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On Contractibility of the Orbit Space of a G-Poset of Brauer Pairs

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Given a p-block b of a finite group G, we show that the G-poset of Brauer pairs strictly containing $(1, b)$ has contractible G-orbit space. A similar result is proved for certain G-posets of p-subgroups. Both results generalise P. Symonds' verification of a conjecture of P. Webb. © 1999 Academic Press

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Symonds [6] proved the conjecture of Webb [9] that, given a finite group G and a prime p dividing $|G|$, then the G-poset $\mathscr{S}_p(G)$ of nontrivial p-subgroups of G has contractible G-orbit space $|\mathscr{S}_p(G)|/G$. More generally, consider a G-poset $\mathscr{S}$ consisting of p-subgroups of G with $\mathscr{S}$ having the property that $P \in \mathscr{S}$ whenever P and Q are p-subgroups of G satisfying $P \geq Q \in \mathscr{S}$. Let $\mathscr{S}_\triangleleft$ denote the G-simplicial subcomplex of $\mathscr{S}$ such that the nonempty simplexes in $\mathscr{S}_\triangleleft$ are the chains of the form $(P_0 \triangleleft \cdots \triangleleft P_n)$ where each $P_i \trianglelefteq P_n$. Symonds' argument shows:

THEOREM 1 (Symonds). *For $\mathscr{S}$ as in the previous text, $|\mathscr{S}_\triangleleft|/G$ is contractible.*

Theorem 1 generalizes the conjectured assertion because Thévenaz–Webb [8, Theorem 2] gives a G-homotopy equivalence $|\mathscr{S}_p(G)_\triangleleft| \simeq_G$

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$|\mathscr{S}_p(G)|$. Using another method, we shall prove a different generalization:

THEOREM 2. *For $\mathscr{S}$ as in the preceding text, $|\mathscr{S}|/G$ is contractible.*

In fact, we prove that a generalization of Webb's conjectured assertion holds for G-posets of Brauer pairs. Some fundamental properties of Brauer pairs (also called subpairs) were established in Alperin−Broué [1] (another account is given in Thévenaz [7, Section 40]). Let F be a field of characteristic p, and let b be a block (idempotent) of FG. Let $\mathscr{T}$ be a G-poset consisting of Brauer pairs on FG containing $(1, b)$ with $(P, e) \in \mathscr{T}$ whenever (P, e) and (Q, f) are Brauer pairs on FG satisfying $(P, e) \geq (Q, f) \in \mathscr{T}$. Let $\mathscr{T}_{\triangleleft}$ be the G-simplicial subcomplex of $\mathscr{T}$ whose nonempty chains are of the form $((P_0, e_0) \triangleleft \cdots \triangleleft (P_n, e_n))$ where each $P_i \trianglelefteq P_n$. We show:

THEOREM 3. *For $\mathscr{T}$ as in the earlier text, $|\mathscr{T}_{\triangleleft}|/G$ is acyclic.*

Now suppose that the block b has a positive defect, let $\mathscr{B}(b)$ be the G-poset of all Brauer pairs strictly containing $(1, b)$, and let $\mathscr{A}$ be any G-subposet of $\mathscr{B}(b)$ such that $\mathscr{A}$ contains all the Brauer pairs $(P, e) \in \mathscr{B}$ such that P is elementary Abelian. The proof of Thévenaz−Webb [8, Theorem 2] generalizes easily to the following result; we sketch the argument in the following text.

THEOREM 4 (Thévenaz−Webb). *For $\mathscr{B}(b)$ and $\mathscr{A}$ as in the foregoing text, there are G-homotopy equivalences,*

$$|\mathscr{B}(b)| \simeq_G |\mathscr{A}| \simeq_G |\mathscr{B}(b)_{\triangleleft}|.$$

In the case of the principal block, the following result is precisely the assertion conjectured by Webb.

THEOREM 5. *Given a positive defect block b of FG, then $|\mathscr{B}(b)|/G$ and $|\mathscr{B}(b)_{\triangleleft}|/G$ are contractible.*

Our technique is based on a certain double chain complex, by means of which, the G-orbit space of a given G-simplicial complex X and the orbit spaces of some simplicial subcomplexes of X are to be compared with the G-orbit space of a carefully chosen G-simplicial complex Y and the orbit spaces of some simplicial subcomplexes of Y. To begin, we must generalize some material in Curtis−Reiner [3, Section 66].

Recall that any finite G-poset W may be regarded as a G-simplicial complex whose simplexes are the totally ordered subsets of W. If W is regular (meaning that $gx = x$ whenever $x, y \in W$ and $g \in G$ with $x \leq y \geq gx$), then the G-orbit poset W/G has underlying polyhedron $|W/G|$ canonically G-homeomorphic to the G-orbit space $|W|/G$.

Let X be a finite G-simplicial complex. The nonempty simplexes in X comprise a G-poset $\mathrm{sd}(X)$ partially ordered by the subchain relation. As a G-simplicial complex, $\mathrm{sd}(X)$ may be identified with the barycentric subdivision of X. It is easy to see that if X happens to be a G-poset, then the G-poset $\mathrm{sd}(X)$ is regular. In general, therefore, $|X|/G$ is G-homeomorphic to $|\mathrm{sd}(\mathrm{sd}(X))/G|$.

Let R be a commutative unital ring of characteristic zero. Recall that the augmented chain complex $\tilde{C}(X, RG)$ of X with coefficients in R is a chain complex of permutation RG-modules, and has G-stable R-basis $\widetilde{\mathrm{sd}}(X) = \bigcup_{n \geq -1} \widetilde{\mathrm{sd}}_n(X)$, where $\widetilde{\mathrm{sd}}_n(X)$ is the set of all simplexes $\underline{x}$ whose dimension $n(\underline{x})$ is equal to n. (Thus the empty simplex $\emptyset$ is the unique element of $\widetilde{\mathrm{sd}}_{-1}(X)$.) Writing M_1^G for the image of the 1-relative trace map $\mathrm{tr}_1^G\colon M \to M^G$ on any RG-module M, then $\tilde{C}(X, RG)_1^G$ is a chain complex of free R-modules. The following result is doubtless well known.

PROPOSITION 6. *Let X be a finite G-simplicial complex, and let R be a commutative unital ring of characteristic zero. Then we have an isomorphism of homology,*

$$\tilde{H}(|X|/G, R) \cong H\big(\tilde{C}(X, RG)_1^G\big).$$

Proof. Because $\tilde{C}(X, RG) \simeq_G \tilde{C}(\mathrm{sd}(\mathrm{sd}(X)), RG)$, we have a homotopy equivalence,

$$\tilde{C}(X, RG)_1^G \simeq \tilde{C}(\mathrm{sd}(\mathrm{sd}(X)), RG)_1^G.$$

So we may assume that X is a regular G-poset. Then an isomorphism,

$$\tilde{C}(X/G, R) \cong \tilde{C}(X, RG)_1^G$$

is specified by the correspondences $\emptyset \leftrightarrow \mathrm{tr}_1^G(\emptyset)$, and

$$\big(\mathrm{Orb}_G(x_0) < \cdots < \mathrm{Orb}(x_n)\big) \leftrightarrow \mathrm{tr}_1^G(x_0 < \cdots < x_n),$$

for $(x_0 < \cdots < x_n) \in \mathrm{sd}(X)$. But $|X/G| \cong |X|/G$, and we are finished. ∎

Let us consider three finite G-simplicial complexes X, Y, Z such that $X \leq Z \geq Y$ and $X * Y \geq Z$; the join $X * Y$ is defined by the identity,

$$\widetilde{\mathrm{sd}}(X * Y) := \widetilde{\mathrm{sd}}(X) \times \widetilde{\mathrm{sd}}(Y).$$

The triple (X, Y, Z) is called a double G-simplicial complex. (Compare with Quillen [5, 1.9], and the notion of a bisimplicial set in Gelfand−Manin

[4, Section I.3]). We write $\underline{x}Z\underline{y}$ to mean that $(\underline{x}, \underline{y}) \in \widetilde{\mathrm{sd}}(Z)$. We let $\underline{x}Z$ be the $N_G(\underline{x})$-simplicial complex such that

$$\widetilde{\mathrm{sd}}(\underline{x}Z) := \left\{ \underline{y} \in \widetilde{\mathrm{sd}}(Y) : \underline{x}Z\underline{y} \right\}.$$

Similarly, we define $Z\underline{y}$ as an $N_G(\underline{y})$-simplicial complex with vertices in X. Note that $X = Z\emptyset$ and $Y = \emptyset Z$.

Let $D = D(X, Y, Z, RG)$ be the double chain complex of permutation RG-modules such that D is a subcomplex of the tensor product double complex $\tilde{C}(X, RG) \otimes_R \tilde{C}(Y, RG)$, and $D_{s,t}$ has R-basis,

$$\widetilde{\mathrm{sd}}_{s,t}(X, Y, Z) := \left(\widetilde{\mathrm{sd}}_s(X) \times \widetilde{\mathrm{sd}}_t(Y)\right) \cap \widetilde{\mathrm{sd}}(Z).$$

Then $\tilde{C}(Z, RG) = [-1]\mathrm{Tot}(D)$, where $[-1]$ denotes the "dimension shift" one place to the right. Therefore:

Remark 7. We have $\tilde{C}(Z, RG)_1^G = [-1]\mathrm{Tot}(D_1^G)$.

LEMMA 8. *Suppose that $\underline{x}Z/N_G(\underline{x})$ and $Z\underline{y}/N_G(\underline{y})$ are R-acyclic for all (nonempty) $\underline{x} \in \mathrm{sd}(X)$ and $\underline{y} \in \mathrm{sd}(Y)$. Then*

$$\tilde{H}(|X|/G, R) \cong \tilde{H}(|Y|/G, R) \cong \tilde{H}(|Z|/G, R).$$

In particular, X/G is R-acyclic if and only if Y/G is R-acyclic.

Proof. Let E be the spectral sequence arising from the column-filtration of the double chain complex D_1^G. By the hypothesis on $Z\underline{y}/N_G(\underline{y})$ and Proposition 6, $E_{s,t}^1 = \tilde{H}_t(|Y|/G, R)$ if $s = -1$, otherwise $E_{s,t}^1 = 0$. Because the E^1-page collapses to a single column,

$$E_{st}^1 \cong \tilde{H}_{s+t}\left(\mathrm{Tot}\left(D_1^G\right)\right).$$

But by Proposition 6 and Remark 7,

$$\tilde{H}_{s+t}\left(\mathrm{Tot}\left(D_1^G\right)\right) \cong \tilde{H}_{s+t+1}(|Z|/G, R).$$

Therefore, $\tilde{H}_*(|X|/G, R) = \tilde{H}_*(|Z|/G, R)$. To complete the argument, we interchange X and Y (in effect, switching to the row-filtration of D_1^G). ∎

Proof of Theorem 2. Because [2, Theorem 3] tells us that $|\mathscr{S}|/G$ is simply connected, it suffices to show that $|\mathscr{S}|/G$ is acyclic (over the rational integers). We may assume that $\mathscr{S}$ contains non-Sylow p-subgroups of G. Let $X = \mathscr{S}$, and let Y be the G-subposet of $\mathscr{S}$ obtained by deleting the G-conjugates of some minimal element of $\mathscr{S}$. Let Z be such that (X, Y, Z) is a double G-simplicial complex and, given (nonempty) $\underline{x} \in \mathrm{sd}(X)$ and $\underline{y} \in \mathrm{sd}(Y)$, then $\underline{x}Z\underline{y}$ provided the maximal vertex y of $\underline{y}$ fixes $\underline{x}$ under conjugation. Then $Z\underline{y}$ is the $N_G(\underline{y})$-poset of y-fixed elements X^y, which is conically $N_G(\underline{y})$-contractible via the composite map $x \mapsto xy \mapsto y$. Meanwhile, $\underline{x}Z$ consists of those p-subgroups of $N_G(\underline{x})$ which belong to Y, and by induction on the number of vertices of X, we may assume that $\underline{x}Z/N_G(\underline{x})$ is acyclic. So Lemma 8 applies. By induction again, we may assume that $|Y|/G$ is acyclic, hence so is $|X|/G$, as required. ∎

Proof of Theorem 3. Again, we shall apply Lemma 8. Because the maximal Brauer pairs containing $(1, b)$ are permuted transitively by G, we may assume that $\mathscr{T}$ contains a nonmaximal Brauer pair. Let $X = \mathscr{T}_{\lhd}$, and let Y be the G-simplicial subcomplex of X obtained by deleting the G-conjugates of some minimal vertex (P^0, e^0) of $\mathscr{T}$. We form a double G-simplicial complex (X, Y, Z) such that, given nonempty simplexes $(\underline{P}, \underline{e}) = ((P_0, e_0) \lhd \cdots \lhd (P_n, e_n))$ of X and $(\underline{Q}, \underline{f}) = ((Q_0, f_0) \lhd \cdots \lhd (Q_m, f_m))$ of Y, then $(\underline{P}, \underline{e})Z(\underline{Q}, \underline{f})$ provided each $(P_i, e_i) \unlhd (Q_j, f_j)$. Fixing $(\underline{Q}, \underline{f})$, then for each $(\underline{P}, \underline{e}) \in \mathrm{sd}(Z(\underline{Q}, \underline{f}))$, let $(\underline{P}, \underline{e})'$ be the element of $\mathrm{sd}(Z(\underline{Q}, \underline{f}))$ obtained from $(\underline{P}, \underline{e})$ by inserting (Q_0, f_0) as the maximal term (if the maximal term is already (Q_0, f_0), then $(\underline{P}, \underline{e})' = (\underline{P}, \underline{e})$). The barycentric subdivision $\mathrm{sd}(Z(\underline{Q}, \underline{f}))$ of $Z(\underline{Q}, \underline{f})$ is $N_G(\underline{Q}, \underline{f})$-contractible via

$$(\underline{P}, \underline{e}) \mapsto (\underline{P}, \underline{e})' \mapsto ((Q_0, f_0)).$$

Therefore, $|Z(\underline{Q}, \underline{f})\,|/N_G(\underline{Q}, \underline{f})$ is contractible, and perforce, acyclic.

By induction on the number of vertices of X, we may assume that $|Y|/G$ is acyclic. So, fixing a nonempty simplex $(\underline{P}, \underline{e})$ as in the previous text, it suffices to show that $|(\underline{P}, \underline{e})Z|/N_G(\underline{P}, \underline{e})$ is acyclic. We need only worry about the case where $(\underline{P}, \underline{e}) = ((P^0, e^0))$, because if $(\underline{P}, \underline{e})$ is not a G-conjugate of $((P^0, e^0))$, then we can consider the element $(\underline{Q}, \underline{f})'$ of $\mathrm{sd}((\underline{P}, \underline{e})Z)$ obtained from $(\underline{Q}, \underline{f})$ by inserting (P_n, e_n) as the minimal term, and the argument proceeds as before. Clearly, $((P^0, e^0))Z$ is nonempty. Also, $((P^0, e^0))Z$ is the $N_G(P^0, e^0)$-simplicial complex $\mathscr{T}^0_{\lhd}$, where $\mathscr{T}^0$ consists of the Brauer pairs on $FN_G(P^0, e^0)$ strictly containing (P^0, e^0). By induction, we may assume that $|\mathscr{T}^0_{\lhd}|/N_G(P^0, e^0))$ is acyclic, and now there is nothing left to prove. ∎

Sketch Proof of Theorem 4. We indicate the modification to be made to the proof of Thévenaz–Webb [8, Theorem 2]. Given $(P, e) \in \mathscr{B}(b) - \mathscr{A}$, then the $N_G(P, e)$-posets,

$$\{(Q, f) : (1, b) < (Q, f) < (P, e)\} \quad \text{and} \quad \{Q : 1 < Q < P\}$$

are isomorphic, and we can apply [8, 1.7] to the inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$, deducing the first asserted G-homotopy equivalence.

To demonstrate the second half of the assertion, we may assume that $\mathscr{A}$ consists of precisely those Brauer pairs (Q, f) such that Q is Abelian. Let $\mathscr{P} := \mathrm{sd}(\mathscr{B}(b)_{\lhd})$ as a G-poset. Let ϕ be the surjective G-poset map $\mathscr{P}^{\mathrm{op}} \to \mathscr{A}$ such that, given $(\underline{P}, \underline{e}) = ((P_0, e_0) \lhd \cdots \lhd (P_n, e_n)) \in \underline{P}$, then $\phi(\underline{P}, \underline{e}) := (A, f)$ where A is the intersection of the centres of the p-subgroups P_i, and $(A, f) \leq (P_0, e_0)$. Let us now fix $(A, f) \in \mathscr{A}$, and let $\mathscr{Q}$ be the $N_G(A, f)$-subposet of $\mathscr{P}$ consisting of the elements $(\underline{Q}, \underline{f})$ such that $\phi(\underline{Q}, \underline{f}) \geq (A, f)$. For such $(\underline{Q}, \underline{f})$, let $(\underline{Q}, \underline{f})'$ be the element of $\mathscr{Q}$ obtained by inserting (A, f) as the minimal term (leaving $(\underline{Q}, \underline{f})$ unchanged if (A, f) is already the minimal term). Then $\mathscr{Q}$ is $\overline{N}_G(A, f)$-contractible via $(\underline{Q}, \underline{f}) \mapsto (\underline{Q}, \underline{f})' \mapsto ((A, f))$, and the assertion holds by [8, Theorem 1(ii)]. ∎

Proof of Theorem 5. By Theorems 3–5, respectively, $|\mathscr{B}(b)_{\lhd}|/G$ is acyclic, $|\mathscr{B}(b)|/G \simeq |\mathscr{B}(b)_{\lhd}|/G$, and $|\mathscr{B}(b)|/G$ is simply connected. ∎

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