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Integrable coupled KdV systems

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We give the conditions for a system of N-coupled Korteweg de Vries (KdV) type of equations to be integrable. We find the recursion operators of each subclass and give all examples for $N=2$. *© 1998 American Institute of Physics.* [S0022-2488(98)03003-5]

I. INTRODUCTION

In Ref. 1 we gave an extension of the recently proposed Svinolupov Jordan KdV[2,3] systems to a class of integrable multicomponent KdV systems and gave their recursion operators. This class is known as the degenerate subclass of the KdV system. In this work we will extend it to a more general KdV type of system equations containing both the degenerate and nondegenerate cases. This is a major step towards the complete classification of KdV systems. In addition we give a new extension of such a system of equations.

Let us consider a system of N nonlinear equations of the form

$$q^i_t = b^i_j q^j_{xxx} + s^i_{jk} q^j q^k_x, \tag{1}$$

where $i,j,k=1,2,...,N$, q^i are functions depending on the variables x, and t, and b^i_j, and S^i_{jk} are constants. The purpose of this work is to find the conditions on these constants so that the equations in (1) are integrable. In general the existence of infinitely many conserved quantities is admitted as the definition of integrability. This implies the existence of infinitely many generalized symmetries. In this work we assume the following definition for integrability:

Definition: A system of equations is said to be integrable if it admits a recursion operator.

The recursion operator (if it exists) of the system of equations given in (1), in general, may take a very complicated form. Let the highest powers of the operators D and D^{-1} be respectively defined by m=degree of R and n=order nonlocality of R. In this work we are interested in a subclass of equations admitting a recursion operator with $m=2$ and $n=1$. Namely, it is of the form

$$R^i_j = b^i_j D^2 + a^i_{jk} q^k + c^i_{jk} q^k_x D^{-1}, \tag{2}$$

where D is the total x derivative, D^{-1} is the inverse operator, and a^i_{jk} and c^i_{jk} are constants with

$$s^i_{jk} = a^i_{kj} + c^i_{jk}. \tag{3}$$

Before starting to the classification of (1) we recall a few fundamental properties of the recursion operator. An operator R^i_j is a recursion operator if it satisfies the following equation

$$R^i_{j,t} = F'^i_k R^k_j - R^i_k F'^k_j, \tag{4}$$

where F'^i_k is the Fréchet derivative of the system (1), which is given by

$$\sigma^i_t = F'^i_j \sigma^j, \tag{5}$$

where σ^i's are called the symmetries of the system (1). The condition (3) implies that Eq. (1) itself is assumed to be in the family of the hierarchy of equations (or flows)

$$q^i_{\tau_n}=\sigma^i_n,$$

where for all $n=0,1,...$, σ^i_n denotes the symmetries of the integrable KdV system (1). For instance, for $n=0,1$ we have respectively the classical symmetries $\sigma^i_0=q^i_x$ and $\sigma^i_1=q^i_t$.

Equation (5) is called the symmetry equation of (1) with

$$F'^{\,i}_j=b^i_jD^3+s^i_{jk}q^k_x+s^i_{kj}q^kD. \tag{6}$$

Recursion operators are defined as operators mapping symmetries to symmetries, i.e.,

$$R^i_j\sigma^j=\lambda\sigma^i, \tag{7}$$

where λ is an arbitrary constant. Equations (5) and (7) imply (4). It is the equation (4) which determines the constants a^i_{jk} and c^i_{jk} in terms of b^i_j and s^i_{jk}'s. The same equation (4) brings severe constraints on b^i_j and s^i_{jk}.

We shall obtain a classification of (1) based on the matrix b^i_j,

(i) det $(b^i_j)=0$,
(ii) det $(b^i_j)\neq 0$,

and also we divide the classification procedure, for each class, into two parts where $s^i_{jk}=s^i_{kj}$ and $s^i_{jk}\neq s^i_{kj}$. For the system of equations admitting a recursion operator we have the following proposition.

Proposition 1: Let $q^i(t,x)$ be functions of t and x satisfying the N KdV equations (1) and admitting a recursion operator R^i_j in (2). Then the constants b^i_j, s^i_{jk}, a^i_{jk}, and c^i_{jk} satisfy [in addition to the (3)] the following relations:

$$b^k_lc^i_{jk}-b^i_kc^k_{jl}=0, \tag{8}$$

$$b^k_la^i_{jk}-b^i_k(a^i_{jl}+3c^k_{jl}-s^k_{jl})=0, \tag{9}$$

$$b^i_k(3a^k_{jl}+3c^k_{jl}-2s^k_{jl}-s^k_{lj})=0, \tag{10}$$

$$c^i_{jk}s^k_{lm}-s^i_{lk}c^k_{jm}=0, \tag{11}$$

$$c^i_{jk}s^k_{lm}+c^i_{jk}s^k_{ml}-c^k_{jm}s^i_{kl}-c^k_{jl}s^i_{km}=0, \tag{12}$$

$$a^i_{jk}s^k_{lm}-s^i_{km}a^k_{jl}-s^i_{lk}a^k_{jm}-s^i_{lk}c^k_{jm}+s^k_{jl}c^i_{km}+a^i_{kl}s^k_{jm}=0, \tag{13}$$

$$c^i_{km}(s^k_{pj}-s^k_{jp})=0. \tag{14}$$

Now we will discuss the problem of classifying the integrable system of equations (1) for the two exclusive cases depending upon the matrix b^i_j.

II. CLASSIFICATION FOR THE CLASS det $(b^i_j)=0$

In this subclass we assume the rank of the matrix b^i_j as $N-1$. Investigation of the subclasses for other ranks of matrix b can be done similarly. For this case we may take $b^i_j=\delta^i_j-k^ik_j$ where k_i is a unit vector, $k^ik_i=1$. In this work we use the Einstein convention, i.e., repeated indices are summed up from 1 to N. We find the following solution of (8)–(14) for the parameters a^i_{jk} and c^i_{jk} for all N

Proposition 2: Let k^i be a constant unit vector and $b^i_j=\delta^i_j-k^ik_j$. Then the complete solutions of the equations (8)–(14) are given by

$$a^i_{lj}=\tfrac{2}{3}s^i_{jl}+\tfrac{1}{3}[k^i(k_j a_l-2k_l n_j)+(-ak^i+b^i)k_l k_j],$$
$$c^i_{jl}=\tfrac{1}{3}s^i_{lj}-\tfrac{1}{3}[k^i(k_l a_j+k_j n_l)+(-ak^i+b^i)k_l k_j]+k^i k_l n_j, \tag{15}$$

where

$$n_j=k^k k_i s^i_{jk}-ak_j, \quad a_j=k^k k_i s^i_{kj}-nk_j, \quad b^i=k^l k^k s^i_{lk}-nk^i, \tag{16}$$

$a=k^n a_n$, $n=k^i n_i$ and s^i_{jk}'s are subject to satisfy the following:

$$c^i_{jk}s^k_{lm}=c^k_{jm}s^i_{lk}, \tag{17}$$

$$s^i_{jk}-s^i_{kj}=k^i[k_j(a_k-n_k)-k_k(a_j-n_j)], \tag{18}$$

$$k_n s^n_{lj}=n_l k_j+k_l a_j, \tag{19}$$

$$k^n s^i_{jn}=k^i n_j+b^i k_j, \tag{20}$$

$$k^n s^i_{nj}=(n-a)k^i k_j+k^i a_j+b^i k_j, \tag{21}$$

$$n[(a_i-n_i)k_j-(a_j-n_j)k_i]=0, \tag{22}$$

$$a_i=\rho n_i+ak_i, \tag{23}$$

where ρ is a constant. At this point we will discuss the classification procedure with respect to the symbol s^i_{jk} whether it is symmetric or nonsymmetric with respect to its lower indices.

A. The symmetric case, $s^i_{jk}=s^i_{kj}$

Among the constraints listed in Proposition 2 the one given in (22) implies that s^i_{jk}'s are symmetric if and only if $a_i-n_i=\alpha k_i$ where $\alpha=a-n$. There are two distinct cases depending on whether $n=0$ or $n\neq 0$. We shall give these two subcases as corollaries of the previous proposition.

Corollary 1: Let $s^i_{jk}=s^i_{kj}$ and $n=0$. Then we have the following solution for all N:

$$a^i_{kj}=\tfrac{2}{3}s^i_{jk}+\tfrac{1}{3}[k^i(k_j a_k-2k_k n_j)+k_k k_j b^i], \tag{24}$$

$$c^i_{jk}=\tfrac{1}{3}s^i_{jk}-\tfrac{1}{3}[k^i(k_j a_k-2k_k n_j)+k_k k_j b^i], \tag{24}$$

where $a=0$, $\rho=1$, and

$$a_l=n_l, n_l=k_i k^j s^i_{lj}, b^i=k^j k^l s^i_{lj}. \tag{25}$$

The vector k^i and s^i_{jk} are not arbitrary; they satisfy the following constraints:

$$s^i_{jk}s^k_{lm}-s^i_{lk}s^k_{jm}=-2(k_j n_l-k_l n_j)(-k^i n_m+b^i k_m),$$
$$k_n s^n_{lj}=n_l k_j+k_l n_j, \quad k^n s^i_{jn}=k^i n_j+b^i k_j. \tag{26}$$

As an illustration we give an example for this case.[1] A particular solution of the equations (26) for $N=2$ is

$$b^i_j=\delta^i_j-y^i y_j=x^i x_j,$$
$$s^i_{jk}=\frac{3}{2}\alpha_1 x^i x_j x_k+\alpha_2 x^i y_j y_k+\frac{\alpha_1}{2}y^i(y_j x_k+y_k x_j), \tag{27}$$

where $i,j=1,2$ and

$$x^i=\delta^i_1\,,y^i=\delta^i_2\,, \tag{28}$$

and

$$k_i=y_i\,,\quad n_i=\tfrac{1}{2}\alpha_1 x_i\,,\quad b^i=\alpha_2 x^i. \tag{29}$$

Constants a^i_{jk} and c^i_{jk} appearing in the recursion operator are given by

$$\begin{aligned} a^i_{jk}&=\alpha_1 x^i x_j x_k+\alpha_2 x^i y_j y_k+\frac{\alpha_1}{2}y^i x_j y_k\,,\\ c^i_{jk}&=\frac{\alpha_1}{2}x^i x_j x_k+\frac{\alpha_1}{2}y^i x_j y_k\,. \end{aligned} \tag{30}$$

Taking $\alpha_1=2$ and $\alpha_2=1$ (without loss of generality) we obtain the following coupled system

$$u_t=u_{xxx}+3uu_x+vv_x\,,\quad v_t=(uv)_x\,. \tag{31}$$

The above system was first introduced by Ito[4] and the multi-Hamiltonian structure studied by Antonowicz and Fordy[5] and by Olver and Rosenau.[6] The recursion operator of this system is given by

$$R=\begin{pmatrix} D^2+2u+u_xD^{-1} & v\\ v+v_xD^{-1} & 0\end{pmatrix}. \tag{32}$$

In Ref. 1 we have another example for $N=3$.

For the case $n\neq 0$ for all N we have the following.

Corollary 2: Let $s^i_{jk}=s^i_{kj}$, $n\neq 0$, and $\rho=0$. Then the solution given in Proposition 2 reduces to

$$\begin{aligned} a^i_{lj}&=\tfrac{2}{3}s^i_{jl}+\tfrac{1}{3}(a-2n)k^ik_jk_l\,,\\ c^i_{jl}&=\tfrac{1}{3}s^i_{lj}-\tfrac{1}{3}(a-2n)k^ik_jk_l\,, \end{aligned} \tag{33}$$

where

$$n_j=nk_j\,,\quad a_j=ak_j\,,\quad b^i=ak^i, \tag{34}$$

and the constraint equations

$$s^i_{kj}s^k_{ml}-s^k_{mj}s^i_{kl}=0,\quad k_ns^n_{lj}=(a+n)k_lk_j\,,\quad k^ns^i_{jn}=(a+n)k^ik_j\,. \tag{35}$$

For this case we point out that solution of (33) and (35) gives decoupled systems.

B. The nonsymmetric case, $s^i_{jk}\neq s^i_{kj}$

In this case the constraints in Proposition 1, in particular (22), implies that we must have $n=0$. In this case we have the following expressions for a^i_{lj} and c^i_{lj} for all N:

$$\begin{aligned} a^i_{lj}&=\tfrac{2}{3}s^i_{jl}+\tfrac{1}{3}[k^i(k_ja_l-2k_ln_j)+(-ak^i+b^i)k_lk_j],\\ c^i_{jl}&=\tfrac{1}{3}s^i_{lj}-\tfrac{1}{3}[k^i(k_la_j+k_jn_l)+(-ak^i+b^i)k_lk_j]+k^ik_ln_j\,, \end{aligned} \tag{36}$$

where

$$\begin{aligned} n_j&=k^kk_is^i_{jk}-ak_j\,,\\ a_j&=k^kk_is^i_{kj}\,,\quad b^i=k^lk^ks^i_{lk}\,, \end{aligned} \tag{37}$$

and the constraint equations among the parameters are

$$c^i_{jk}c^k_{lm}-c^i_{lk}c^k_{jm}=0, \tag{38}$$

$$(a^i_{jk}-c^i_{jk})s^k_{lm}+(c^i_{km}-a^i_{km})s^k_{lj}+(s^i_{mk}-s^i_{km})a^k_{jl}+(s^k_{jm}-s^k_{mj})a^i_{kl}=0. \tag{39}$$

For $N=2$ we will give an example. Constants a^i_{jk} and c^i_{jk} appearing in the recursion operator are given by

$$\begin{aligned} a^i_{jk}&=\alpha_1 x^i x_j x_k+\alpha_2 x^i y_j y_k+\alpha_3 y^i x_j y_k, \\ c^i_{jk}&=\frac{\alpha_1}{2}x^i x_j x_k+\alpha_1 y^i x_j y_k, \end{aligned} \tag{40}$$

where α_1, α_2, and α_3 are arbitrary constants and

$$k_i=y_i, \quad n_i=\alpha_3 x_i, \quad b^i=\alpha_2 x^i, \quad a_i=\alpha_1 x_i. \tag{41}$$

We obtain the following coupled system:

$$\begin{aligned} u_t&=u_{xxx}+3\alpha_1 u u_x, \\ v_t&=\alpha_3 u_x v+\alpha_1 u v_x, \end{aligned} \tag{42}$$

which is equivalent to the symmetrically coupled KdV system[7]

$$\begin{aligned} u_t&=u_{xxx}+v_{xxx}+6uu_x+4uv_x+2u_x v, \\ u_t&=u_{xxx}+v_{xxx}+6vv_x+4vu_x+2v_x u, \end{aligned} \tag{43}$$

and the recursion operator for this integrable system of equations (42) is

$$R=\begin{pmatrix} D^2+\alpha_1(2u+u_x D^{-1}) & 0 \\ \alpha_3 v+\alpha_1 v_x D^{-1} & 0 \end{pmatrix}. \tag{44}$$

III. CLASSIFICATION FOR THE CASE det $(b^i_j)\neq 0$

As in the degenerate case det $(b^i_j)=0$, we have two subcases, symmetric and nonsymmetric. Before these we have the following proposition.

Proposition 3: Let det $(b^i_j)\neq 0$. Then the solution of equations given Proposition 1 is given as follows:

$$\begin{aligned} a^i_{jl}&=\tfrac{2}{9}(s^i_{lj}+2s^i_{jl})-\tfrac{1}{9}C^k_l b^i_m K^m_{jk}, \\ c^i_{lj}&=\tfrac{1}{9}(7s^i_{lj}-4s^i_{jl})+\tfrac{1}{9}C^k_l b^i_m K^m_{jk}, \end{aligned} \tag{45}$$

where

$$K^i_{lj}=s^i_{lj}-s^i_{jl} \tag{46}$$

and the constraint equations

$$b^k_l c^i_{jk}-b^i_k c^k_{jl}=0, \tag{47}$$

$$5C^m_i K^i_{rj}-C^l_r K^m_{lj}-C^l_j K^m_{rl}=0, \tag{48}$$

$$c^k_{jm}K^i_{lk}+c^k_{jl}K^i_{mk}=0, \tag{49}$$

$$c^i_{jk}s^k_{lm}-c^k_{jm}s^i_{lk}=0, \tag{50}$$

$$K^k_{lj}c^i_{km}=0, \tag{51}$$

$$(a^i_{jk}-c^i_{jk})s^k_{lm}+(c^i_{km}-a^i_{km})s^k_{lj}+K^i_{mk}a^k_{jl}+K^k_{jm}a^i_{kl}=0. \tag{52}$$

where C^i_r is the inverse of b^r_i.

A. The nonsymmetric case, $s^i_{jk}\neq s^i_{kj}$

Equations (47)–(52) define an over-determined system for the components of s^i_{jk}. Any solution of this system leads to the determination of the parameters a^i_{jl} and c^i_{jl} by (45).

As an example we give the following, for $N=2$, coupled system

$$v_t=av_{xxx}+2bvv_x,$$

$$u_t=4au_{xxx}+2bu_xv+buv_x, \tag{53}$$

where a and b are arbitrary constants. This system, under a change of variables, is equivalent to the KdV equation with the time evolution part of its Lax equation.[8] The recursion operator of the system (53) is

$$R=\begin{pmatrix} \frac{4}{3}(3aD^2+bv) & \frac{b}{3}(3u+2u_xD^{-1}) \\ 0 & \frac{1}{3}(3aD^2+4bv+2bv_xD^{-1}) \end{pmatrix}. \tag{54}$$

Hence the KdV equation coupled to time evolution part of its Lax-pair is integrable and its recursion operator is given above. This is the only new example for $N=2$ system.

B. The symmetric case, $s^i_{jk}=s^i_{kj}$

When the symbol s^i_{jk} is symmetric with respect to subindices, the parameters K^i_{jk} vanish. Then the equations (45)–(52) reduce to

$$a^i_{jk}=\tfrac{2}{3}s^i_{jk}, \quad c^i_{jk}=\tfrac{1}{3}s^i_{jk}, \tag{55}$$

where the parameters b^i_k and s^i_{jk} satisfy

$$b^k_l s^i_{jk}-b^i_k s^k_{jl}=0, \tag{56}$$

$$s^i_{jk}s^k_{lm}-s^i_{lk}s^k_{jm}=0. \tag{57}$$

We shall not study this class in detail, because in Ref. 1 some examples of this class are given for $N=2$. Here we give another example which correspond to the perturbation expansion of the KdV equation. Let $q^i=\delta^i u$, where $i=0,1,2,...,N$, and u satisfies the KdV equation $u_t=u_{xxx}+6uu_x$. The q^i's satisfy a system of KdV equations which belong to this subclass:

$$q^0_t=q^0_{xxx}+6q^0q^0_x, \tag{58}$$

$$q^1_t=q^1_{xxx}+6(q^0q^1)_x, \tag{59}$$

$$q^2_t=q^2_{xxx}+6[(q^1)^2+q^0q^2]_x, \tag{60}$$

$$\cdots . \tag{61}$$

$$q^N_t=q^N_{xxx}+3\sum_{i=1}^{N}[\delta^i(q^0)^2]_x. \tag{62}$$

IV. FOKAS–LIU EXTENSION

The classification of the KdV system given in this work with respect to the symmetries can be easily extended to the following simple modification of (1):

$$q_t^i = b_j^i q_{xxx}^j + s_{jk}^i q^j q_x^k + \chi_j^i q_x^j, \tag{63}$$

where χ_j^i's are arbitrary constants. Equation (63) without the last term will be called the principle part of that equation. Hence the equation (1) we have studied so far is the principle part of its modification (63). We assume the existence of a recursion operator corresponding to the above system in the form

$$R_j^i = b_j^i D^2 + a_{jk}^i q^k + c_{jk}^i q_x^k D^{-1} + w_j^i, \tag{64}$$

where w_j^i's are constants. We have the following proposition corresponding to the integrability of the above system.

Proposition 4: The operator given in (64) is the recursion operator of the KdV system (63) if in addition to the equations listed in Proposition 1 [(8)–(14)] the following constraints on the constants χ_j^i and w_j^i are satisfied:

$$\chi_l^k a_{jk}^i - \chi_k^i a_{jl}^k - \chi_k^i c_{jl}^k + \chi_j^k c_{kl}^i - w_j^k s_{kl}^i + w_k^i s_{jl}^k = 0, \tag{65}$$

$$\chi_l^k c_{jk}^i - \chi_k^i c_{jl}^k = 0, \tag{66}$$

$$\chi_k^i w_j^k - \chi_j^k w_k^i = 0, \tag{67}$$

$$(\chi_j^k - w_j^k) b_k^i - (\chi_k^i - w_k^i) b_j^k = 0, \tag{68}$$

$$\chi_j^k a_{kl}^i - \chi_k^i a_{jl}^k + w_k^i s_{lj}^k - w_j^k s_{lk}^i = 0. \tag{69}$$

Since the constraints (8)–(14) are enough to determine the coefficients a_{jk}^i and c_{jk}^i with some constraints on the given constants b_j^i and s_{jk}^i, we have the following corollary of the above proposition.

Corollary 3: The KdV system in (63) is integrable if and only if its principle part is integrable.

The principle part of (63) is obtained by ignoring the last term (the term with χ_j^i). Hence the proof of this corollary follows directly by observing that the constraints on the constants χ_j^i and w_j^i listed in (65)–(69) are independent of the constraints on the constants of the principle part listed in (8)–(14). Before the application of this corollary let us go back to the Proposition 4 and ask the question whether the KdV system (63) admits a recursion operator with $w_j^i = 0$.

Corollary 4: The KdV system (63) admits a recursion operator of the principle part. Then the last term $\chi_j^i q_x^j$ is a symmetry of the principle part. If $w_j^i = 0$, the above equations (65)–(69) reduce to

$$\chi_l^k a_{jk}^i - \chi_k^i a_{jl}^k - \chi_k^i c_{jl}^k + \chi_j^k c_{kl}^i = 0, \tag{70}$$

$$\chi_l^k s_{jk}^i - \chi_k^i s_{jl}^k = 0, \tag{71}$$

$$\chi_j^k b_k^i - \chi_k^i b_j^k = 0, \tag{72}$$

$$\chi_l^k c_{jk}^i - \chi_k^i c_{jl}^k = 0. \tag{73}$$

In order that the term $\chi_j^i q_x^j$ be a symmetry of the principle part, the constants χ_j^i are subject to satisfy the following equations:

$$\chi_l^k s_{jk}^i - \chi_k^i s_{jl}^k = 0, \tag{74}$$

$$\chi_j^k b_k^i - \chi_k^i b_j^k = 0, \tag{75}$$

$$\chi_l^k K_{jk}^i + \chi_j^k K_{lk}^i = 0. \tag{76}$$

These equations simply follow from the set of equations (70)–(73), and hence the quantities $\sigma^i = \chi_j^i q_x^j$ are symmetries of the principle part.

By the application of Corollary 3, the full classification of the system (63) with the recursion operator (64) such that q_t^i (i.e., the system of equations themselves) belong to the symmetries of this system is possible. To each subclass given in the previous sections there exists a Fokas–Liu extension such that $w_j^i = \chi_j^i$ with the following constraints:

$$\chi_k^i c_{jl}^k - \chi_l^k c_{jk}^i = 0, \tag{77}$$

$$\chi_l^k a_{jk}^i - \chi_j^k a_{lk}^i - \chi_k^i (a_{jl}^k - a_{lj}^k) = 0. \tag{78}$$

The above constraints are identically satisfied for the class [det $(b_j^i) \neq 0$, symmetrical case] when $\chi_j^i = \alpha \delta_j^i + \beta b_j^i$. Hence the Fokas–Liu extension of the nondegenerate symmetrical case is straightforward with this choice of χ_j^i. Here α and β are arbitrary constants.

For the degenerate case the set of equations (77) and (78) must solved for a given principle part, b_j^i and a_{jk}^i. Recently a system of integrable KdV system with $N=2$ has been introduced by Fokas and Liu.[9] This system is a nice example for the application of Corollary 3. We shall give this system in its original form first and then simplify:

$$\begin{aligned} u_t + v_x + (3\beta_1 + 2\beta_4)\beta_3 u u_x + (2 + \beta_1\beta_4)\beta_3 (uv)_x + \beta_1\beta_3 v v_x \\ + (\beta_1 + \beta_4)\beta_2 u_{xxx} + (1 + \beta_1\beta_4)\beta_2 v_{xxx} = 0, \end{aligned} \tag{79}$$

$$\begin{aligned} v_t + u_x + (2 + 3\beta_1\beta_4)\beta_3 v v_x + (\beta_1 + 2\beta_4)\beta_3 (uv)_x + \beta_1\beta_3\beta_4 u u_x \\ + (\beta_1 + \beta_4)\beta_2\beta_4 u_{xxx} + (1 + \beta_1\beta_4)\beta_2\beta_4 v_{xxx} = 0, \end{aligned} \tag{80}$$

where β_1, β_2, β_3, and β_4 are arbitrary constants. The recursion operator of this system is given in Ref. 8. Consider now a linear transformation

$$u = m_1 r + n_1 s, \quad v = m_2 r + n_2 s, \tag{81}$$

where m_1, m_2, n_1, and n_2 are constants, and s and r are new dynamical variables, $q^i = (s, r)$. Choosing these constants properly, the Fokas–Liu system reduces to a simpler form

$$\begin{aligned} r_t &= (rs)_x + \alpha_1 r_x + \alpha_2 s_x, \\ s_t &= \gamma_1 s_{xxx} + \gamma_2 r r_x + 3 s s_x + \alpha_3 r_x + \alpha_4 s_x, \end{aligned} \tag{82}$$

where we are not giving the coefficients $\alpha_1, \alpha_2, \alpha_3$, and α_4 in terms of the parameters of the original equation given above, because these expressions are quite lengthy. The only condition on the parameters α_i is given by $\alpha_3 = \gamma_2 \alpha_2$. This guarantees the integrability of the above system (82). On the other hand, the transformation parameters are given by

$$m_2 = -\frac{\beta_1 + \beta_4}{1 + \beta_1\beta_4} m_1, \quad n_2 = \beta_4 n_1, \tag{83}$$

$$n_1 = -\frac{1}{\delta\beta_3}, \quad \delta = \beta_1 (1 + \beta_4^2) + 2\beta_4. \tag{84}$$

The principle part of the Fokas–Liu system (82) is exactly the Ito system given in (31) and hence the recursion operator is the sum of the one given in (32) and χ_j^i which are given by $\chi_1^1 = \alpha_4$, $\chi_2^1 = \alpha_3$, $\chi_1^2 = \alpha_2$, $\chi_2^2 = \alpha_1$. That is,

$$R = \begin{pmatrix} \gamma_1 D^2 + 2s + s_x D^{-1} + \alpha_4 & \gamma_2 r + \alpha_3 \\ r + r_x D^{-1} + \alpha_2 & \alpha_1 \end{pmatrix}. \tag{85}$$

Another example was given very recently[10] in a very different context for $N=2$:

$$u_t=\tfrac{1}{2}v_{xxx}+2uv_x+u_x v, \quad v_t=3vv_x+2au_x. \tag{86}$$

The principle part of these equations is transformable to the Fuchssteiner system given in (42). Taking $\alpha_1=2$, $\alpha_3=4$, and scaling x and t properly we obtain (without losing any generality)

$$r_t=2r_{xxx}+6rr_x, \quad s_t=2s_x r+4sr_x. \tag{87}$$

The transformation between the principle part ($a=0$) of (86) and (87) is simply given by $u=m_1 s+\frac{1}{2}r$, $v=2r$. Then the recursion operator of the system (86) is given by

$$R=\begin{pmatrix} 0 & \frac{1}{2}D^2+2u+u_x D^{-1} \\ 2a & 2v+v_x D^{-1} \end{pmatrix}. \tag{88}$$

V. CONCLUSION

We have given a classification of a system of KdV equations with respect to the existence of a recursion operator. This is indeed a partial classification. Although we have found all conditions for each subclass, we have not presented them explicitly. We obtained three distinct subclasses for all values of N and gave the corresponding recursion operators. We also gave an extension of such systems by adding a linear term containing the first derivative of dynamical variables. We called such systems the Fokas–Liu extensions. We proved that these extended systems of KdV equations are also integrable if and only if their principle parts are integrable. For $N=2$, we have given all subclassess explicitly. Among these the recursion operator of the KdV coupled to the time evolution part of its Lax pair seems to be new. Here we would like to add that when $N=2$ recursion operators, including the Fokas–Liu extensions, are hereditary.[11]

Our classification crucially depends on the form of the recursion operator. The recursion operators used in this work were assumed to have degree two (highest degree of the operator D in R) and nonlocality order one (highest degree of the operator D^{-1} in R). The next work in this program should be the study on the classification problems with respect to the recursion operators with higher degree and higher nonlocalities. For instance, when $N=2$, Hirota–Satsuma, Boussinesq, and Bogoyavlenskii coupled KdV equations admit recursion operators with $m=4$ and $n=1$.[12] Hence these equations do not belong to our classification given in this work.

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[1] M. Gürses and A. Karasu, Phys. Lett. A **214**, 21 (1996).
[2] S. L. Svinolupov, Teor. Mat. Fiz. **87**, 391 (1991).
[3] S. I. Svinolupov, Funct. Anal. Appl. **27**, 257 (1993).
[4] M. Ito, Phys. Lett. A **91**, 335 (1982).
[5] M. Antonowicz and A. P. Fordy, Physica D **28**, 345 (1987).
[6] P. J. Olver and P. Rosenau, Phys. Rev. E **53**, 1900 (1996).
[7] B. Fuchssteiner, Prog. Theor. Phys. **65**, 861 (1981).
[8] M. Ablowitz and H. Segur, *Solutions and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[9] A. S. Fokas and Q. M. Liu, Phys. Rev. Lett. **77**, 2347 (1996).
[10] L. Bonora, Q. P. Liu, and C. S. Xiong, Commun. Math. Phys. **175**, 177 (1996).
[11] B. Fuchssteiner and A. S. Fokas, Physica D **4**, 47 (1981).
[12] M. Gürses and A. Karasu, *Coupled KdV Systems: Recursion Operators of Degree Four* (submitted).