Relaxation of multidimensional variational problems with constraints of general form

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This paper is devoted to further development of an idea of a well-known theorem of Bogolubov [2]. Here we construct a relaxation of multidimensional variational problems with constraints of rather general form on gradients of admissible functions; it is assumed that the gradient of an admissible function belongs to an arbitrary bounded set. This relaxation involves as a class of admissible functions the closure of the class of admissible functions of the original problem in the topology of uniform convergence, and uses a theorem characterizing this closure, which is proved in [15]. The case when the gradient of an admissible function is constrained within a bounded closed convex body is studied in the works [13,15,19].

The study of multidimensional variational problems was started in 1970s by Ekeland and Temam [13]. The existing literature on relaxation of variational problems, including two monographs by Buttazzo [3] and Dacorogna [9], and the review paper by Marcellini [18] containing a considerable list of references, is quite rich. However, the author failed to find a setting similar to that of the paper. For the most recent results on relaxation and related topics see [1,4–8,11,14].

This paper deals with the case where an integrand depends on a scalar function of several variables. At the end of the paper we will make a conjecture on generalization of the main relaxation result of the paper to the case of an integrand depending on a vector function of several variables. We also make a conjecture on generalization of

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the above-mentioned theorem on characterization of the closure, which is an important tool in the proof of the main result, for the vectorial case.

$R^n$ will stand for $n$-dimensional Euclidean space of points $t = (t_1, \ldots, t_n)$. Let $\Omega$ be an arbitrary bounded open set in $R^n$. Denote by $C(\Omega)$ the space of all real continuous functions on $\Omega$ with the norm

$$\|x(\cdot)\|_{C(\Omega)} = \max_{t \in \Omega} |x(t)|.$$ 

Denote by $W^1_{\infty}(\Omega)$ the Sobolev space of all essentially bounded measurable functions on $\Omega$, with essentially bounded first generalized partial derivatives. It is well known that a function $x(\cdot)$ from $W^1_{\infty}(\Omega)$ is continuous on $\Omega$ and possesses the ordinary first derivatives $\partial x/\partial t_i$ ($i = 1, \ldots, n$) almost everywhere (a.e.) on $\Omega$ (see [13,20]). If domain $\Omega$ satisfies additional conditions (e.g., if $\Omega$ is Lipschitzian), then $W^1_{\infty}(\Omega) \subset C(\Omega)$. Let $\overline{W}^1_{\infty}(\Omega) = W^1_{\infty}(\Omega) \cap C(\Omega)$. So, if $\Omega$ is sufficiently regular, then $\overline{W}^1_{\infty}(\Omega) = W^1_{\infty}(\Omega)$.

Denote by $B_r(0)$ a ball in $R^n$ with the center at the origin and radius $r$. Given a set $V \subset R^n$ and a positive number $r$ let $V_r = \{v \in V : \text{dist}(v, \partial V) \geq r\}$, where $\partial V$ is the boundary of $V$.

Recall that function $x(\cdot) : \overline{\Omega} \to R$ is said to be piece-wise affine, if it is continuous and there exists a partition of $\overline{\Omega}$ into a subset of measure zero and a finite number of open sets, on which $x(\cdot)$ is affine. A continuous function on $\overline{\Omega}$ is said to be almost piece-wise affine, if its restriction to an arbitrary strict interior subdomain of $\Omega$ is piece-wise affine.

Let $X$, $Y$ be topological spaces, and $I, J$ be functionals defined on $X$ and $Y$, respectively. The variational problem $\inf\{J(y) : y \in Y\}$ is said to be a relaxation of the problem $\inf\{I(x) : x \in X\}$, if there exists a continuous mapping $i : X \to Y$, such that: (i) $i(X)$ is dense in $Y$, (ii) $J(i(x)) \leq I(x)$ for each $x \in X$, and (iii) for an arbitrary $y \in Y$ there exists a sequence $x_k \in X$ ($k \in N$) such that $i(x_k) \to y$ and $J(y) \geq \lim_{k \to \infty} I(x_k)$. Moreover, if functional $J$ is lower semicontinuous, then a relaxation is called a lower semicontinuous relaxation (see [16]).

Let $f : \overline{\Omega} \times R \times R^n \to R$ be a continuous function, $U$ be an arbitrary bounded set in $R^n$ with an affine hull $R^n$, $\Gamma \subset \partial \Omega$ and $\phi : \Gamma \to R$ be some fixed function. Consider the following problem of multidimensional variational calculus, which we will refer to as problem (P):

$$J(x(\cdot)) = \int_{\Omega} f(t, x(t), \text{grad} \, x(t))d(t) \to \inf,$$

$$\text{grad} \, x(t) \in U \quad \text{a.e. in } \Omega,$$

$$x(t) = \phi(t) \quad \text{for } t \in \Gamma,$$

where $x(\cdot) \in \overline{W}^1_{\infty}(\Omega)$. The case when $\Gamma = \emptyset$, i.e., when the boundary condition (3) is absent, will be referred to as problem (P$_0$).

A function $x(\cdot) \in \overline{W}^1_{\infty}(\Omega)$ is called admissible in problem (P)((P$_0$)), if it satisfies conditions (2), (3) ((2)). The set of all admissible functions in problem (P)((P$_0$)) will
be denoted by $E(U, \phi)(E(U))$. Thus

$$E(U) = \{ x(\cdot) \in \overline{W}_1^\infty(\Omega) : \text{grad} x(t) \in U \text{ a.e. in } \Omega \},$$

$$E(U, \phi) = \{ x(\cdot) \in E(U) : x(\cdot)|_\Gamma = \phi \}.$$

The space $\overline{W}_1^\infty(\Omega)$ and its subsets $E(U)$, $E(U, \phi)$ will be considered with the metric of uniform convergence.

Along with problem (P) we consider the following problem (problem (PR)):

$$J_R(x(\cdot)) = \int_\Omega f_U^{**}(t, x(t), \text{grad} x(t))d(t) \to \inf,$$

$$x(t) = \phi(t) \quad \text{for } t \in \Gamma,$$

(1')

where $\overline{co} U$ is the closed convex hull of $U$ and $f_U^{**}(t, x, \cdot) = (f(t, x, \cdot) + \delta(\cdot|U))^{**}$. Here

$$\delta(u|U) = \begin{cases} 0 & \text{for } u \in U, \\ +\infty & \text{for } u \in R^n \setminus U \end{cases}$$

is the indicator function of $U$, and ** designates the operation of taking second Young–Fenchel conjugate (see [17, p. 183]). In case of $\Gamma = \emptyset$ problem (PR) will be denoted as (P0R).

The above-mentioned assertion on closure consists of the following:

$$\overline{E}(U) = E(\overline{co} U),$$

i.e. the closure in the uniform metric of a class of functions continuous on $\Omega$ with gradients from the bounded set $U$ coincides with the class of functions continuous on $\overline{\Omega}$ and with gradients from the closed convex hull of $U$. Moreover, if condition (4) of Theorem 1 below is satisfied, then Theorem 1' from Hüsseinov [15] implies the following coincidence

$$E(U, \phi) = E(\overline{co} U, \phi).$$

**Theorem 1.** Let $U \subset R^n$ be an arbitrary bounded set in $R^n$ with the affine hull $R^n$. Suppose that there exists an admissible function $y_0(\cdot) \in E(\overline{co} U, \phi)$ in problem (PR) such that

$$\text{grad} y_0(t) \in U_0 \quad \text{a.e. in } \Omega,$$

(4)

where $U_0$ is a closed set contained in the interior of $\overline{co} U$. Then, for an arbitrary function $x(\cdot) \in E(\overline{co} U, \phi)$ admissible in problem (PR), there exists a sequence of functions $x_k(\cdot)$ ($k \in N$), admissible in problem (P), uniformly converging to $x(\cdot)$, and such that

$$\lim_{k \to \infty} J(x_k(\cdot)) = J_R(x(\cdot)).$$

In particular, when the boundary condition (3) is absent, i.e. for problem (P0), condition (4) in Theorem 1 is satisfied automatically.
The following lemma will be used in the proof of Theorem 1.

**Lemma.** Let $T$ be a topological space, $U$ be an arbitrary bounded set in $\mathbb{R}^n$, $U_0 \subset U$ be a compact set contained in the interior of $\overline{co} \ U$ or a segment, and $f : T \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. Then a restriction of function $f_U^**(\tau, u)$ to $T \times U_0$ is continuous.

**Proof.** Since $f_U^* = f_T^*$, we suppose, without loss of generality, that $U$ is closed. Fix a point $(\tau_0, u_0) \in T \times U_0$ and a positive number $\varepsilon$. It is easily seen that, there exists a neighborhood $S(\tau_0)$ of point $\tau_0$ such that

$$|f(\tau, u) - f(\tau_0, u)| < \varepsilon \quad \text{for} \quad \tau \in S(\tau_0), \quad u \in \overline{co} \ U. \quad (5)$$

It is well known that

$$f_U^* = \min \left\{ \sum_{i=1}^{n+1} \lambda_i f(\tau, u_i) : \sum_{i=1}^{n+1} \lambda_i u_i = u, \ u_i \in U, \ \sum_{i=1}^{n+1} \lambda_i = 1, \ \lambda_i \geq 0 \right\}. \quad (6)$$

From this and (5) we obtain that

$$f_U^*(\tau, u) = \sum_{i=1}^{n+1} \tilde{\lambda}_i f(\tau, \tilde{u}_i) \geq \sum_{i=1}^{n+1} \tilde{\lambda}_i f(\tau_0, \tilde{u}_i) - \varepsilon \geq f_U^*(\tau_0, u) - \varepsilon.$$

Symmetrically,

$$f_U^*(\tau_0, u) \geq f_U^*(\tau, u) - \varepsilon.$$

Consequently,

$$|f_U^*(\tau, u) - f_U^*(\tau_0, u)| < \varepsilon \quad \text{for} \quad \tau \in S(\tau_0), \quad u \in \overline{co} \ U. \quad (7)$$

Since $f_U^*(\tau_0, \cdot)$ is a convex and lower semicontinuous it is continuous on $U$ (in both the cases stipulated in the lemma). Therefore, there exists a number $\delta > 0$ such that

$$|f_U^*(\tau, u) - f_U^*(\tau_0, u_0)| < \varepsilon \quad \text{for} \quad u \in U_0, \quad \|u - u_0\| < \delta. \quad (8)$$

The last two inequalities imply that

$$|f_U^*(\tau, u) - f_U^*(\tau_0, u_0)| < 2\varepsilon$$

for $\tau \in S(\tau_0), \quad \|u - u_0\| < \delta$. Therefore, function $f_U^*|_{T \times U_0}$ is continuous at the point $(\tau_0, u_0)$.

**Proof of Theorem 1.** Let $x(\cdot) \in E(\overline{co} \ U, \phi)$ be an admissible function in problem (PR) and $\varepsilon > 0$. Consider the sequence of functions $x_k(t) = ((k - 1)/k)x(t) + (1/k)y_0(t)$ ($k \in \mathbb{N}$). Clearly, $x_k(\cdot) \in E(\overline{co} \ U, \phi)$ and

$$x_k(\cdot) \rightarrow k x(\cdot) \quad \text{uniformly on} \ \Omega, \quad (6)$$

$$\text{grad} \ x_k(t) \rightarrow \text{grad} \ x(t) \quad \text{for a.a.} \ t \in \Omega, \quad (7)$$

$$\text{grad} \ x_k(t) + B_{r_k}(0) \subset U \quad \text{for a.a.} \ t \in \Omega, \quad (8)$$

where $r_k$, ($k \in \mathbb{N}$) are positive numbers.
It follows from relations (6), (8) and the lemma that
\[
\|x_k(\cdot) - x_2(\cdot)\|_{C(\Omega)} \leq \frac{\varepsilon}{4},
\]
\[
|J_R(x_k(\cdot)) - J_R(x(\cdot))| \leq \frac{\varepsilon}{4}
\]
(9)
for sufficiently large indices \(k\).

Let \(k_0\) be such that (9) holds for \(k_0\). Let \(\bar{x}(\cdot) = x_{k_0}(\cdot), r = r_{k_0}/2\). By Theorem 1' from Hüsseinov [15], there exists a sequence of almost piece-wise affine functions \(y_k(\cdot) \in E(\overline{\varnothing} U, \phi)\) uniformly converging to \(x(\cdot)\). Then the sequence of vector functions \(y_k(\cdot) (k \in N)\) weakly converges to vector function \(\text{grad} \bar{x}(\cdot)\) in Banach space \(L^2(\Omega)\) of summable \(n\)-vector functions on domain \(\Omega\). By Mazur’s Theorem (Corollary 3.14 from Dunford and Schwartz [12, p. 457]) it follows that there exist convex combinations \(z_m(\cdot) = \sum_{k=1}^{N_m} \alpha_k z_k(\cdot) (m \in N)\) of functions \(y_k(\cdot) (k \in N)\), where \(\alpha_k \geq 0, \sum_{k=1}^{N_m} \alpha_k = 1\) and \(N_m (m \in N)\) is a strictly increasing sequence of integers such that
\[
\text{grad} z_m(t) \to \text{grad} \bar{x}(t) \quad \text{for a.a. } t \in \Omega.
\]
(10)
Thus, the functions \(z_m(\cdot)\) are almost piece-wise affine, \(z_m(\cdot) \in E((\overline{\varnothing} U)_r, \phi)\) \((m = 1, 2, \ldots)\), the sequence \(z_m(\cdot) (m \in N)\) uniformly converges to \(\bar{x}(\cdot)\), and condition (10) is satisfied. From that we obtain
\[
\|z_m(\cdot) - \bar{x}(\cdot)\|_{C(\Omega)} \leq \frac{\varepsilon}{4},
\]
\[
|J_R(z_m(\cdot)) - J_R(\bar{x}(\cdot))| \leq \frac{\varepsilon}{4}
\]
(11)
for sufficiently large \(m\). Fix one of such indices \(m_0\) and denote \(\tilde{z}(\cdot) = z_{m_0}(\cdot)\). We obtain from relations (9) with \(k = k_0\) and (11) with \(m = m_0\)
\[
\|\tilde{z}(\cdot) - x(\cdot)\|_{C(\Omega)} \leq \frac{\varepsilon}{2},
\]
\[
|J_R(\tilde{z}(\cdot)) - J_R(x(\cdot))| \leq \frac{\varepsilon}{2}
\]
(12)
So, function \(\tilde{z}(\cdot)\) is almost piece-wise affine, \(\tilde{z}(\cdot) \in E((\overline{\varnothing} U)_r, \phi)\) and satisfies relations (12).

Denote \(M = 1 + \max \{|x(t)|\}\). Since integrand \(f\) is continuous on compact \(K = \overline{\Omega} \times [-M,M] \times \overline{U}\), there exists a positive number \(\delta_0 < \varepsilon/2\) such that
\[
|f(t_1, x_1, u_1) - f(t_2, x_2, u_2)| \leq \frac{\varepsilon}{2}
\]
(13)
for \((t_1, x_1, u_1), (t_2, x_2, u_2) \in K, \|t_1 - t_2\| < \delta_0, \|u_1 - u_2\| < \delta_0\).

In sequel, we shall omit the index \(U\) in notation \(f^{**}_U\). By the lemma function \(f^{**}\) is continuous on compact \(K_r = \overline{\Omega} \times [-M,M] \times (\overline{\varnothing} U)_r\). Hence, there exists \(\delta_0 \in (0, \delta_0)\) such that
\[
|f^{**}(t_1, x_1, u_1) - f^{**}(t_1, x_1, u_1)| < \frac{\varepsilon}{2}
\]
(14)
for \((t_1, x_1, u_1), (t_2, x_2, u_2) \in K, ||t_1 - t_2|| < \delta_0, ||u_1 - u_2|| < \delta_0\). Since the functions \(x(\cdot)\) and \(\tilde{z}(\cdot)\) are continuous on \(\Omega\), there exists \(\delta \in (0, \delta_0/2)\) such that

\[
|x(t_1) - x(t_2)| < \delta_0, \quad |\tilde{z}(t_1) - \tilde{z}(t_2)| < \frac{\delta_0}{2} \quad \text{for} \quad ||t_1 - t_2|| < \delta. \quad (15)
\]

Denote by \(\Delta_j (j \in N)\) the simplices of affineness of function \(\tilde{z}(\cdot)\), \(a_j = \text{grad} z(t)\) for \(t \in \text{int} \Delta_j (j \in N)\). Without loss of generality, we assume that \(\text{diam} \Delta_j < \delta (j \in N)\). Fix \(t_j \in \Delta_j (j \in N)\). It is well known that

\[
f^{**}(t_j, \tilde{z}(t_j), a_j)
= \inf \left\{ \sum_{i=1}^{n+1} \alpha_i^j f(t_j, \tilde{z}(t_j), v_i^j) : \sum_{i=1}^{n+1} \alpha_i^j = a_j, v_i^j \in U, \sum_{i=1}^{n+1} \alpha_i^j \geq 0 \right\}.
\]

Then for some numbers \(\alpha_i^j > 0 (i=1,2,\ldots,n+1), \sum_{i=1}^{n+1} \alpha_i^j = 1\) and affinely independent vectors \(v_i^j (i=1,2,\ldots,n+1)\) from \(U\)

\[
\left| f^{**}(t_j, \tilde{z}(t_j), a_j) - \sum_{i=1}^{n+1} \alpha_i^j f(t_j, \tilde{z}(t_j), v_i^j) \right| < \frac{\epsilon}{2},
\]

\[
\sum_{i=1}^{n+1} \alpha_i^j v_i^j = a_j.
\]  

(16)

Put \(u_i^j = v_i^j - a_j (i=1,2,\ldots,n+1)\) and denote \(\sum_j = \text{co}\{u_1^j, \ldots, u_{n+1}^j\}\). Since, vectors \(u_i^j (i=1,2,\ldots,n+1)\) are affinely independent and \(\sum_{i=1}^{n+1} \alpha_i^j v_i^j = 0\), where \(\alpha_i^j > 0 (i=1,2,\ldots,n+1)\) then \(\sum_j\) is an \(n\)-dimensional simplex with the interior containing zero.

Denote \(D_j = \sum_0^j\) polar of the simplex \(\sum_j\), \(s_j(\cdot)\) – support function of set \(\{u_1^j, \ldots, u_{n+1}^j\}\).

Partition simplex \(\Delta_j\) into at most countably many simplices \(\Delta_k^j, \Delta_2^j, \ldots\), homothetic to \(D_j\) and such that \(\text{diam} \Delta_k^j < \delta \text{diam} D_j\). Denote by \(d_k^j\) the similarity coefficients of simplices \(\Delta_k^j\) and \(D_j\) and put

\[
s_k^j (t) = \begin{cases} s(t - t_k^j) - d_k^j & \text{for} \ t \in \Delta_k^j, \\ 0 & \text{for} \ t \in \Omega \setminus \Delta_k^j \end{cases}
\]

and \(\sigma_i(\Delta_k^j) = \{t \in \Delta_k^j : s_k^j(t) = (t - t_k^j, u_k^j) - d_k^j\} (i = 1,2,\ldots,n+1)\), for arbitrary indices \(j,k\), where \(t_k^j \in \Delta_k^j\) is the image of the origin under the homothety \(D_j \to \Delta_k^j\). Obviously, function \(s_k^j(\cdot)\) is piece-wise affine and

\[-\delta \leq s_k^j(t) \leq 0.\]  

(17)

Put

\[
s(t) = \sum_{j,k} s_k^j(t) \quad \text{and} \quad z(t) = \tilde{z}(t) + s(t).
\]

Since

\[
\text{grad} z(t) = \text{grad} \tilde{z}(t) + u_i^j = a_j + u_i^j = v_i^j \in U \quad \text{for} \ t \in \sigma_i(\Delta_k^j)
\]
and simplices $\sigma_i(\Delta^j_k)$ ($i = 1, 2, \ldots, n + 1$; $j, k \in N$) cover domain $\Omega$, then function $z(\cdot)$ is admissible in problem (P), i.e. $z(\cdot) \in E(U, \phi)$.

Utilizing inequalities (15)–(17) and Proposition 2 from Hüsseinov [15] we estimate the difference
\[
\left| \int_{\Delta^j_k} f^{**}(t, \tilde{z}(t), \text{grad} \tilde{z}(t)) \, dt - \int_{\Delta^j_k} f(t, z(t), \text{grad} z(t)) \, dt \right|
\]
\[
= \left| \int_{\Delta^j_k} f^{**}(t, \tilde{z}(t), \text{grad} \tilde{z}(t)) \, dt - \sum_{i=1}^{n+1} \int_{\sigma_i(\Delta^j_k)} f(t, \tilde{z}(t) + s^j_i(t), v^j_i) \, dt \right|
\]
\[
\leq \left| \text{mes}(\Delta^j_k) f^{**}(t_j, \tilde{z}(t_j), a_j) - \sum_{i=1}^{n+1} \alpha^j_i \text{mes}(\Delta^j_k) f(t_j, \tilde{z}(t_j), v^j_i) \right| + \varepsilon \text{mes}(\Delta^j_k)
\]
\[
= \text{mes}(\Delta^j_k) \left[ f^{**}(t_j, \tilde{z}(t_j), a_j) - \sum_{i=1}^{n+1} \alpha^j_i f(t_j, \tilde{z}(t_j), v^j_i) \right] \leq 2 \varepsilon \text{mes}(\Delta^j_k).
\] (18)

Summing up inequalities (18) by $j, k$ we obtain
\[
|J_{f^{**}}(\tilde{z}(\cdot)) - J(z(\cdot))| < 2 \varepsilon \text{mes}(\Omega).
\] (19)

It is clear from (17) that
\[
\|\tilde{z}(\cdot) - z(\cdot)\|_{C(\overline{\Omega})} < \frac{\varepsilon}{2}.
\]

From this and from the first of inequalities (12) it follows that
\[
\|z(\cdot) - x(\cdot)\|_{C(\overline{\Omega})} < \varepsilon,
\]
and from (19) and from the second of inequalities (12) that
\[
|J_R(x(\cdot)) - J(z(\cdot))| < \varepsilon[1 + 2 \text{mes}(\Omega)].
\]

The theorem is proved. \Box

Theorem 1 and Lemma 4 from Hüsseinov [15] imply the following result.

**Theorem 2.** Let $U$ be a bounded set in $R^n$ with an affine hull $R^n$, and assumption (4) of Theorem 1 be satisfied. Then problem (PR) is a lower semicontinuous relaxation of problem (P).

For $U \subset R^{m \times n}$ the closure of the quasiconvex hull is defined as (see [10, Definition 2.2]):
\[
\overline{Qco} \ U = \{ \hat{\zeta} \in R^{m \times n} : f(\hat{\zeta}) \leq 0, \forall f : R^{m \times n} \to R, \text{quasiconvex and } f|_U = 0 \}.
\]

We denote for $U \subset R^{m \times n}$
\[
E(U) = \{ x(\cdot) \in W^1_{\infty}(\Omega; R^n) : Dx(t) \in U \text{ a.e. in } \Omega \},
\]
where \(Dx(t)\) denotes the Jacobi matrix of \(x(\cdot)\) at \(t\). We conjecture the following coincidence: 
\[
\overline{E(U)} = E(\overline{QcoU}),
\]
where \(\overline{E(U)}\) denotes the closure of \(E(U)\) in uniform metric of \(W^1_\infty(\Omega; \mathbb{R}^m)\).

Consider the following two variational problems. The first is the problem \((P)\) obtained from \((P)\) by treating \(f\) as a function \(R^{m \times n} \rightarrow \mathbb{R}\), \(\text{grad} x(t)\) replaced by \(Dx(t)\) the Jacobi matrix of \(x(\cdot) : \Omega \rightarrow \mathbb{R}^m\) at \(t\), and \(\phi(\cdot) : \Gamma \rightarrow \mathbb{R}^m\). The second problem is
\[
J_R(x(\cdot)) = \int_{\Omega} Qf_U(t,x(t),Dx(t)) \, dt \rightarrow \inf,
\]
where \(Qf_U(t,x,\cdot)\) is the quasiconvex envelope (i.e. the maximal quasiconvex function not exceeding \(f\)) of the function \(f(t,x,\cdot) + \delta(\cdot|U)\), \(\delta(\cdot|U)\) being the indicator function of \(U\).

**Conjecture.** Let \(U \subset R^{m \times n}\) be an arbitrary bounded set with \(\overline{QcoU}\) having an interior point. Suppose that there exists an admissible function \(y_0(\cdot) \in E(\overline{QcoU}, \varphi)\) in problem \((\mathcal{PR})\) such that \(Dy_0(t) \in U_0\) a.e. in \(\Omega\), where \(U_0\) is a closed set contained in the interior of \(\overline{QcoU}\), then for an arbitrary vector function \(x(\cdot) \in E(\overline{QcoU}, \varphi)\) admissible in problem \((\mathcal{PR})\), there exists a sequence of vector-functions \(x_k(\cdot) (k \in \mathbb{N})\) admissible in problem \((\mathcal{P})\), uniformly converging to \(x(\cdot)\) and such that
\[
\lim J(x_k(\cdot)) = J_R(x(\cdot)).
\]

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**References**


