

This theorem shows that the family $\mathscr{F}$ matches the statistics of the unitary family of matrices. Hence, we say that the family $\mathscr{F}$ has unitary symmetry type.

1.0.1 Structure of the Thesis

In the next chapter, we give necessary background on cubic characters and corresponding family of L-functions and eventually define the family $\mathscr{F}$. Then in the third chapter, we give the explicit formula, which is the main tool we use to prove Theorem 1.0.2, and in the final chapter we prove Theorem 1.0.2.

Chapter 2

Preliminaries

In this chapter, we describe the family $\mathscr{F}$ of characters χ defined over $K = \mathbb{Q}(\omega)$, where $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$. Before we define the family $\mathscr{F}$, we first list well-known properties of the ring of integers $\mathcal{O}_K = \mathbb{Z}[\omega]$ of K, then investigate cubic residue characters, and then introduce more general characters; namely, Dirichlet characters, and lastly we show that we can regard cubic residue characters as Dirichlet characters.

2.0.1 The ring of integers $\mathcal{O}_K$

$\mathcal{O}_K$ is a principal ideal domain consisting of complex numbers of the form $a + b\omega$, with $a, b \in \mathbb{Z}$. For any $\alpha = a + b\omega \in \mathcal{O}_K$, the norm of α is given by $\mathrm{N}(\alpha) = \alpha\overline{\alpha} = a^2 - ab + b^2$.

Proposition 2.0.1 ([15, IX.Prop.9.1.1])**.** *The units of $\mathcal{O}_K$ are $\pm 1, \pm\omega, \pm\omega^2$.*

Two elements α_1 and α_2 of $\mathcal{O}_K$ are called associate if they differ by a unit. We classify the primes of $\mathcal{O}_K$ in the following proposition.

Proposition 2.0.2 ([15, IX.Prop.9.1.4])**.** *Let p be a rational prime. Then,*

1. *If* $p \equiv 1 \bmod 3$*, then* $p = \pi\overline{\pi}$*, where* π *is a complex prime in* $\mathcal{O}_K$*.*

2. *If* $p \equiv 2 \bmod 3$*, then* p *remains prime in* $\mathcal{O}_K$*.*

3. $3 = -\omega^2(1-\omega)^2$ *and* $1-\omega$ *is prime in* $\mathcal{O}_K$*.*

Therefore, the primes of $\mathcal{O}_K$ *are the complex primes* $\pi, \overline{\pi}$ *satisfying* (1)*, and ordinary primes* $p \equiv 2 \bmod 3$*, and* $1-\omega$ *and their associates.*

Given a prime π that is not associate to $1-\omega$, and any α with $\pi \nmid \alpha$, $\alpha^{\frac{\mathrm{N}\pi-1}{3}}$ is congruent to exactly one of $1, w, w^2$ modulo π. The cubic residue symbol $(\alpha/\pi)_3$ is defined as the third root of unity for which $\alpha^{\frac{\mathrm{N}\pi-1}{3}} \equiv (\alpha/\pi)_3 \bmod \pi$ holds when $\pi \nmid \alpha$ and is 0 if $\pi \mid \alpha$. We put $\chi_\pi(\alpha) = (\alpha/\pi)_3$.

In general, for $c \in \mathcal{O}_K$ with $c = \pi_1^{k_1}\pi_2^{k_2}\ldots\pi_l^{k_l}$, and coprime to 3, we define the cubic residue character χ_c multiplicatively:

$$\chi_c(\alpha) = \prod_{1\leq i\leq l} (\chi_{\pi_i}(\alpha))^{k_i}. \tag{2.1}$$

Lemma 2.0.3 ([15, IX.Theorem 1], The Law of Cubic Reciprocity)**.** *Let* $\alpha_1, \alpha_2 \in \mathcal{O}_K$ *and* $\alpha_1, \alpha_2 \equiv \pm 1 \bmod 3$*. Then,*

$$\chi_{\alpha_1}(\alpha_2) = \chi_{\alpha_2}(\alpha_1).$$

Lemma 2.0.4 ([15, IX.Theorem 1'], The Supplementary Law of Cubic Reciprocity)**.** *Let* $\pi \neq 1-\omega$ *be a prime of* $\mathcal{O}_K$*. If* $\pi = q$ *is rational, write* $q = 3m-1$*. If* $\pi = a + b\omega$ *complex and* $\pi \equiv 2 \bmod 3$*, write* $a = 3m-1$*. Then,*

$$\chi_\pi(1-\omega) = \omega^{2m}.$$

2.0.2 Hecke Characters

Hecke was led to the concept of Hecke characters while searching for the most general class of characters χ with the associated L-function having a functional equation.

Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$ and let $J^{\mathfrak{m}}$ be the group of fractional ideals relatively prime to $\mathfrak{m}$. A homomorphism

$$\chi : J^{\mathfrak{m}} \to \mathbb{C}^\times \tag{2.2}$$

is called a *Hecke character* modulo $\mathfrak{m}$ if there are characters

$$\chi_f : \left(\mathcal{O}_K/\mathfrak{m}\right)^\times \to S^1, \qquad \chi_\infty : \mathbb{C}^\times \to S^1$$

such that

$$\chi((\alpha)) = \chi_f(\alpha)\chi_\infty(\alpha)$$

holds for any $\alpha \in \mathcal{O}_K$ coprime to $\mathfrak{m}$. The L-function associated with a Hecke character is defined by

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs over the integral ideals of K and $\chi(\mathfrak{a}) = 0$ when $(\mathfrak{a}, \mathfrak{m}) \neq 1$.

Proposition 2.0.5 ([16, VII.Prop.8.1])**.** *$L(s,\chi)$ converges absolutely and uniformly when $\mathrm{Re}(s) \geqslant 1+\epsilon$ for any $\epsilon > 0$, and has the Euler product*

$$L(s,\chi) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1},$$

where $\mathfrak{p}$ varies over the prime ideals of K.

The completed Hecke L-function $\Lambda(s,\chi)$ is defined by

$$\Lambda(s,\chi) = (|d_K|\mathrm{N}(\mathfrak{m}))^{s/2}(2\pi)^{-s}\Gamma(s)L(s,\chi),$$

where $d_K = -3$ is the discriminant of K. The following theorem is due to Hecke.

Proposition 2.0.6 ([16, VII. Cor. 8.6])**.** *The completed L-series $\Lambda(s,\chi)$ is holomorphic on $\mathbb{C}$ when χ is primitive and $\mathfrak{m} = (m) \neq \mathcal{O}_K$, and satisfies the functional equation*

$$\Lambda(s,\chi) = W(\chi)(\mathrm{N}\mathfrak{m})^{-1/2}\Lambda(1-s,\overline{\chi}), \tag{2.3}$$

where

$$W(\chi) = \sum_{\substack{x \bmod \mathfrak{m} \\ (x,\mathfrak{m})=1}} \chi(x) e^{2\pi i\, \mathrm{Tr}(x/m\sqrt{-3})},$$

is the associated Gauss sum, and x runs over a system of representatives of $(\mathcal{O}_K/\mathfrak{m})^\times$. Here, $\mathrm{Tr}(z) = z + \bar{z}$.

Definition 2.0.7. *A Dirichlet character* mod $\mathfrak{m}$ is defined as a character of the ray class group $J^{\mathfrak{m}}/P^{\mathfrak{m}}$, where $P^{\mathfrak{m}}$ is the subgroup of principal fractional ideals (a) such that $a \equiv 1 \bmod \mathfrak{m}$.

Dirichlet characters are shown to be special Hecke characters in the following proposition.

Proposition 2.0.8 ([16, VII. Prop. 6.9])**.** *The Dirichlet characters χ modulo $\mathfrak{m}$ are precisely the Hecke characters modulo $\mathfrak{m}$ such that*

$$\chi(a\mathcal{O}_K) = \chi_f(a)$$

for some character χ_f of $(\mathcal{O}_K/\mathfrak{m})^\times$. Furthermore, the conductor of the Dirichlet character equals the conductor of the corresponding Hecke character.

2.0.3 The Family $\mathscr{F}$

Lemma 2.0.9. *Given any prime $\pi \in \mathcal{O}_K$ with $\mathrm{N}(\pi) \equiv 1 \bmod 9, \chi_\pi$ can be regarded as a Hecke character modulo $\pi\mathcal{O}_K$.*

Proof. We define the character $\chi : J^{(\pi)} \to \mathbb{C}^\times$ by

$$\chi((\alpha/\beta)) = \chi_\pi(\alpha\beta^{-1})$$

for $(\alpha\beta, \pi) = 1$. It can be easily verified that χ is a homomorphism. Note that the kernel of χ is exactly $P^{(\pi)}$. Furthermore, $\chi_\pi(\omega) = \omega^{(\mathrm{N}(\pi)-1)/3} = 1$. Thus, χ_π is trivial on units, and the result follows by Proposition 2.0.8. □

Now we are ready to define our family $\mathscr{F}$:

Definition 2.0.10. Let $\mathscr{F}$ be the family of primitive cubic Dirichlet characters determined by the cubic residue symbols χ_π, where $\pi \in \mathcal{O}_K$ runs over the primes with $\pi \equiv 1 \bmod 9$.

Chapter 3

Explicit Formula

In this section, the explicit formula we use to prove our theorem is given. The following estimates are needed to prove the explicit formula:

Lemma 3.0.1 ([17, Appendix C.1])**.** *Let $\lambda > 0$ be given and*

$$R(\lambda) = \{s \in \mathbb{C} : |s| \geqslant \lambda, |\arg(s)| < \pi - \lambda\}.$$

Then,

$$(\Gamma'/\Gamma)(s) = \log s + O(1/|s|),$$
$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s}(1 + O(1/|s|))$$

uniformly for $s \in R(\lambda)$.

Lemma 3.0.2 ([3, Lemma 2.4])**.** *Let $\chi_\pi \in \mathscr{F}$. Then, uniformly for $s = \sigma + it$, where $1/2 < \sigma \leqslant 2$, or $|t| \geqslant 1$ and $-1 \leqslant \sigma \leqslant 2$,*

$$\frac{L'}{L}(s, \chi_\pi) = \sum_{|\gamma - t| \leqslant 1} \frac{1}{s - \rho} + O\big(\log\big(\mathrm{N}(\pi)(3 + |t|)\big)\big) \tag{3.1}$$

where the sum varies over the zeros $\rho = \beta + i\gamma$ of $\Lambda(s, \chi_\pi)$ counted with multiplicity.

Lemma 3.0.3 ([3, Lemma 2.5])**.** *Let $\chi_\pi \in \mathscr{F}$. Then, for any $T \geqslant 1$, there exists some $T_1 \in [T, T+1]$ which does not correspond to the ordinate of any zero such that uniformly for $-1 \leqslant \sigma \leqslant 2$,*

$$\frac{L'}{L}(\sigma \pm iT_1, \chi_\pi) \ll \log^2\big(\mathrm{N}(\pi)(3 + T)\big). \tag{3.2}$$

3.0.1 Explicit Formula

The explicit formula given below relates the non-trivial zeros of each L-function $L(s,\chi)$ for $\chi \in \mathscr{F}$ to a character sum over the prime ideals.

Lemma 3.0.4 (Explicit Formula). *Let $\chi_\pi \in \mathscr{F}$, and $\phi(x)$ be an even Schwartz function whose Fourier transform is supported in $(-v,v)$. Then,*

$$\sum_\rho \phi\Big(\frac{(\rho-1/2)\log X}{2\pi i}\Big) = \widehat{\phi}(0)\frac{\log \mathrm{N}(\pi)}{\log X} + O\Big(\frac{1}{\log X}\Big)$$
$$- \sum_{\mathfrak{p}} \sum_{1\leqslant k\leqslant 2} \big(\chi_\pi(\mathfrak{p}^k) + \chi_\pi(\mathfrak{p}^{2k})\big)\,\widehat{\phi}\left(\frac{k\log \mathrm{N}(\mathfrak{p})}{\log X}\right)\frac{\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{k/2}\log X},$$

where the sum on the left runs over the non-trivial zeros of $L(s,\chi_\pi)$ with multiplicity, and the implied constant depends only on ϕ.

Proof. First define the function

$$G(s) := \phi\Big(\frac{(s-1/2)\log X}{2\pi i}\Big),$$

which is holomorphic in $-1 \leqslant \mathrm{Re}(s) \leqslant 2$ and satisfies

$$G(s) = G(1-s), \qquad s^2 G(s) \ll 1. \tag{3.3}$$

Let T be a sufficiently large real number so that there exists $T_1 \in [T, T+1]$ as in Lemma 3.0.3. Let R be the rectangle with vertices $2 \pm iT_1, -1 \pm iT_1$. By Cauchy's integral formula, we obtain

$$2\pi i \sum_\rho G(\rho) = \int_R G(s)\frac{\Lambda'}{\Lambda}(s,\chi_\pi)ds,$$

where the integral is taken counter-clockwise around R.

Recall that

$$\Lambda(s,\chi_\pi) = (-3\mathrm{N}(\pi))^{s/2}(2\pi)^{-s}\Gamma(s)L(s,\chi_\pi). \tag{3.4}$$

On taking the logarithmic derivative of $\Lambda(s,\chi_\pi)$ we obtain

$$\frac{\Lambda'}{\Lambda}(s,\chi_\pi) = \frac{L'}{L}(s,\chi_\pi) + \frac{\Gamma'}{\Gamma}(s) + \frac{1}{2}\log(3\mathrm{N}(\pi)) - \log(2\pi). \tag{3.5}$$

Since $(L'/L)(s,\chi_\pi) \ll \log^2(\mathrm{N}(\pi)(3+T))$ by Lemma 3.0.3, $(\Gamma'/\Gamma)(s) \ll \log T$ by Lemma 3.0.1 and that $G(s) \ll \frac{1}{T^2}$ on the horizontal lines, the contribution of the horizontal integrals is $\ll T^{-2}\log^2(T\mathrm{N}(\pi))$. Thus, using the functional equations for $G(s)$ in (3.3), and $\Lambda(s,\chi_\pi)$ in (2.3), we obtain

$$\begin{aligned} 2\pi i\sum_\rho G(\rho) &= \int_{2-iT}^{2+iT} G(s)\left(\frac{\Lambda'}{\Lambda}(s,\chi_\pi)+\frac{\Lambda'}{\Lambda}(s,\overline{\chi_\pi})\right)ds + O(T^{-2}\log^2(T\mathrm{N}(\pi))) \\ &= \int_{2-iT}^{2+iT} G(s)\left(\frac{L'}{L}(s,\chi_\pi)+\frac{L'}{L}(s,\overline{\chi_\pi})+2\frac{\Gamma'}{\Gamma}(s)+\log\frac{3\mathrm{N}(\pi)}{4\pi^2}\right)ds \\ &\quad + O(T^{-2}\log^2(T\mathrm{N}(\pi))). \end{aligned} \tag{3.6}$$

By taking the logarithmic derivative of $L(s,\chi_\pi)$, we obtain

$$-\frac{L'}{L}(s,\chi_\pi) = \sum_{n\geqslant 1,\mathfrak{p}} \frac{\chi_\pi(\mathfrak{p}^n)\log\mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{sn}}.$$

Also,

$$\begin{aligned} \sum_{n\geqslant 1,\mathrm{N}(\mathfrak{p})>y} \frac{\chi_\pi(\mathfrak{p}^n)\log\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{sn}} &= \sum_{\mathrm{N}(\mathfrak{p})>y}\log\mathrm{N}(\mathfrak{p})\sum_{n\geqslant 1}\left(\frac{\chi_\pi(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}\right)^n \\ &= \sum_{\mathrm{N}(\mathfrak{p})>y}\log\mathrm{N}(\mathfrak{p})\left[\frac{1}{1-\frac{\chi_\pi(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}}\right]\frac{\chi_\pi(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s} \\ &\ll \sum_{\mathrm{N}(\mathfrak{p})>y}\frac{\log\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^2}. \end{aligned} \tag{3.7}$$

By Riemann-Stieltjes Integration,

$$\begin{aligned} \sum_{\mathrm{N}(\mathfrak{p})>y}\frac{\log\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^2} &= \int_y^\infty \frac{1}{x^2}d\Big[\sum_{y<\mathrm{N}(\mathfrak{p})\leqslant x}\log\mathrm{N}(\mathfrak{p})\Big] \\ &= \lim_{x\to\infty}\frac{1}{x^2}\sum_{y<\mathrm{N}(\mathfrak{p})\leqslant x}\log\mathrm{N}(\mathfrak{p}) - \int_y^\infty\Big[\sum_{y<\mathrm{N}(\mathfrak{p})\leqslant x}\log\mathrm{N}(\mathfrak{p})\Big](-2x^{-3})dx, \end{aligned} \tag{3.8}$$

Since

$$\sum_{\mathrm{N}\mathfrak{p}\leqslant x}\log\mathrm{N}\mathfrak{p} \leqslant 4\sum_{p\leqslant x}\log p \ll x,$$

it follows that

$$(3.8) \ll \lim_{x\to\infty} 1/x + \int_y^\infty 2x^{-2}dx \ll 1/y.$$

Therefore, using (3.3) we have

$$\int_{2-iT}^{2+iT} G(s)\frac{L'}{L}(s,\chi_\pi)ds = -\int_{2-iT}^{2+iT} G(s)\sum_{n\geqslant 1,\mathfrak{p}} \frac{\chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{sn}}ds$$
$$= -\int_{2-iT}^{2+iT} G(s)\sum_{\substack{n\geqslant 1\\ \mathrm{N}(\mathfrak{p})\leqslant y}} \frac{\chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{sn}}ds + O(y^{-1}). \tag{3.9}$$

Now we shift the contour to $\sigma = 1/2$ to get

$$\begin{aligned} &-\int_{2-iT}^{2+iT} G(s)\sum_{n\geqslant 1,\mathrm{N}(\mathfrak{p})\leqslant y} \frac{\chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{sn}}ds \\ &= \left[\int_{1/2+iT}^{1/2-iT} + \int_{2\pm iT}^{1/2\pm iT}\right] G(s)\sum_{n\geqslant 1,\mathrm{N}(\mathfrak{p})\leqslant y} \frac{\chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{sn}}ds. \end{aligned} \tag{3.10}$$

Since

$$\int_{2\pm iT}^{1/2\pm iT} G(s)\left(\mathrm{N}(\mathfrak{p})\right)^{-ns}\log \mathrm{N}(\mathfrak{p})ds \ll T^{-2}\left(\mathrm{N}(\mathfrak{p})\right)^{-n/2},$$

the contribution of horizontal integrals is $\ll y^{1/2}T^{-2}$. Therefore,

$$\begin{aligned} &-\int_{2-iT}^{2+iT} G(s)\sum_{n\geqslant 1,\mathrm{N}(\mathfrak{p})\leqslant y} \frac{\chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{sn}}ds + O(y^{-1}) \\ &= -\frac{1}{2\pi i}\sum_{\substack{n\geqslant 1,\mathfrak{p}\\ \mathrm{N}(\mathfrak{p})\leqslant y}} \chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})\int_{1/2-iT}^{1/2+iT} G(s)(\mathrm{N}(\mathfrak{p}))^{-ns}ds \\ &\quad + O(y^{1/2}T^{-2} + y^{-1}), \end{aligned} \tag{3.11}$$

Then using the definition of $G(s)$ and substituting $2\pi iu = (s-1/2)\log X$, we obtain

$$\begin{aligned} \int_{2-iT}^{2+iT} G(s)\frac{L'}{L}(s,\chi_\pi)ds = &-\sum_{\substack{n\geqslant 1,\mathfrak{p}\\ \mathrm{N}(\mathfrak{p})\leqslant y}} \frac{\chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{n/2}\log X} \\ &\cdot \int_{-\frac{T\log X}{2\pi}}^{\frac{T\log X}{2\pi}} \phi(u)e\Big(\frac{-un\log \mathrm{N}(\mathfrak{p})}{\log X}\Big)du + O(\frac{y^{1/2}}{T^2} + y^{-1}) \end{aligned} \tag{3.12}$$

By taking limit as $T\to\infty$, we deduce that for any $y>0$,

$$\frac{1}{2\pi i}\int_{\sigma=2} G(s)\frac{L'}{L}(s,\chi_\pi)ds = -\sum_{\substack{n\geqslant 1\\ \mathrm{N}(\mathfrak{p})\leqslant y}} \frac{\chi_\pi(\mathfrak{p}^n)\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{n/2}\log X}\widehat{\phi}\Big(\frac{n\log \mathrm{N}(\mathfrak{p})}{\log X}\Big) + O(y^{-1}).$$

The same holds for $\bar{\chi}_\pi$. For the other terms in (3.6), we have

$$\int_{1\pm iT}^{2\pm iT} G(s)\frac{\Gamma'}{\Gamma}(s)ds \ll \frac{\log T}{T^2},$$

and

$$\int_{1\pm iT}^{2\pm iT} G(s)\log\frac{3\mathrm{N}(\pi)}{4\pi^2}ds \ll \frac{1}{T^2}.$$

Shifting the contour to $\sigma = 1/2$ for these integrals, and combining our results, we conclude that

$$\sum_\rho G(\rho) = \frac{\widehat{\phi}(0)}{\log X}\log\frac{3\mathrm{N}(n)}{4\pi^2} + \frac{1}{\pi i}\int_{\sigma=1/2} G(s)\frac{\Gamma'}{\Gamma}(s)ds$$
$$- \sum_{\mathfrak{p}}\sum_{k\geqslant 1}\left(\chi_\pi(\mathfrak{p}^k) + \chi_\pi(\mathfrak{p}^{2k})\right)\widehat{\phi}\left(\frac{k\log\mathrm{N}(\mathfrak{p})}{\log X}\right)\frac{\log\mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{k/2}\log X}.$$

By the approximate formula (cf. [18, 8.363.3])

$$\frac{\Gamma'}{\Gamma}(a+ib) + \frac{\Gamma'}{\Gamma}(a-ib) = 2\frac{\Gamma'}{\Gamma}(a) + O\left((b/a)^2\right)$$

and substituting $2\pi iu = (s-1/2)\log X$, we obtain

$$\frac{1}{\pi i}\int_{\sigma=1/2} G(s)\frac{\Gamma'}{\Gamma}(s)ds = \frac{2}{\log X}\int_{-\infty}^{\infty}\phi(t)\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+i\frac{2\pi t}{\log X}\right)dt$$
$$= \frac{2(\Gamma'/\Gamma)(1/2)}{\log X}\widehat{\phi}(0) + O\left((\log X)^{-3}\right).$$

Furthermore,

$$\sum_{\mathfrak{p}}\sum_{k>2}\left(\chi_\pi(\mathfrak{p}^k) + \chi_\pi(\mathfrak{p}^{2k})\right)\widehat{\phi}\left(\frac{k\log\mathrm{N}(\mathfrak{p})}{\log X}\right)\frac{\log\mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{k/2}} \ll \sum_p p^{-3/2}\log p \ll 1.$$

Therefore, we arrive at the desired result.

□

Lemma 3.0.5 (Perron's formula [19, Ch.17 p.105])**.** *For $T, a, y > 0$,*

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}ds - \delta(y)\right| < \begin{cases} y^a\min\left(1, T^{-1}|\log y|^{-1}\right) & \text{if } y \neq 1 \\ aT^{-1} & \text{if } y = 1, \end{cases} \qquad (3.13)$$

where

$$\delta(y) = \begin{cases} 0 & \text{if } 0 < y < 1 \\ 1/2 & \text{if } y = 1 \\ 1 & \text{if } y > 1. \end{cases}$$

Lemma 3.0.6. *Let ψ be a Dirichlet character modulo $\mathfrak{m}$. Assuming GRH, the estimate*

$$\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} \psi(\mathfrak{p}) \log \mathrm{N}(\mathfrak{p}) \ll x^{1/2} \log x \log^2(x\mathrm{N}(\mathfrak{m}))$$

holds for $x > 1$.

Proof. Since no ideal has norm less than 3, we can assume $x \geqslant 3$. Let χ be the primitive character of modulus $\mathfrak{f}$ that induces ψ. Then we have

$$L(s,\psi) = \prod_{\mathfrak{p}\nmid\mathfrak{m}} \left(1-\chi_c(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}\right)^{-1} = \prod_{\mathfrak{p}|d}\left(1-\chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}\right)L(s,\chi),$$

where $\mathfrak{m} = (m), \mathfrak{f} = (f)$, and d is the product of primes dividing m, but not f.

Let $T > 1$ and fix $T_1 \in [T, T+1]$ as in Lemma 3.0.3 for $L(s,\chi)$. Then we have for $a = 1 + (2\log x)^{-1}$,

$$\frac{1}{2\pi i}\int_{a-iT_1}^{a+iT_1} -\frac{L'}{L}(s,\chi)\frac{x^s}{s}ds = \frac{1}{2\pi i}\int_{a-iT_1}^{a+iT_1} -\sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}\frac{x^s}{s}ds.$$

Here, $\frac{L'}{L}(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}$ runs over the integral ideals $\mathfrak{a}$ of K. By rearranging the summation and applying Perron's formula 3.0.5 with $y = x/\mathrm{N}(\mathfrak{a})$, we get

$$-\sum_{\mathfrak{a}} \chi(\mathfrak{a})\Lambda(\mathfrak{a})\frac{1}{2\pi i}\int_{a-iT_1}^{a+iT_1}\left(\frac{x}{\mathrm{N}(\mathfrak{a})}\right)^s s^{-1}ds = \sum_{\mathrm{N}(\mathfrak{a})<x}\chi(\mathfrak{a})\Lambda(\mathfrak{a}) - \frac{1}{2}\sum_{\mathrm{N}(\mathfrak{a})=x}\chi(\mathfrak{a})\Lambda(\mathfrak{a})$$
$$+ O\Bigg(\sum_{\mathrm{N}(\mathfrak{a})\neq x}\Lambda(\mathfrak{a})\left(\frac{x}{\mathrm{N}(\mathfrak{a})}\right)^a \min\left\{1, \frac{1}{T|\log(x/\mathrm{N}(\mathfrak{a}))|}\right\}\Bigg) + O(T^{-1}).$$

Here, Λ is von Mangoldt's function defined on K by

$$\Lambda(\mathfrak{a}) = \begin{cases} \log \mathrm{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^k, \\ 0 & \text{otherwise.}\end{cases}$$

We will work on the terms one by one to simplify the estimate. First note that the second and the forth terms are needed only when there is a prime power $\mathfrak{a}$ such that $\mathrm{N}(\mathfrak{a}) = x$ and there are at most two such ideals. So their contribution to the estimate will be negligible. Furthermore,

$$\sum_{\mathrm{N}(\mathfrak{a})\leqslant x}\chi(\mathfrak{a})\Lambda(\mathfrak{a}) = \sum_{\mathrm{N}(\mathfrak{p})\leqslant x}\chi(\mathfrak{p})\log\mathrm{N}(\mathfrak{p}) + O(x^{1/2})$$

since

$$\sum_{\mathrm{N}(\mathfrak{p})^k\leqslant x, k\geqslant 2}\chi(\mathfrak{p})\log \mathrm{N}(\mathfrak{p}) \ll \sum_{\mathrm{N}(\mathfrak{p})^k\leqslant x, k\geqslant 2}\log \mathrm{N}(\mathfrak{p})$$
$$= \sum_{\mathrm{N}(\mathfrak{p})\leqslant\sqrt{x}}\sum_{k\leqslant\frac{\log x}{\log \mathrm{N}(\mathfrak{p})}}\log \mathrm{N}(\mathfrak{p}) \leqslant \sum_{\mathrm{N}(\mathfrak{p})\leqslant\sqrt{x}}\log \mathrm{N}(\mathfrak{p})\Big(\frac{\log x}{\log \mathrm{N}(\mathfrak{p})}\Big) \ll \sqrt{x}.$$

By Riemann Stieltjes integration,

$$\begin{aligned}
-(\zeta'/\zeta)(a) &= \sum_{n=1}^{\infty}\Lambda(n)n^{-1-1/2\log x}\\
&= \int_1^{\infty}\frac{1}{t^{1+1/2\log x}}d\Big[\sum_{n\leqslant t}\Lambda(n)\Big]\\
&= \left(1+\frac{1}{2\log x}\right)\int_1^{\infty}\frac{1}{t^{1+1/2\log x}}\Big(\sum_{n\leqslant t}\Lambda(n)\Big)dt \ll \log x
\end{aligned}$$

since $\sum_{n\leqslant t}\Lambda(n)\ll t$. Then, for $\mathrm{N}(\mathfrak{a})\leqslant x/2$ (similarly for $\mathrm{N}(\mathfrak{a})\geqslant 3x/2$), since $\sum_{n=1}^{\infty}\Lambda(n)n^{-s} = -\zeta'(s)/\zeta(s)$ and $x^a = x\sqrt{e}$, we obtain

$$\sum_{\mathrm{N}(\mathfrak{a})\leqslant x/2}\Big(\frac{x}{\mathrm{N}(\mathfrak{a})}\Big)^a\frac{\Lambda(\mathfrak{a})}{T|\log(x/\mathrm{N}(\mathfrak{a}))|} \ll xT^{-1}(-(\zeta'/\zeta)(a)) \ll \frac{x\log x}{T}.$$

For $x/2 < \mathrm{N}(\mathfrak{a}) < x$, let $\langle x\rangle$ denote the rational prime power closest to x but is different from x. Assume there is some ideal $\mathfrak{a}$ with $\mathrm{N}(\mathfrak{a})\in(x/2,x)$ and $\mathrm{N}(\mathfrak{a}) = \langle x\rangle$. Note that there are at most two such ideals (see Proposition 2.0.2) and for these ideals we have

$$|\log(x/\mathrm{N}(\mathfrak{a}))| = -\log\left(1-\frac{x-\langle x\rangle}{x}\right) \geqslant \frac{x-\langle x\rangle}{x}.$$

For the rest of prime powers in $(x/2,\langle x\rangle)$, similarly we use the inequality

$$|\log(x/\mathrm{N}(\mathfrak{a}))| \geqslant \frac{\langle x\rangle - \mathrm{N}(\mathfrak{a})}{\langle x\rangle}.$$

Thus, the total contribution in this interval is

$$\begin{aligned}
&\ll \Big(\frac{x}{\langle x\rangle}\Big)^a\Lambda(\langle x\rangle)\min\left(1,\frac{x}{T(x-\langle x\rangle)}\right) + \sum_{x/2<n<\langle x\rangle}\Big(\frac{x}{n}\Big)^a\frac{\langle x\rangle}{\langle x\rangle - n}\frac{1}{T}\sum_{\mathrm{N}(\mathfrak{a})=n}\Lambda(\mathfrak{a})\\
&\ll \log x\min\left(1,\frac{x}{T(x-\langle x\rangle)}\right) + \frac{x\log^2 x}{T}.
\end{aligned}$$

This estimate also holds in the interval $(x, 3x/2)$. Now, we shift the contour to $b = 1/2 + (2\log x)^{-1}$ so that

$$\frac{1}{2\pi i}\int_{a-iT_1}^{a+iT_1} -\frac{L'}{L}(s,\chi)\frac{x^s}{s}ds = \left(\int_{a+iT_1}^{b+iT_1} + \int_{b-iT_1}^{a-iT_1} + \int_{b+iT_1}^{b-iT_1}\right)\left(\frac{L'}{L}(s,\chi)\frac{x^s}{s}\right)ds.$$

By (3.2) the contribution of the horizontal integrals is

$$\ll x\log^2(\mathrm{N}(\mathfrak{f})(3+T))/(T\log x). \tag{3.14}$$

We split the vertical integral and use (3.1) for $|t| \leqslant 1$ and (3.2) for $|t| > 1$ to get

$$\int_{b+iT_1}^{b-iT_1} \frac{L'}{L}(s,\chi)\frac{x^s}{s}ds \ll x^{1/2}\left(\log x\log(3\mathrm{N}(\mathfrak{f})) + \log T\log^2(\mathrm{N}(\mathfrak{f})(3+T))\right).$$

Combining all the estimates above we conclude that

$$\sum_{\mathrm{N}(\mathfrak{p})\leqslant x}\chi(\mathfrak{p})\log\mathrm{N}(\mathfrak{p}) \ll \frac{x\log^2 x}{T} + \frac{x\log^2(3\mathrm{N}(\mathfrak{f})T)}{T\log x} + x^{1/2}\left(\log x\log(3\mathrm{N}(\mathfrak{f})) + \log T\log^2(\mathrm{N}(\mathfrak{f})(3+T))\right).$$

Choosing $T = x$ and adding the contribution of the finite product $\prod_{\mathfrak{p}|d}(1 - \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s})$ gives the desired estimate. □

3.0.2 Averaging over the family $\mathscr{F}$

Let $\mathscr{F}$ be the family of L-functions that is defined by 2.0.10. Recall that

$$\mathcal{D}(X;\phi,\mathscr{F}) = \frac{1}{\mathscr{A}_{\mathscr{F}}(X)}\sum_{\chi\in\mathscr{F}} w\left(\frac{n_\chi}{X}\right)\sum_{\substack{\gamma\\ L(1/2+i\gamma,\chi)=0}}\phi\Big(\frac{\gamma\log X}{2\pi}\Big),$$

where n_χ is the norm of the conductor of the primitive character χ and

$$\mathscr{A}_{\mathscr{F}}(X) = \sum_{\chi\in\mathscr{F}} w\left(\frac{n_\chi}{X}\right).$$

Then using the explicit formula in Lemma 3.0.4 gives

$$\begin{aligned}\mathcal{D}(X;\phi,\mathscr{F}) = \frac{1}{\mathscr{A}_{\mathscr{F}}(X)}\sum_{\chi\in\mathscr{F}} w\left(\frac{n_\chi}{X}\right)\Bigg[\widehat{\phi}(0)\frac{\log\mathrm{N}(\pi)}{\log X} + O\Big(\frac{1}{\log X}\Big) \\ - \sum_{\mathfrak{p}\nmid 3}\sum_{1\leqslant k\leqslant 2}\left(\chi(\mathfrak{p}^k) + \chi(\mathfrak{p}^{2k})\right)\widehat{\phi}\left(\frac{k\log\mathrm{N}(\mathfrak{p})}{\log X}\right)\frac{\log\mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{k/2}\log X}\Bigg].\end{aligned} \tag{3.15}$$

We will show that the first term can be replaced by $\hat{\phi}(0)$ with an admissible error, which will then determine the symmetry type. The remaining terms will determine how large the support of $\hat{\phi}$ should be. We deal with the estimate of the sums over prime ideals for the rest of the thesis.

Chapter 4

Proof of the Main Result

The next result will be used in several lemmas below.

Lemma 4.0.1. *There exists absolute constants* $c, c_1 > 0$ *such that*

$$\#\{(\pi) : \pi \textit{ prime}, \mathrm{N}(\pi) \leqslant x, \pi \equiv 1 \bmod 9\} = \frac{1}{9}\,\mathrm{Li}(x) + O(x \exp(-c\sqrt{\log x})).$$

for $x > c_1$.

Proof. Note that these primes correspond to the elements of $P^{(9)}$. Ray class group $J^{(9)}/P^{(9)}$ is isomorphic to the Galois group $\mathrm{Gal}(K^{(9)}/K)$ given by the Artin map $\left[\frac{K^{(9)}/K}{\cdot}\right]$ [16, VI. Theorem 7.1]. We apply Chebotarev density theorem [20, Theorem 1.3] to this Galois extension with identity automorphism as the conjugacy class, and the pull-back gives the primes in the kernel. Noting that $|J^{(9)}/P^{(9)}| = 9$ and that a possible Siegel zero has real part < 1 (cf. [21]), the result follows. □

The next result provides an asymptotic formula for the size of the family $\mathscr{A}_{\mathscr{F}}(X)$.

Lemma 4.0.2.

$$\mathscr{A}_{\mathscr{F}}(X) = \sum_{\pi \equiv 1(9)} w\left(\mathrm{N}(\pi)/X\right) = \frac{\tilde{w}(1)}{9}\frac{X}{\log X} + O\left(\frac{X}{\log^2 X}\right),$$

where $\tilde{w}(s) = \int_0^\infty x^{s-1} w(x)dx$ *is the Mellin transform of* w.

Proof. We first divide the summation $\sum_{\pi\equiv 1(9)} w(\mathrm{N}(\pi)/X)$ into two parts, according as $\mathrm{N}(\pi) \leqslant X^{1/2}$ and $\mathrm{N}(\pi) > X^{1/2}$. The first part is $O_w(X^{1/2}/\log X)$. For the second part of the summation, using Riemann Stieltjes integration together with Lemma 4.0.1 yields

$$\begin{aligned}\sum_{\substack{\mathrm{N}(\pi)\geqslant X^{1/2}\\ \pi\equiv 1(9)}} w(\mathrm{N}(\pi)/X) &= \int_{X^{1/2}}^\infty w(t/X)d\Bigg[\sum_{\substack{\mathrm{N}(\pi)\leqslant t\\ \pi\equiv 1(9)}} 1\Bigg] \\ &\ll \int_{X^{1/2}}^\infty w(t/X)d\left[\frac{1}{9}\int_2^t \frac{dx}{\log x} + E(t)\right] \\ &= \frac{1}{9}\int_{X^{1/2}}^\infty \frac{w(t/X)}{\log t}dt + \int_{X^{1/2}}^\infty w(t/X)d(E(t)).\end{aligned}$$

For the error part, we have

$$\int_{X^{1/2}}^\infty w(t/X)d(E(t)) = [w(t/X)E(t)]\Big|_{X^{1/2}}^\infty - \frac{1}{X}\int_{X^{1/2}}^\infty E(t)w'(t/X)dt,$$

and

$$\frac{1}{X}\int_{X^{1/2}}^\infty E(t)w'(t/X)dt = \frac{1}{X}\int_X^\infty E(t)w'(t/X)dt + O(X\exp(-c\sqrt{\log X}))$$

since

$$\frac{1}{X}\int_{X^{1/2}}^X t\exp(-c\sqrt{\log t})w'(t/X)dt \ll X\exp(-c\sqrt{\log X}).$$

Using $w'(t) \ll 1/t^2$ and then substituting $u = (\log t)^{1/2}$, we see that

$$\begin{aligned}\frac{1}{X}\int_X^\infty E(t)w'(t/X)dt &\ll X\int_{(\log X)^{1/2}}^\infty e^{-cu}udu \\ &\ll X(\log X)^{1/2}\exp(-c\sqrt{\log X}),\end{aligned}$$

where we used integration by parts for the second line. Thus, we have shown so far that

$$\sum_{\chi\in\mathscr{F}} w(\mathrm{N}(\pi)/X) = \int_{X^{1/2}}^\infty \frac{w(t/X)}{\log t}dt + O(X(\log X)^{1/2}\exp(-c\sqrt{\log X})).$$

Substituting $u = t/X$ in the integral gives

$$\int_{X^{1/2}}^\infty \frac{w(t/X)}{\log t}dt = \frac{X}{\log X}\int_{1/X^{1/2}}^\infty \frac{w(u)}{1+\frac{\log u}{\log X}}du.$$

Replacing $1/(1+\log u/\log X)$ by $1+O(|\log u|/\log X)$ for u in $[X^{-1/2}, X^{1/2}]$, the above integral can be written as

$$\frac{X}{\log X}\int_{1/X^{1/2}}^{X^{1/2}} w(u)\left[1+O\left(\frac{|\log u|}{\log X}\right)\right]du + X\int_{X^{1/2}}^{\infty}\frac{w(u)}{\log u+\log X}du.$$

We have

$$\begin{aligned}\frac{X}{\log^2 X}\int_{1/X^{1/2}}^{X^{1/2}} w(u)|\log u|du &\ll \frac{X}{\log^2 X}\int_{1/X^{1/2}}^{1}|\log u|du + \frac{X}{\log^2 X}\int_{1}^{X^{1/2}}\frac{|\log u|}{u^2}du \\ &= O\left(\frac{X}{\log^2 X}\right),\end{aligned}$$

and

$$X\int_{X^{1/2}}^{\infty}\frac{w(u)}{\log u+\log X}du \ll \frac{X}{\log X}\int_{X^{1/2}}^{\infty}\frac{1}{u^2}du = O\left(\frac{\sqrt{X}}{\log X}\right).$$

Since

$$\begin{aligned}\int_{1/X^{1/2}}^{X^{1/2}} w(u)du &= \tilde{w}(1) - \int_{0}^{1/X^{1/2}} w(u)du - \int_{X^{1/2}}^{\infty} w(u)du \\ &= \tilde{w}(1) + O(1/X^{1/2}),\end{aligned}$$

the result follows. □

We also use the following lemmas to get $\hat{\phi}(0)$ as the main term of $\mathcal{D}(X;\phi,\mathscr{F})$.

Lemma 4.0.3.

$$\sum_{\pi\equiv 1(9)} w(\mathrm{N}(\pi)/X)\log \mathrm{N}(\pi) = \frac{X}{9}\tilde{w}(1) + O(X(\log X)^{3/2}\exp(-c\sqrt{\log X})).$$

Proof. As in the proof of Lemma 4.0.2, we first divide the summation into two sums with $\mathrm{N}(\pi) \leqslant X^{1/2}$ and $\mathrm{N}(\pi) > X^{1/2}$. The first sum is $\ll X^{1/2}\log X$. For the second sum, using partial integration yields

$$\begin{aligned}\sum_{\substack{\mathrm{N}(\pi)>X^{1/2}\\ \pi\equiv 1(9)}} w(\mathrm{N}(\pi)/X)\log \mathrm{N}(\pi) &= \int_{X^{1/2}}^{\infty} w(t/X)\log t\, d\Bigg[\sum_{\substack{\mathrm{N}(\pi)\leqslant t\\ \pi\equiv 1(9)}} 1\Bigg] \\ &\ll \int_{X^{1/2}}^{\infty} w(t/X)\log t\left(d\left[\frac{1}{9}\int_{2}^{t}\frac{dt}{\log t}\right] + dE(t)\right) \\ &= \frac{1}{9}\int_{X^{1/2}}^{\infty} w(t/X)dt + \int_{X^{1/2}}^{\infty} w(t/X)\log t\, dE(t)\end{aligned}$$

by Lemma 4.0.1. Then

$$\frac{1}{9}\int_{X^{1/2}}^{\infty} w(t/X)dt \ll \frac{X}{9}\int_{1/X^{1/2}}^{\infty} w(u)du = \frac{X}{9}\tilde{w}(1) + O(X^{1/2}).$$

For the error term, using integration by parts gives

$$\int_{X^{1/2}}^{\infty} w(t/X)\log t\, dE(t) = [w(t/X)\log t\, E(t)]\Big|_{X^{1/2}}^{\infty} - \int_{X^{1/2}}^{\infty} E(t)\big(w'(t/X)\log t/X + w(t/X)/t\big)dt.$$

The first term on the right is $O(X^{1/2}\exp(-c\sqrt{\log X})\log X)$ by Lemma 4.0.1. Using the bound $w'(t/X) \ll X^2/t^2$, we have

$$\frac{1}{X}\int_{X}^{\infty} E(t)w'(t/X)\log t\, dt \ll X\int_{X}^{\infty} \frac{1}{t}\exp(-c\sqrt{\log t})\log t\, dt.$$

Substituting $u = \sqrt{\log t}$ and then applying integration by parts, this integral simplifies to

$$2X\int_{\sqrt{\log X}}^{\infty} e^{-cu}u^3 du = O\left(X(\log X)^{3/2}\exp(-c\sqrt{\log X})\right).$$

Combining the estimates we arrive at the desired result. □

Lemma 4.0.4. *Assuming GRH,*

$$\sum_{\pi\equiv 1(9)} w(\mathrm{N}(\pi)/X)\chi_\pi(\mathfrak{p}) = O(X^{1/2}\log^2(X\mathrm{N}(\mathfrak{p})))$$

for $\mathfrak{p} \neq (1-\omega)$.

Proof. We write $\mathfrak{p} = (\tau)$ with $\tau \equiv 1 \bmod 3$. Then, using ray class characters ξ modulo 9 to remove the condition that $\pi \equiv 1 \bmod 9$ and using cubic reciprocity (see Lemma 2.0.4) we can write

$$\sum_{\pi\equiv 1(9)} w(\mathrm{N}(\pi)/X)\chi_\pi(\mathfrak{p}) = \frac{1}{9}\sum_{\xi\in G}\sum_{\pi\equiv 1(3)} w(\mathrm{N}(\pi)/X)\chi_\pi(\tau)\xi(\pi),$$

where G is the dual of the group $J^{(9)}/P^{(9)}$. Applying Riemann Stieltjes integration, the above double summation becomes

$$\frac{1}{9}\sum_{\xi\in G}\int_{3}^{\infty}\frac{w(t/X)}{\log t}d\Bigg[\sum_{\pi\equiv 1(3),\mathrm{N}(\pi)\leqslant t}(\chi_\tau\xi)(\pi)\log \mathrm{N}(\pi)\Bigg].$$

We can regard $\chi_\tau \xi$ as a Dirichlet character modulo (9τ). By Lemma 3.0.6 and using integration by parts, the above integral is

$$\ll \int_3^\infty \sqrt{t}\log t \log^2(t\mathrm{N}(\tau))\frac{|w'(t/X)|}{X\log t}dt + \int_3^\infty \log^2(t\mathrm{N}(\tau))\frac{w(t/X)}{\sqrt{t}\log t}dt. \quad (4.1)$$

We estimate these integrals for the rest of the proof. For the first integral, we have

$$\frac{1}{X}\int_3^\infty \sqrt{t}\log^2(t\mathrm{N}(\tau))|w'(t/X)|dt \ll X^{1/2}\log^2(\mathrm{N}(\tau)X).$$

This estimation follows by dividing the integral into two integrals according as $3 < t \leqslant X$ and $X < t < \infty$: When $3 < t \leqslant X$, we have

$$\frac{1}{X}\int_3^X \sqrt{t}\log^2(t\mathrm{N}(\tau))|w'(t/X)|dt \ll X^{1/2}\log^2(X\mathrm{N}(\tau)).$$

When $X < t < \infty$, we have

$$\begin{aligned}\frac{1}{X}\int_X^\infty \sqrt{t}\log^2(t\mathrm{N}(\tau))|w'(t/X)|dt &\ll X\int_X^\infty \frac{\sqrt{t}}{t^2}\log^2(\mathrm{N}(\tau)t)dt \\ &\ll X^{1/2}\log^2(X\mathrm{N}(\tau))\end{aligned}$$

since

$$\begin{aligned}\int_X^\infty \frac{1}{t^{3/2}}\log^2(t\mathrm{N}(\tau))dt &= \int_X^\infty \frac{1}{t^{3/2-0.1}}\frac{\log^2(t\mathrm{N}(\tau))}{t^{0.1}}dt \\ &\ll \frac{\log^2(X\mathrm{N}(\tau))}{X^{0.1}}\int_X^\infty \frac{1}{t^{3/2-0.1}}dt \\ &\ll \log^2(X\mathrm{N}(\tau))X^{-1/2}.\end{aligned}$$

For the second integral, we have

$$\int_3^\infty \frac{1}{\sqrt{t}\log t}\log^2(t\mathrm{N}(\tau))w(t/X)dt \ll X^{1/2}\log^2(\mathrm{N}(\tau)X).$$

Then collecting the estimates, (4.1) is $\ll X^{1/2}\log^2(X\mathrm{N}(\tau))$. □

4.0.1 Conclusion

Now, we are ready to bound

$$\sum_{k=1}^{2}\sum_{\mathfrak{p}\nmid 3}\hat{\phi}\left(\frac{k\log \mathrm{N}\mathfrak{p}}{\log X}\right)\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{k/2}}\sum_{\pi\equiv 1(9)} w(\mathrm{N}(\pi)/X)\chi_\pi(\mathfrak{p}).$$

The case $k = 2$ is negligible since the resulting estimate is certainly smaller than the case $k = 1$. Therefore, we only work with $k = 1$:

$$\begin{aligned}
&\sum_{\mathfrak{p}\nmid 3} \hat{\phi}\left(\frac{\log \mathrm{N}\mathfrak{p}}{\log X}\right) \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1/2}} \sum_{\pi \equiv 1(9)} w(\mathrm{N}(\pi)/X)\chi_\pi(\mathfrak{p}) \\
&\qquad = \int_3^{X^v} \hat{\phi}\left(\frac{\log t}{\log X}\right) d\left[\sum_{\mathfrak{p}\nmid 3, 3<\mathrm{N}\mathfrak{p}\leqslant t} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1/2}} \sum_{\pi \equiv 1(9)} w(\mathrm{N}(\pi)/X)\chi_\pi(\mathfrak{p})\right] \\
&\qquad = -\frac{1}{\log X}\int_3^{X^v} \hat{\phi}'\left(\frac{\log t}{\log X}\right) \frac{1}{t}\left[\sum_{\mathfrak{p}\nmid 3, 3<\mathrm{N}\mathfrak{p}\leqslant t} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1/2}} \sum_{\pi \equiv 1(9)} w(\mathrm{N}(\pi)/X)\chi_\pi(\mathfrak{p})\right] dt.
\end{aligned}$$

By Lemma 4.0.4, this integral is

$$\begin{aligned}
&\ll \frac{X^{1/2}}{\log X}\int_3^{X^v} \frac{1}{t}\left[\sum_{\mathfrak{p}\nmid 3, 3<\mathrm{N}\mathfrak{p}\leqslant t} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1/2}} \log^2(X\mathrm{N}(\mathfrak{p}))\right] dt \\
&\qquad \ll \frac{X^{1/2}}{\log X}\int_3^{X^v} \frac{\log^2(Xt)}{t}\left[\sum_{\mathfrak{p}\nmid 3, 3<\mathrm{N}\mathfrak{p}\leqslant t} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1/2}}\right] dt.
\end{aligned}$$

Using the estimate

$$\sum_{\mathrm{N}\mathfrak{p}\leqslant t} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1/2}} \ll t^{1/2},$$

we conclude that

$$\begin{aligned}
\sum_{\mathfrak{p}\nmid 3} \hat{\phi}\left(\frac{\log \mathrm{N}\mathfrak{p}}{\log X}\right) \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1/2}} \sum_{\pi \equiv 1(9)} w(\mathrm{N}(\pi)/X)\chi_\pi(\mathfrak{p}) &\ll \frac{X^{1/2}}{\log X}\int_3^{X^v} \frac{\log^2(Xt)}{t^{1/2}} dt \\
&\ll X^{v/2+1/2}\log X.
\end{aligned}$$

Using Lemmas 4.0.2, 4.0.3 together with this estimate in 3.15, and letting $X \to \infty$, Theorem 1.0.2 follows provided that $v < 1$.

Bibliography

[1] N. M. Katz and P. Sarnak, "Zeroes of zeta functions and symmetry," *Bull. Amer. Math. Soc. (N.S.)*, vol. 36, no. 1, pp. 1–26, 1999.

[2] N. M. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, vol. 45 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1999.

[3] C. David and A. M. Güloğlu, "One-Level Density and Non-Vanishing for Cubic L-Functions Over the Eisenstein Field," *International Mathematics Research Notices*, vol. 2022, pp. 18833–18873, 09 2021.

[4] S. Chowla, "The Riemann hypothesis and Hilbert's tenth problem," *Norske Vid. Selsk. Forh. (Trondheim)*, vol. 38, pp. 62–64, 1965.

[5] J. S. Ellenberg, W. Li, and M. Shusterman, "Nonvanishing of hyperelliptic zeta functions over finite fields," *Algebra Number Theory*, vol. 14, no. 7, pp. 1895–1909, 2020.

[6] A. E. Özlük and C. Snyder, "On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis," *Acta Arith.*, vol. 91, no. 3, pp. 209–228, 1999.

[7] K. Soundararajan, "Nonvanishing of quadratic Dirichlet L-functions at $s = \frac{1}{2}$," *Ann. of Math. (2)*, vol. 152, no. 2, pp. 447–488, 2000.

[8] P. Gao and L. Zhao, "One level density of low-lying zeros of quadratic Hecke L-functions to prime moduli," *Hardy-Ramanujan J.*, vol. 43, pp. 173–187, 2020.

[9] C. David, A. Florea, and M. Lalin, "Nonvanishing for cubic L-functions," *Forum Math. Sigma*, vol. 9, pp. Paper No. e69, 58, 2021.

[10] E. Fouvry and H. Iwaniec, "Low-lying zeros of dihedral l-functions," *Duke Mathematical Journal*, vol. 116, 03 2003.

[11] C. P. Hughes and S. J. Miller, "Low-lying zeros of l-functions with orthogonal symmetry," *Duke Mathematical Journal*, vol. 136, jan 2007.

[12] C. David, A. Florea, and M. Lalin, "Nonvanishing for cubic l-functions," *Forum of Mathematics, Sigma*, vol. 9, p. e69, 2021.

[13] G. Ricotta and E. Royer, "Statistics for low-lying zeros of symmetric power l-functions in the level aspect," vol. 23, no. 5, pp. 969–1028, 2011.

[14] S. Drappeau, K. Pratt, and M. Radziwill, "One-level density estimates for Dirichlet L-functions with extended support." working paper or preprint, Feb. 2020.

[15] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, vol. 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second ed., 1990.

[16] J. Neukirch, *Algebraic number theory*, vol. 322 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[17] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, vol. 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[18] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. Academic Press, Inc., San Diego, CA, sixth ed., 2000. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger.

[19] H. Davenport, *Multiplicative number theory*, vol. 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second ed., 1980. Revised by Hugh L. Montgomery.

[20] J.-P. Serre, "Quelques applications du théorème de densité de Chebotarev," *Inst. Hautes Études Sci. Publ. Math.*, no. 54, pp. 323–401, 1981.

[21] H. Stark, "Some effective cases of the brauer-siegel theorem.," *Inventiones mathematicae*, vol. 23, pp. 135–152, 1974.