Journal of Economic Theory **87**, 434–449 (1999)
Article ID jeth.1999.2547, available online at http://www.idealibrary.com on IDEAL®

Boundary Behavior of Excess Demand and Existence of Equilibrium

Farhad Hüsseinov*

Department of Economics, Bilkent University, 06533 Bilkent, Ankara, Turkey
farhad@bilkent.edu.tr

Received October 24, 1997; revised April 16, 1999

This paper presents a market equilibrium existence theorem that generalizes and unifies many well-known results. The importance of the theorem is illustrated by applications to large exchange economies. A further extension of Aumann's Existence theorem is obtained which dispenses with the monotony assumption on preferences. In addition the market equilibrium results of Grandmont and Neuefeind are compared. It is shown that the boundary condition of Grandmont's result is equivalent to some natural relaxation of Neuefeind's. In the case of two commodities they are equivalent but for a greater number of commodities Neuefeind's condition is stronger. *Journal of Economic Literature* Classification Numbers: C62, D51. © 1999 Academic Press

1. INTRODUCTION

Finding reasonable conditions on an excess demand function which guarantee that the excess demand vanishes for some price vector has an important generalizing effect. Indeed, if such conditions are found, they can be used in order to learn if an equilibrium exists for an economic environment, by checking whether an excess demand function generated by that environment satisfies one of these conditions. The classical results on such conditions are due to Debreu, Gale, and Nikaido (see Debreu [5, Chapter 5]) and Arrow and Hahn [1, Chapter 2], and comparatively recent ones are due to Grandmont [8, 9] and Neuefeind [12]. Following Debreu [5] we will refer to these results as the *market equilibrium results*.

Here we present a new market equilibrium result and apply it to large economies. This result strengthens and generalizes all the paradigmatical

* I am thankful to a referee and an associate editor for helpful comments.

0022-0531/99 $30.00

results of the sort listed above except the Debreu–Gale–Nikaido (D–G–N) lemma. It is equivalent to the D–G–N lemma and could be considered as yet another form of the Kakutani theorem, convenient for economic applications. Neuefeind [12] notes in respect to its prototypes from his paper that it constitutes "a very convenient technical tool for existence of equilibrium proofs" and shed light on the many existing of equilibrium existence proofs. Furthermore, it greatly simplifies the proof of many competitive equilibrium existence results and leads naturally to useful extensions to them.

The usefulness of the theorem is demonstrated by showing how it can be used to extend the equilibrium results for large economies. Applying it to large exchange economies we prove a further improvement of Aumann's existence theorem [3]. Schmeidler [15] extends this theorem to the case of large economies with the incomplete preferences of agents. We prune further this theorem: we show that on dispensing with the monotony assumption on preferences we will still have the existence of competitive quasi-equilibria. It should be emphasized that here the monotony assumption is dispensed with completely, i.e., it is not replaced by any variant of insatiability. Hence this result would apply for economies where agents are satiated in their consumption set. From here we conclude that with the assumption of strictly positive individual initial endowments or alternatively, under an additional very slight (global) insatiability assumption there exists a competitive equilibrium. The first variant simultaneously generalizes Aumann's auxiliary theorem [3]. Schmeidler's auxiliary theorem is also generalized to the case of nonmonotonic preferences. Note that both of these theorems are of some independent interest.

Neuefeind [12], concluding his paper, emphasizes that his technique, i.e. the market equilibrium lemmas in his paper, is confined to the case of an excess demand defined on an open set and that it would be desirable to extend this technique to a more general case of excess demand defined not on an open set, e.g., the case treated in Arrow and Hahn [1, Chapter 2]. We demonstrate that the equilibrium results of Arrow and Hahn [1, Chapter 2], and even their much more general versions, easily follow from the market equilibrium result provided in the present paper.

Neuefeind, comparing his result with that of Grandmont, expresses strong belief that Grandmont's assumptions are weaker and gives an example of excess demand function for the case where the number of commodities is odd and is not less than three, demonstrating his claim for that case. However, the relationship between these two results was not fully clarified. We show that that the assumption of Grandmont's market equilibrium lemma on boundary behavior of excess demand is equivalent to some natural relaxation of its counterpart from Neuefeind's market equilibrium lemma. This equivalence reveals congeniality of two market equilibrium lemmas and gives a clear geometric sense to the Grandmont condition.

The remainder of the paper is organized as follows. In Section 2 we formulate a market equilibrium theorem and derive from this theorem strengthening and generalization of market equilibrium existence results due to Grandmont [9] and Neuefeind [12]. We show also how the case of an excess demand correspondence, which is not necessarily defined on an open set, can be handled by utilizing this theorem. In Section 3 we apply this result to study competitive equilibria in large markets with incomplete and nonmonotonic preferences. In the Appendix we clarify the relationship between Grandmont's and Neuefeind's market equilibrium lemmas.

2. MARKET EQUILIBRIUM THEOREM

Let R^l be the commodity space and

$$S=\left\{p\in R^l \;\middle|\; \sum_{i=1}^{l} p^i=1, p^i>0, i=1, ..., l\right\}$$

be an open price simplex. All topological notions in price simplex refer to the relative topology of the $l-1$-dimensional hyperplane containing S. The nonnegative and nonpositive orthants in R^l will be denoted by Ω and Ω_-, respectively. For any two vectors $x=(x^1, ..., x^l)$, $y=(y^1, ..., y^l)\in R^l$ we will write $x\geqslant y$, $x>y$, and $x\gg y$, iff $x^i\geqslant y^i$ for each i, $x\geqslant y$ but not $x=y$, and $x^i>y^i$ for each i, respectively.

THEOREM 1. (a) *Let D be a nonempty, convex, open subset of S. Let $\Phi\colon D\mapsto R^l$ be an upper hemicontinuous, compact-, convex-valued correspondence fulfilling the Walras law (in the strong form), i.e. $p\cdot\Phi(p)=0$ for each $p\in D$, and let there be $K\subset D$-compact such that the set*

$$\bar{K}=\{p\in D \mid \exists x\in\Phi(p),\ p'\cdot x<0, \forall p'\in K\}$$

has a positive distance from ∂D. Then there exists $p^*\in D$ such that $0\in\Phi(p^*)$.

(b) *If $\Phi\colon D\mapsto R^l$ satisfies the Walras law in the weak form, $p\cdot\Phi(p)\leqslant 0$ for each $p\in D$, then there exists $p^*\in D$ such that $\Phi(p^*)\cap\Omega_-\neq\varnothing$.*

Proof. (a) Suppose, on the contrary, $0\notin\Phi(p)$ for each $p\in D$. Let K_1 be a convex compact subset of D such that $K\cup\bar{K}\subset\operatorname{int} K_1$. Applying the

Debreu–Gale–Nikaido lemma (see Debreu [5, Chapter 7]) to the restriction of Φ to K_1 we obtain

$$p^* \in K_1 \qquad \text{and} \quad x^* \in \Phi(p^*), \quad x^* \in K_1^0, \tag{1}$$

where K_1^0 is the polar of K_1. Since $x^* \in K_1^0$, and $x^* \neq 0$ by the assumption, and $K \subset \operatorname{int} K_1$ we have

$$p \cdot x^* < 0, \qquad \forall p \in K.$$

By the definition of $\bar{K}$, $p^* \in \bar{K}$ and then $p^* \in \operatorname{int}\ K_1$. Denoting $H(p^*) = \{x \in R^l \mid p^* \cdot x = 0\}$ a hyperplane through the origin orthogonal to p^*, we obtain from the last inclusion

$$K_1^0 \cap H(p^*) = \{0\}. \tag{2}$$

Since $x^* \in \Phi(p^*)$, by the Walras law $x^* \in H(p^*)$. Since also $x^* \in K_1^0$ by (1), then (2) implies $x^* = 0$. This contradicts our assumption.

(b) Denote by Pr_p the orthogonal projection operator onto hyperplane $H(p)$. Put $\Phi'(p) = \text{Pr}_p(\Phi(p))$ the image of $\Phi(p)$ under Pr_p. Then, Φ' satisfies all of the assumptions of point (a). In particular,

$$\bar{K}' = \{p \in D \mid \exists x \in \Phi'(p),\, p' \cdot x < 0, \forall p' \in K\}$$

has a positive distance from ∂D, because since $p \gg 0$, if $x \notin \operatorname{int} K^0$-polar of K, then $\text{Pr}_p x \notin\ \operatorname{int}\ K^0$. Applying the already proved case (a) to Φ' we obtain $p^* \in D$ such that $0 \in \Phi'(p^*)$. Then, there exists $x^* \in \Phi(p)$ such that $Pr_p(x^*) = 0$. Since, $p \gg 0$ and $p \cdot x^* \leqslant 0$, it follows that $x^* \in \Omega_-$. Thus, $x^* \in \Phi(p) \cap \Omega_-$, and the theorem is proved.

Remark 1. Since any compact in S is contained in the convex hull of some finite set in D, a finite set will do as well in Theorem 1.

Remark 2 The inequality in the definition of $\bar{K}$ in point (a) can be reversed, i.e., we could assume that near the boundary of D the excess demand has a positive p-value for at least one $p \in K$. But this can not be done in point (b).

Remark 3. The assumption of Theorem 1 on existence of a compact K can be reformulated in the following way: there exists compact $K \subset D$, and a neighborhood V of the boundary ∂D in D such that for an arbitrary $p \in V$ there exists $p' \in K$ and $x \in \Phi(p)$, such that $p' \cdot x \geqslant 0$, i.e., p'-value of excess demand x is nonnegative. Clearly $V = D \backslash \bar{K}$.

Remark 4. In Theorem 1 the domain D of excess demand could be assumed to be a nonempty, convex, open subset of a hyperplane H in R^l that does not contain the origin.

In the particular case of an excees demand function satisfying the Walras law in the strong form, Theorem 1(a) implies the following generalization of Neuefeind's basic lemma [12, p. 1832]:

Let D be a nonempty, convex, open subset of S. Let $f\colon D \to R^l$ be a continuous function satisfying the Walras law, i.e., $p \cdot f(p) = 0$, for all $p \in D$, and let there be a compact subset $K \subset D$ such that the set $\bar{K} = \{p \in D \mid p' \cdot f(p) < 0, \forall p' \in K\}$ has a positive distance from ∂D. Then, there exists $p^* \in D$ such that $f(p^*) = 0$.

Notice that the assumption of Theorem 1 involving the compact K, weakens its prototype from Neuefeind's lemma. This weakening is twofold: first, K is not a singleton, and second, the inequality in the definition of $\bar{K}$ is strict. Therefore, the geometric meaning of that prototype is that near the boundary of D, excess demand belongs to an open half-space with normal $\bar{p}$ in D whereas, that of the condition of Theorem 1 is that near the boundary of D excess demand does not belong to an open cone, which contains the closed negative orthant except the origin.

Yet another particular case of Theorem 1 is obtained when K is a singleton:

Let D be a nonempty, convex open subset of S. Let $\Phi : D \mapsto R^l$ be an upper hemicontinuous, compact-, convex-valued correspondence fulfilling the Walras law, i.e., $p \cdot \Phi(p) = 0$ for each $p \in D$, and let there exists $\bar{p} \in D$ and a neighborhood V of ∂D in D such that $\bar{p} \cdot \Phi(p) \geqslant 0$ for each $p \in V$. Then, there exists $p^* \in D$ such that $0 \in \Phi(p^*)$.

This result implies the following

COROLLARY 6 (Debreu [6, Theorem 8]). *Let $\Phi : S \mapsto R^l$ be convex-valued, bounded below, upper hemicontinuous, and let it satisfy the Walras law and the following boundary condition: if $p_k \in S$ converges to $p \in \partial S$, then $\inf\{\|x\| \mid x \in \Phi(p_k)\} \to \infty$. Then, there is $p^* \in S$ such that $0 \in \Phi(p^*)$.*

Proof. Let $a \in R^l$ be such that $\Phi(S) \subset \Omega_a = a + \Omega$. Fix $\bar{p} \in D$ and denote $H(\bar{p}) = \{x \in R^l \mid \bar{p} \cdot x < 0\}$. Since $\bar{p} \gg 0$ the intersection $\Omega_a \cap H(\bar{p})$ is bounded. It follows from the boundary condition and inclusion $\Phi(S) \subset \Omega_a$ that there exists a neighborhood V of ∂S in S such that $\bar{p} \cdot \Phi(p) \geqslant 0$ for each $p \in V$. Then, by the previous result, there exists $p^* \in S$ such that $0 \in \Phi(p^*)$.

Remark 1, Theorem 1(a), Assertion 2, and Remark 7 from the Appendix imply the following strengthening of the market equilibrium lemma of Grandmont [8, p. 543, and 9, pp. 159–160].

COROLLARY 2. *Let D be a nonempty, convex, open subset of S, and let $\Phi: D \mapsto R^l$ be an upper hemicontinuous, compact-, convex-valued correspondence satisfying the Walras law. For any sequence of price vectors converging to the boundary of the domain D, let there exist a price vector (which may depend on the sequence) such that the value of excess demand for this price vector is nonnegative for infinitely many price vectors in the sequence. Then, there exists $p^* \in D$ such that $0 \in \Phi(p^*)$.*

Note that the trivial case of an excess demand function which is zero everywhere is not encompassed by the Grandmont lemma. The difference between the Grandmont lemma and Corollary 2 is that instead of the word "positive," the word "nonnegative" is used, that is, we permit the $\bar{p}$-value of excess demand to be zero near the boundary.

The next corollary of Theorem 1 generalizes the market equilibrium existence results from Arrow and Hahn [1, Chapter 2] which treat the case of excess demand defined not on an open set. Arrow and Hahn first prove the existence of equilibrium for a production economy, where the "preferred" actions of agents give rise to an excess demand function Φ defined and continuous on the closed price simplex $\bar{S}$ (Theorem 1). Then, they weaken the continuity condition assuming that an excess demand function Φ is defined and continuous on a subset of $\bar{S}$, containing S, and if Φ is not defined for $p^0 \in \partial \bar{S}$, then

$$\lim_{p \to p^0} \sum_i \Phi_i(p) = +\infty.$$

Moreover, they assume that Φ is bounded from below (Theorem 3). The following corollary generalizes and strengthens this theorem. In particular, the condition of boundedness from below is dispensed with.

COROLLARY 3. *Let $D \subset \bar{S}$ be a convex set with nonempty relative interior D^0. Let $\Phi: D \mapsto R^l$ be a nonempty-, compact-, convex-valued, upper hemicontinuous correspondence satisfying the Walras law. Let there be a compact $K \subset D^0$ such that for every $p^0 \in \bar{D} \backslash D$ there exists $q \in K$ such that*

$$\liminf_{p \to p^0} \{q \cdot x \mid x \in \Phi(p)\} > 0. \tag{3}$$

Then, there exists $p^* \in D$ such that $\Phi(p^*) \cap \Omega_- \neq \varnothing$.

Proof. Fix an arbitrary $p^0 \in \bar{D} \backslash D$. By condition (3) there exist a $q \in K$ and an open neighborhood $V(p^0)$ of p^0 in $\bar{D}$ such that

$$\inf\{q \cdot x \mid x \in \Phi(p),\ p \in V(p^0) \cap D\} > 0.$$

Put $D_1 = \bar{D} \setminus V$, where $V = \cup \{ V(p) \mid p \in \bar{D} \setminus D \}$. Since D_1 is compact and Φ is upper hemicontinuous, $\Phi(D_1)$ is compact. Thus $\Phi(D) = \Phi(D_1) \cup \Phi(V)$ with $\Phi(D_1)$ compact and $\Phi(V)$ not intersecting a polar K^0 of K.

Now assuming the contrary, $\Phi(D) \cap \Omega_- = \emptyset$, we will have that there exists a convex closed cone C such that $\Omega_- \setminus \{0\} \subset \operatorname{int} C$, and $\Phi(D) \cap C = \emptyset$, i.e., Φ satisfies the condition of Theorem 1(a). Applying this theorem we obtain $p^* \in D$ such that $0 \in \Phi(p^*)$. So, $\Phi(p^*) \cap \Omega_- \neq \emptyset$. The corollary is proved.

The main mathematical tool in the proof of Theorem 1 was the D–G–N lemma. The following assertion shows that the D–G–N lemma itself follows from Theorem 1.

ASSERTION 1. *Theorem* 1 *implies the Debreu–Gale–Nikaido lemma.*

Proof. Let $\Psi: \bar{S} \mapsto R^l$ be a correspondence satisfying the assumptions of D–G–N lemma, i.e., bounded, upper hemicontinuous, convex-, closed-valued and satisfying the Walras law in a weak form. Assume $\Psi(\bar{S}) \cap \Omega_- = \emptyset$. Since Ψ is upper hemicontinuous and compact-valued, its range is compact. Clearly then, there exists a cone C such that $\Psi(\bar{S}) \cap C = \emptyset$, and $\operatorname{int} C \supset \Omega_- \setminus \{0\}$. Put $K = S \cap C^\circ$, where C° is the polar of cone C. Clearly K is a compact subset of S. Put $D = S$, and $\Phi = \Psi|_S$ the restriction of correspondence Ψ into S in Theorem 1(b). Then, by this theorem there exists $p^* \in D$ such that $\Phi(p^*) \cap \Omega_- \neq \emptyset$. This contradicts our assumption.

3. COMPETETIVE EQUILIBRIUM IN LARGE MARKETS WITH INCOMPLETE AND NONMONOTONIC PREFERENCES

In this section we provide an example of how the theorem of the previous section can be used to extend competitive equilibrium existence results for a large economy[1]. The proofs given below are substantially simpler than the proofs of original existence results.

We consider the model of a large exchange economy introduced by Aumann [3]. Schmeidler [15] extended Aumann existence theorem to the case where preferences of agents are incomplete. We aim to dispense with the assumption of monotony of preferences. It will be shown that a competitive quasi-equilibrium still exists if we allow nonmonotonic preferences. A competitive quasi-equilibrium is a competitive equilibrium in the case of

[1] For additional examples see F. Hüsseinov, "Boundary Behavior of Excess Demand and Existence of Equilibrium," discussion paper, 97-2, Bilkent University, 1997.

strictly positive initial endowments of agents (the definitions of these con cepts are given below). So, in this case we obtain the existence of a competitive equilibrium with incomplete and nonmonotonic preferences. Furthermore, adopting a very slight (global) insatiability assumption, we show that there exists a competitive equilibrium. Since we allow for satiable preferences equilibria of this section are free-disposal equilibria. Clearly, assuming that preferences are locally insatiable, we will have a no free-disposal equilibria.

The set of traders is the closed unit interval $T=[0,1]$. The terms measure, measurable, integral and integrable are to be understood in the sense of Lebesgue. Note that any atomless finite measure space will do as well. $\int x$ means $\int_T x(t)\,d(t)$, and $\int_A x$ means $\int_A x(t)\,dt$ for a measurable set $A \subset T$. The commodity space is the nonnegative orthant Ω. Initial assignment is denoted by ω. For each trader $t \in T$, $\succ_t$ denotes its preference relation on Ω. We assume

(A.1) $\int \omega \gg 0$. This assumption asserts that all commodities are present in the market. Actually, passing to the subspace, it can be considered to be satisfied always.

(A.2) *transitivity*: for each t, $x \succ_t y$ and $y \succ_t z$ implies $x \succ_t z$,

(A.3) *irreflexivity*: for each t, and for each $x \in \Omega$, not $x \succ_t x$,

(A.4) *continuity*: for each t, the set $\{(x, y) \in \Omega \times \Omega \mid x \succ_t y\}$ is open in $\Omega \times \Omega$.

(A.5) *measurability*: if x and y are assignments, then the set $\{t \mid x(t) \succ_t y(t)\}$ is measurable.

Definition 9. A pair (x, p) consisting of an allocation x and a price vector $p \in \bar{S}$ is said to be *a competitive quasi-equilibrium* if (1) $x(t)$ is feasible, i.e., $\int x \leqslant \int \omega$, (2) $x(t)$ belongs to the budget set $B_p(t) = \{x \in \Omega \mid p \cdot x \leqslant p \cdot \omega(t)\}$ for almost all $t \in T$, and (3) bundle $x(t)$ is maximal with respect to $\succ_t$ in the budget set $B_p(t)$ if $p \cdot \omega(t) > 0$. A pair (x, p) is said to be *a competitive equilibrium*, if (1) x is feasible, (2) bundle $x(t)$ is maximal with respect to $\succ_t$ in the budget set $B_p(t)$ for almost all t.

Theorem 2. *Under the assumptions* (A.1)–(A.5) *there exists a competitive quasi-equilibrium.*

Proof. Define *an individual demand* $d_p(t) = \{x \in B_p(t) \mid \nexists z \in B_p(t), z \succ_t x\}$ as the set of all maximal with respect to $\succ_t$ consumption plans in budget set $B_p(t)$.

We will use the following propositions.

PROPOSITION 1. *For each* $p \in S$ and $t \in T$ *the individual demand correspondence* $d_p(t)$ *is nonempty-, compact-valued, and for each fixed* p, $d_p(t)$ *is Borel-measurable.*

PROPOSITION 2. The average demand correspondence $d(p) = \int_T d_p(t)\, d(t)$ *is nonempty-, compact-, convex-valued.*

PROPOSITION 3. *For each agent* t *his demand correspondence* $d_p(t)$ *is upper hemicontinuous.*

By Lemma 5.3 from Aumann [3], Proposition 3 implies

PROPOSITION 4. *The average demand correspondence* $d(p)$ *is upper hemicontinuous.*

Define *the (average) excess demand correspondence* as $\Phi(p) = d(p) - \int \omega$ for $p \in S$.

PROPOSITION 5. *Excess demand correspondence* Φ *is upper hemicontinuous and satisfies the Walras law in a weak form, i.e.,*

$$p \cdot \Phi(p) \leqslant 0, \qquad \forall p \in S.$$

PROPOSITION 6. *Excess demand correspondence* Φ *is bounded below.*

Two cases are possible:

Case 1. $\overline{\Phi(S)} \cap \Omega_- = \varnothing$. Since $\overline{\Phi(S)}$ is closed and bounded below (Proposition 6) there exists a convex closed cone C such that $\Omega \setminus \{0\} \subset \operatorname{int} C$ and $C \cap \overline{\Phi(S)} = \varnothing$. Put $K = C^\circ \cap S$, where C° is the polar of C. Clearly $K \subset \operatorname{int} S$ is a compact satisfying the boundary condition of Theorem 1. So, by this theorem we obtain $p^* \in S$ such that $\Phi(p^*) \cap \Omega_- \neq \varnothing$, i.e., p^* is an equilibrium.

Case 2. $\overline{\Phi(S)} \cap \Omega_- \neq \varnothing$. In this case there exist sequences $(p_k) \subset S$ and $x_k \in \Phi(p_k)(k \in N)$ such that

$$\lim x_k = x \in \Omega_-. \tag{4}$$

Let f_k be a measurable selection of individual excess demand correspondence $\Phi_p(t) = d_p(t) - \omega(t)$, such that $\int f_k = x_k$. Since $\bar{S}$ is compact we assume, without loss of generality, that $p_k \to p^*$. If $p^* \in S$, then by Proposition 5, $x \in \Phi(p^*)$, and then $\Phi(p^*) \cap \Omega_- \neq \varnothing$, i.e., p^* is an equilibrium price vector. Suppose $p^* \in \partial S$. Applying Fatou's Lemma in several dimensions (see Hildenbrand [10, p. 69]), we obtain an integrable function $f\colon T\to R^l$, such that $f\geqslant -\omega$ and

$$\text{(a)}\qquad f(t)\in Ls(f_k(t))\qquad \text{a.e. in } T,$$

$$\text{(b)}\qquad \int f\leqslant \lim_k \int f_k.$$

It follows from (b) and (4) that $\int f\in\Omega_-$. We will show that

$$f(t)+\omega(t)\in d_{p^*}(t)\qquad \text{a.e. on}\quad T(p^*)=\{t\in T\mid p^*\cdot\omega(t)>0\},\tag{5}$$

and

$$f(t)+\omega(t)\in B_{p^*}(t)\qquad \text{a.e. on } T.\tag{6}$$

Obviously, in this case the pair $(f+\omega, p^*)$ will be a competitive quasi-equilibrium.

Actually, (6) follows directly from the upper hemicontinuity of correspondence $p\mapsto B_p(t)$ on $\bar{S}$.

Fix $t\in T(p^*)$ satisfying (a). Then, there exists an increasing sequence of positive integers $k(j)$, $j\in N$ such that

$$f_{k(j)}(t)\to f(t).\tag{7}$$

Assume that (5) is not satisfied. Then, there exists $g(t)\in \operatorname{int} B_{p^*}(t)$ such that

$$g(t)\succ_t f(t)+\omega(t),\tag{8}$$

on the set of positive measure in $T(p^*)$. Since correspondence $p\mapsto B_p(t)$ is continuous at p^* for $t\in T(p^*)$, it follows from (7) that

$$g(t)\in B_{p_{k(j)}}(t)\text{ for large enough } j.$$

By (7) and (8) and continuity of preferences $\succ_t$, we obtain that $f_{k(j)}(t)+\omega(t)$ is not maximal with respect to $\succ_t$ in $B_{p_{k(j)}}$. This contradicts the definition of $f_{k(j)}$. So, inclusion (5), and thereby the theorem is proved.

Remark 5. A preference relation $\succ$ is said to be *acyclic*, if it has no cycle. A *cycle* is a finite set of points $x_1, ..., x_m\in\Omega$ such that $x_1\succ x_2\succ\cdots\succ x_m\succ x_1$. Theorem 2 remains valid for acyclic and depending on prices preferences. The same is true for all of the other results of this section.

COROLLARY 4 (Strengthening of Schmeidler's theorem). *If in addition to the assumptions of Theorem* 2 *the following insatiability assumption is satisfied, then there exists a competitive equilibrium: a.e. in T for each $p\in\partial S$ and $\alpha>0$ the "budget set" $B(p,\alpha)=\{x\in\Omega\mid p\cdot x\leqslant\alpha\}$ has no maximal element with respect to $\succ_t$.*

Proof. By Theorem 2 there exists a competitive quasi-equilibrium (f, p^*). Assume $p^* \in \partial S$. Since $\int \omega \gg 0$, $T(p^*)$ has a positive measure, and $f(t)$ is maximal in the budget set $B_{p^*}(t)$ with respect to $\succ_t$ for all $t \in T(p^*)$. But $B_{p^*}(t)$ has no maximal element with respect to $\succ_t$ for each $t \in T(p^*)$ according to the insatiability assumption. Since $T(p^*)$ is of positive measure we have a contradiction. So $p^* \gg 0$. Then $p^* \cdot \omega(t) \gg 0$, for each t such that $\omega(t) \neq 0$. Hence, $f(t)$ is maximal with respect to $\succ_t$ in $B_{p^*}(t)$ for each such t. If $\omega(t) = 0$, then since $p^* \gg 0$, $B_{p^*}(t) = \{0\}$. Obviously then, $f(t) = 0$ is maximal with respect to $\succ_t$ in $B_{p^*}(t)$. So (f, p^*) is a competitive equilibrium.

Remark 6. The insatiability assumption in Corollary 4 is satisfied for example in the following case: for each $x \in \Omega$, $z > 0$, $x + \lambda z \succ_t x$, for some $\lambda > 0$. Clearly, this condition is much weaker than the usual monotony assumption assumed in Schmeidler's Theorem.

COROLLARY 5. *If the assumption* (A.1) *of Theorem* 2 *is replaced by the stronger assumption*

$$(\text{A.1}') \quad \omega(t) \gg 0, \qquad \textit{a.e. in } T,$$

then there exists a competitive equilibrium.

Proof. Obviously, under the assumption (A.1') the competitive quasi-equilibrium of Theorem 2 is an equilibrium.

Corollary 5 can be treated as another generalization of Aumann's existence theorem. It assumes neither completeness, nor monotony of preferences, but instead strictly positive individual initial endowments. Corollary 5 simultaneously strengthens Aumann's auxiliary Theorem. It requires neither any sort of monotony (e.g. Weak Desirability assumption in this theorem), nor saturation, as opposed to Aumann's theorem. Let for $k > 0$, $A_k : T \mapsto \Omega$ be a measurable convex-, closed-valued correspondence, such that $A_k(t)$ contains a cube $\{x \in \Omega \mid x \leqslant k(\sum_{i=1}^{l} \omega^i(t))e\}$. Define a "$A_k$-bounded budget set" $C_p(t) = B_p(t) \cap A_k(t)$. Here $e = (1, ..., 1) \in \Omega$.

DEFINITION 2. *An A_k-bounded competitive quasi-equilibrium* is a pair (x, p), where x is an allocation, $p \in \bar{S}$, and for each t such that $p \cdot \omega(t) > 0$, the point $x(t)$ is maximal with respect to $\succ_t$ in $C_p(t)$, and $x(t) \in C_p(t)$ otherwise.

GENERALIZATION OF SCHMEIDLER'S AUXILIARY THEOREM. *For $k > 1$, under the conditions* (A.1)–(A.5) *there exists a A_k-bounded competitive quasi-equilibria.*

Proof follows the proof of Theorem 2. Note that Propositions 1–6 remain valid if to change $d_p(t)$ to $d'_p(t)$-the set of all maximal with respect to $\succ_t$ elements in "A_k-bounded budget set" $C_p(t)$.

APPENDIX

1. The Grandmont condition (G) on the boundary behavior of excess demand referred to above is formulated in the following way:

> *For any sequence of price systems converging to the boundary of the domain D there exists a price system (which may depend on the sequence) such that the value of excess demand for this price system is positive for infinitely many price systems in the sequence.*

The following condition, a somewhat weaker version of which has been assumed in Corollary 1, we call the relaxed Neuefeind (RN) condition:

> *there exists compact* $K \subset D$ *such that* $\bar{K} = \{p \in D \mid p' \cdot f(p) \leqslant 0, \forall p' \in K\}$ *has a positive distance from* ∂D.

This condition relaxes the corresponding condition from Neuefeind's lemma, which is formulated in the following way: there exists $\bar{p} \in D$ such that the set $\bar{K} = \{p \in D \mid \bar{p} \cdot f(p) \leqslant 0\}$ has a positive distance from ∂D. We name this condition the Neuefeind condition (N). We show here that conditions (G) and (RN) are equivalent. However, we formulate and prove a rather general fact including the equivalence of two conditions. It turns out that neither continuity nor satisfying the Walras law is needed for that equivalence.

ASSERTION 2. *Let H be a hyperplane in* R^l *that does not contain the origin and let* $D \subset H$ *be a nonempty, convex, open in H set, and let* $\Phi: D \mapsto R^l$ *be an arbitrary correspondence. Then, the following two conditions are equivalent*:

(G) *for an arbitrary sequence* (q^n) *of elements of D that either tends to a vector* $q \in H \backslash D$, *or is unbounded, and for an arbitrary sequence* $z_n \in \Phi(q_n)$ $(n \in N)$ *there exists* $\bar{q} \in D$ *such that* $\bar{q} \cdot z_n > 0$ *for infinitely many n.*

(RN) *There exists a compact subset K* of *D such that the set* $\bar{K} = \{q \in D \mid \Phi(q) \cap K^0 \neq \varnothing\}$, *where* $K^0 = \{x \in R^l \mid p \cdot x \leqslant 0, \forall p \in K\}$ *is a polar of K, is bounded and has a positive distance from* ∂D.

Proof. Assume (RN) is not satisfied. Let K_n be a sequence of compact sets such that $K_n \subset \operatorname{int} K_{n+1}$ and $D = \bigcup_n K_n$. Take such a sequence

$q_n \in \bar{K}_n$, $(n \in N)$, that either tends to a vector $q \in H \backslash D$, or is unbounded. Then, there exists $z_n \in \Phi(q_n) \cap K_n^0$, i.e.,

$$z_n \in \Phi(q_n) \quad \text{and} \quad p \cdot z_n \leqslant 0, \qquad \forall p \in K_n, \quad \forall n. \tag{9}$$

Let $\bar{q}$ be an arbitrary point in D. Then, there exists n_0 such that $\bar{q} \in K_n$, for all $n \geqslant n_0$. Hence, by (9)

$$\bar{q} z_n \leqslant 0 \qquad \text{for all} \quad n \geqslant n_0 .$$

Thus, for the chosen $(q_n) \subset D$ and $z_n \in \Phi(q_n)$ for no $q \in D$ condition (G) is satisfied.

Show that (RN) implies (G). Assume (RN) is satisfied. Without loss of generality assume that K is finite. Indeed, if (RN) is satisfied for some compact K, then it is satisfied for any finite set in D convex hull of which contains K.

Let $(q_n) \subset D$ either tend to a vector $q \in H \backslash D$, or be unbounded, and $z_n \in \Phi(q_n)$ for each n. Then, $q_n \notin \bar{K}$ starting some index n_0, i.e.,

$$\Phi(q^n) \cap K^0 = \varnothing \qquad \text{for all} \quad n \geqslant n_0.$$

Therefore, for each $n \geqslant n_0$, there exists $p_n \in K$ such that $p_n \cdot z_n > 0$. Since K is finite, $p_n = \bar{q}$ for infinitely many n, and for $\bar{q} \in K$, i.e., condition (G) is satisfied.

Remark 7. The conditions (G_0) and (RN_0) obtained from conditions (G) and (RN) by replacing the strict inequality in (G) with nonstrict one, and by replacing K^0 in the definition of set $\bar{K}$ in (RN) with int K^0 respectively, are equivalent. This can be shown exactly in the same way as done above for conditions (G) and (RN).

Clearly Neuefeind condition (N) can be reformulated in the following way (compare Remark 3): there exists $\bar{p} \in D$, and a neighborhood V of boundary ∂D in D such that for an arbitrary $p \in V$, $\bar{p} \cdot f(p) > 0$, i.e., $\bar{p}$-value of excess demand $f(p)$ is positive. It follows at once that the Neuefeind condition (N) is as strong as the Grandmont condition (G). This point was already observed in Border [4]. However, the relationship between the two conditions was not fully explored.

We denote (N_0) the condition obtained from (N) by replacing the strict inequality with the nonstrict one in this condition. We assume in the following two assertions that $D = S$.

ASSERTION 3. *In the case of two commodities conditions* (G) *and* (N) *are equivalent.*

Proof. Since condition (N) is stronger than condition (RN), and (RN) is equvalent to (G) by Assertion 2 it follows that condition (N) implies condition (G). Assume condition (G) is satisfied. Then by Remark 1 and Assertion 2 there exist $p_1, p_2 \in S$, and a neighborhood V of ∂S in S such that for each $p \in V$ there exists $i \in \{1, 2\}$ such that

$$p_i \cdot f(p) > 0. \tag{10}$$

Let $p \in S$ be close to $\partial S = \{(1, 0), (0, 1)\}$, say to $(1, 0)$. Then, inequality (10) is satisfied for some $i \in \{1, 2\}$ and $p_i - p = (-c, c)$ for some $c > 0$. So, we have from inequality (10) and the Walras law that $(p_i - p) \cdot f(p) > 0$ and then $-f^1(p) + f^2(p) > 0$. Since p is close to $(1, 0)$, by the Walras law $|f^2(p)| > |f^1(p)|$ and it follows from the last two inequalities that $f^2(p) > 0$ and $f^1(p) < 0$. So $f^2(p) > -f^1(p) > 0$ and then $f^1(p) + f^2(p) > 0$. Putting $\bar{p} = (\frac{1}{2}, \frac{1}{2})$ we will have that $\bar{p} \cdot f(p) > 0$. Similar reasoning shows that $\bar{p} \cdot f(p) > 0$ also for p close to $(0, 1)$. So condition (N) is satisfied.

ASSERTION 4. *In the case of more than two commodities there exists an excess demand function satisfying condition* (G) *but not* (N).

Proof. We start with the case of three commodities. Denote by $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$; $p_1 = (1 - 2\varepsilon, \varepsilon, \varepsilon)$, $p_2 = (\varepsilon, 1 - 2\varepsilon, \varepsilon)$, $p_3 = (\varepsilon, \varepsilon, 1 - 2\varepsilon)$, where $\varepsilon = 10^{-2}$, and $C = \{x \in R^3 \mid p_i \cdot x \leqslant 0, i = 1, 2, 3\}$. Denote $H_p = \{x \in R^3 \mid p \cdot x = 0\}$ a hyperplane through the origin with the normal $p \neq 0$, and $S_\delta = \{p \in S \mid \text{dist}(p, \partial S) \geqslant \delta\}$ for $\delta > 0$. Fix one of the two orientations of R^3 and denote it by θ.

Clearly, there exists a positive number $\delta > 0$ such that for each $p \in S_\delta$ hyperplane H_p intersects the interior of cone C. The intersection $H_p \cap \partial C$ consists of two (different) rays. Denote by $h_1(p)$ and $h_2(p)$ unit vectors from these rays and assume that the notation is such that the basis $(p, h_1(p), h_2(p))$ of R^3 belongs to θ. Put

$$f(p) = \begin{cases} d(p)\, h_1(p) & \text{for} \quad p \in S \backslash S_\delta\,, \\ 0 & \text{for} \quad p \in S_\delta\,. \end{cases}$$

Here $d(p) = \text{dist}(p, \partial S_\delta)$. Since $f(p) \in H_p$ for each $p \in S$ the Walras law is satisfied. Since $d(p)$ and $h_1(p)$ are continuous in $S \backslash S_\delta$, $f(p)$ is continuous there. Since $d(p)$ tends to zero for p tending to ∂S_δ it follows that f is continuous also on ∂S_δ, So f is continuous function on S satisfying the Walras law i.e., the excess demand function. Put $\bar{p}_1 = (1 - 2\varepsilon', \varepsilon', \varepsilon')$, $\bar{p}_2 = (\varepsilon', 1 - 2\varepsilon', \varepsilon')$, $\bar{p}_3 = (\varepsilon', \varepsilon', 1 - 2\varepsilon')$ for some $0 < \varepsilon' < \varepsilon$. Clearly, for each $x \notin \text{int}\, C$, $x \neq 0$, $\bar{p}_i \cdot x > 0$ for some $i \in \{1, 2, 3\}$. Since $f(p) \notin \text{int}\, C$ and $f(p) \neq 0$ for each $p \in S \backslash S_\delta$, we have that for each $p \in S \backslash S_\delta$ there exists $i \in \{1, 2, 3\}$ such that $\bar{p}_i \cdot f(p) > 0$. So, excess demand function f satisfies condition (G). It is easily seen that f does not satisfy condition (N_0) and therefore condition (N). The above construction of function f fits also for high dimensions.

2. In this point we bring a lemma and use it along with some known facts to sketch crisply the proof of Propositions 1–6 from Section 3.

A binary relation $\succ$ on Ω is said to be *lower hemicontinuous*, if for each $y \in \Omega$ the lower contour set $L(y) = \{x \in \Omega \mid y \succ x\}$ is open in Ω.

LEMMA . *Let a binary relation $\succ$ on Ω be irreflexive, acyclic and lower hemicontinuous, and B be a compact subset of Ω. Then, the set of maximal with respect to $\succ$ elements in B is nonempty and compact.*

Proof. The proof of the existence of a maximal element goes along the lines of the proof of Lemma 2 from [15]. Suppose on the contrary, that for each $x \in B$ there exists $y \in B$ such that $y \succ x$. Then, the lower contour sets $L(y)$, $y \in B$ cover B, i.e., $B = \bigcup_{y \in B} L(y)$. So, there is a finite subcover $L(y_1), ..., L(y_m)$. By acyclicity of $\succ$, in the finite set $\{y_1, ..., y_m\}$ there exists a maximal element with respect to $\succ$, say y_1. Then $y_1 \notin L(y_i)$, $i = 1, ..., m$. So, $y_1 \notin B$, a contradiction. Show that the set of all maximal elements is compact. Let $x_k \in B$, $k \in N$ be a sequence of maximal elements converging to x. Suppose x is not maximal with respect to $\succ$ in B. Then there exists $y \in B$ such that $y \succ x$, i. e., $x \in L(y)$. Since $L(y)$ is open, then $x_k \in L(y)$ starting some index k_0, i.e., $y \succ x_k$ for $k \geqslant k_0$. Contradiction with the maximality of points x_k $(k \geqslant k_0)$.

Now we sketch the proof of Propositions 1–6 from Section 3. Nonempty-, compact-valuedness of the individual demand correspondence $d_p(t)$ follows from the just proved lemma. Since individual demand and individual quasi-demand coincide for $p \in S$, Borel-measurability of $d_p(t)$ follows from Lemma 3 in [15], the proof of which in turn follows the arguments given in the proof of Lemma 5.6 from [3]. Since Schmeidler's proof uses neither monotony nor transitivity of preferences, it is also valid in our case. Note that dispensing with k-truncation in Lemma 3 of [15] does not affect its proof. The proof of Proposition 2 uses Proposition 1 and by now, standard results on correspondences (see [2, 3]). In particular, convex-valuedness of $d(p)$ follows from the Lyapunov–Richter convexity theorem [11, 14]. Proposition 3 follows from Theorem 3 of [10, p. 29], and Proposition 4 follows from Proposition 3 and Lemma 5.3 from [3], as noted in the text. Upper hemicontinuity of excess demand correspondence φ in Proposition 5 follows directly from Proposition 4. The assertion that φ satisfies the Walras law in Proposition 5 and boundedness from below (Proposition 6) are trivial.

REFERENCES

1. K. J. Arrow and F. H. Hahn, “General Competitive Analysis,” Holden Day, San Francisco, CA, 1971.
2. R. J. Aumann, Integrals of set-valued functions, *J. Math. Anal. Appl.* **12** (1965), 1–12.
3. R. J. Aumann, Existence of competitive equilibria in markets with a continuum of traders, *Econometrica*, **34** (1966), 1–17.
4. K. Border, “Fixed Point Theorems with Applications to Economics and Game Theory,” Cambridge Univ. Press, Cambridge, MA, 1985.
5. G. Debreu, “Theory of Value,” Wiley, New York, 1959.
6. G. Debreu, Existence of competitive equilibrium, *in* “Handbook of Mathemetical Economics” (K. J. Arrow and M. D. Intriligator, Eds.), Vol. 2, North-Holland, Amsterdam, 1982.
7. G. Debreu, Market equilibrium, *in* “Mathematical Economics: Twenty Papers of Gerard Debreu” (F. Hahn, Ed.), Cambridge Univ. Press, Cambridge, MA, 1983.
8. J. M. Grandmont, Temporary general equilibrium theory, *Econometrica*, **45** (1977), 535–572.
9. J. M. Grandmont, “Money and Value,” Maison des Sciences de L’Homme/Cambridge Univ. Press, Paris, France, 1994.
10. W. Hildenbrand, “Core and Equilibria of a Large Economy,” Princeton Univ. Press, Princeton, NJ, 1974.
11. A. Lyapunov, Sur les fonctions-vecteures complètement additives, *Bull. Acad. Sci. USSR Ser. Math.* **4** (1940), 465–478.
12. W. Neuefeind, Notes on existence of equilibrium proofs and the boundary behavior of supply, *Econometrica*, **48** (1980), 1831–1837.
13. H. Nikaido, “Convex Structures and Mathemathical Economics,” Academic Press, New York, 1968.
14. H. Richter, Verallgemeinerung eines in der Statistik benötigten Satzes der Masstheorie, *Math. Ann.* **150** (1963), 85–90.
15. D. Schmeidler, Competitive equilibria in markets with a continuum of traders and incomplete preferences, *Econometrica* **37** (1969), 578–585.