On Contractibility of the Orbit Space of a G-Poset of Brauer Pairs

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Communicated by Michel Broué

Received April 11, 1998

Given a p-block b of a finite group G, we show that the G-poset of Brauer pairs strictly containing (1, b) has contractible G-orbit space. A similar result is proved for certain G-posets of p-subgroups. Both results generalise P. Symonds' verification of a conjecture of P. Webb.

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Key Words: double simplicial complex; orbit space; poset of Brauer pairs.

Symonds [6] proved the conjecture of Webb [9] that, given a finite group G and a prime p dividing |G|, then the G-poset \( \mathscr{P}_p(G) \) of nontrivial p-subgroups of G has contractible G-orbit space \( \mathcal{X}_p(G)/G \). More generally, consider a G-poset \( \mathcal{S} \) consisting of p-subgroups of G with \( \mathcal{S} \) having the property that \( P \in \mathcal{S} \) whenever \( P \) and \( Q \) are p-subgroups of G satisfying \( P \preceq Q \in \mathcal{S} \). Let \( \mathcal{S}_\alpha \) denote the G-simplicial subcomplex of \( \mathcal{S} \) such that the nonempty simplexes in \( \mathcal{S}_\alpha \) are the chains of the form \( (P_0 < \cdots < P_n) \) where each \( P_i \preceq P_{i+1} \). Symonds' argument shows:

**Theorem 1** (Symonds). For \( \mathcal{S} \) as in the previous text, \( |\mathcal{S}_\alpha|/G \) is contractible.

Theorem 1 generalizes the conjectured assertion because Thévenaz–Webb [8, Theorem 2] gives a \( G \)-homotopy equivalence \( |\mathcal{P}_p(G)_\alpha| \approx_G \).

* This work was carried out during a visit to the Friedrich-Schiller-Universität-Jena. The author was on leave from Bilkent University, and was funded by the Alexander-von-Humboldt Foundation.
Using another method, we shall prove a different generalization:

**Theorem 2.** For \( \mathcal{P} \) as in the preceding text, \( |\mathcal{P}|/G \) is contractible.

In fact, we prove that a generalization of Webb’s conjectured assertion holds for \( G \)-posets of Brauer pairs. Some fundamental properties of Brauer pairs (also called subpairs) were established in Alperin–Broué [1] (another account is given in Thévenaz [7, Section 40]). Let \( F \) be a field of characteristic \( p \), and let \( b \) be a block (idempotent) of \( FG \). Let \( \mathcal{I} \) be a \( G \)-poset consisting of Brauer pairs on \( FG \) containing \((1, b)\) with \((P, e) \in \mathcal{I}\) whenever \((P, e)\) and \((Q, f)\) are Brauer pairs on \( FG \) satisfying \((P, e) \trianglerighteq (Q, f) \in \mathcal{I}\). Let \( \mathcal{I}_d \) be the \( G \)-simplicial subcomplex of \( \mathcal{I} \) whose nonempty chains are of the form \(((P_0, e_0) \triangleleft \cdots \triangleleft (P_n, e_n))\) where each \( P_i \trianglerighteq P_{i+1} \). We show:

**Theorem 3.** For \( \mathcal{I} \) as in the earlier text, \( \mathcal{I}_d/G \) is acyclic.

Now suppose that the block \( b \) has a positive defect, let \( \mathcal{B}(b) \) be the \( G \)-poset of all Brauer pairs strictly containing \((1, b)\), and let \( \mathcal{A} \) be any \( G \)-subposet of \( \mathcal{B}(b) \) such that \( \mathcal{A} \) contains all the Brauer pairs \((P, e) \in \mathcal{B}(b)\) such that \( P \) is elementary Abelian. The proof of Thévenaz–Webb [8, Theorem 2] generalizes easily to the following result; we sketch the argument in the following text.

**Theorem 4 (Thévenaz–Webb).** For \( \mathcal{B}(b) \) and \( \mathcal{A} \) as in the foregoing text, there are \( G \)-homotopy equivalences,

\[
|\mathcal{B}(b)| \simeq_G |\mathcal{A}| \simeq_G |\mathcal{B}(b)_d|.
\]

In the case of the principal block, the following result is precisely the assertion conjectured by Webb.

**Theorem 5.** Given a positive defect block \( b \) of \( FG \), then \( |\mathcal{B}(b)|/G \) and \( |\mathcal{B}(b)_d|/G \) are contractible.

Our technique is based on a certain double chain complex, by means of which, the \( G \)-orbit space of a given \( G \)-simplicial complex \( X \) and the orbit spaces of some simplicial subcomplexes of \( X \) are to be compared with the \( G \)-orbit space of a carefully chosen \( G \)-simplicial complex \( Y \) and the orbit spaces of some simplicial subcomplexes of \( Y \). To begin, we must generalize some material in Curtis–Reiner [3, Section 66].

Recall that any finite \( G \)-poset \( W \) may be regarded as a \( G \)-simplicial complex whose simplexes are the totally ordered subsets of \( W \). If \( W \) is regular (meaning that \( gx = x \) whenever \( x, y \in W \) and \( g \in G \) with \( x \leq y \leq gx \)), then the \( G \)-orbit poset \( W/G \) has underlying polyhedron \( |W/G| \) canonically \( G \)-homeomorphic to the \( G \)-orbit space \( |W|/G \).
Let $X$ be a finite $G$-simplicial complex. The nonempty simplexes in $X$ comprise a $G$-poset $\text{sd}(X)$ partially ordered by the subchain relation. As a $G$-simplicial complex, $\text{sd}(X)$ may be identified with the barycentric subdivision of $X$. It is easy to see that if $X$ happens to be a $G$-poset, then the $G$-poset $\text{sd}(X)$ is regular. In general, therefore, $|X|/G$ is $G$-homeomorphic to $\text{sd}(\text{sd}(X))/G$.

Let $R$ be a commutative unital ring of characteristic zero. Recall that the augmented chain complex $\tilde{C}(X, RG)$ of $X$ with coefficients in $R$ is a chain complex of permutation $RG$-modules, and has $G$-stable $R$-basis $\tilde{sd}(X) = \bigcup_{n \geq -1} \tilde{sd}_n(X)$, where $\tilde{sd}_n(X)$ is the set of all simplexes $x$ whose dimension $n(x)$ is equal to $n$. (Thus the empty simplex $\emptyset$ is the unique element of $\tilde{sd}_{-1}(X)$.) Writing $M^G_1$ for the image of the 1-relative trace map $\text{tr}^G_1 : M \to M^G$ on any $RG$-module $M$, then $\tilde{C}(X, RG)_1^G$ is a chain complex of free $R$-modules. The following result is doubtless well known.

**Proposition 6.** Let $X$ be a finite $G$-simplicial complex, and let $R$ be a commutative unital ring of characteristic zero. Then we have an isomorphism of homology,

$$\tilde{H}(|X|/G, R) \cong H(\tilde{C}(X, RG)_1^G).$$

**Proof.** Because $\tilde{C}(X, RG) \equiv_G \tilde{C}(\text{sd}(X), RG)$, we have a homotopy equivalence,

$$\tilde{C}(X, RG)_1^G \equiv \tilde{C}(\text{sd}(X), RG)_1^G.$$ 

So we may assume that $X$ is a regular $G$-poset. Then an isomorphism,

$$\tilde{C}(X/G, R) \equiv \tilde{C}(X, RG)_1^G$$

is specified by the correspondences $\emptyset \leftrightarrow \text{tr}^G_1(\emptyset)$, and

$$(\text{Orb}_G(x_0) < \cdots < \text{Orb}(x_n)) \leftrightarrow \text{tr}^G_1(x_0 < \cdots < x_n),$$

for $(x_0 < \cdots < x_n) \in \text{sd}(X)$. But $|X/G| \equiv |X|/G$, and we are finished. 

Let us consider three finite $G$-simplicial complexes $X, Y, Z$ such that $X \leq Z \geq Y$ and $X \ast Y \geq Z$; the join $X \ast Y$ is defined by the identity,

$$\tilde{sd}(X \ast Y) := \tilde{sd}(X) \times \tilde{sd}(Y).$$

The triple $(X, Y, Z)$ is called a double $G$-simplicial complex. (Compare with Quillen [5, 1.9], and the notion of a bisimplicial set in Gelfand–Manin
We write \( x \bar{Z} y \) to mean that \((x, y) \in \bar{sd}(Z)\). We let \( xZ \) be the \( N_G(x) \)-simplicial complex such that

\[
\bar{sd}(xZ) := \{ y \in \bar{sd}(Y) : x \bar{Z} y \}.
\]

Similarly, we define \( YZ \) as an \( N_G(Y) \)-simplicial complex with vertices in \( X \). Note that \( X = Z \varnothing \) and \( Y = \varnothing Z \).

Let \( D = D(X, Y, Z, RG) \) be the double chain complex of permutation \( RG \)-modules such that \( D \) is a subcomplex of the tensor product double complex \( \mathcal{C}(X, RG) \otimes_R \mathcal{C}(Y, RG) \), and \( D_{s,t} \) has \( R \)-basis,

\[
\bar{sd}_{s,t}(X, Y, Z) := (\bar{sd}_s(X) \times \bar{sd}_t(Y)) \cap \bar{sd}(Z).
\]

Then \( \mathcal{C}(Z, RG) = [-1] \text{Tot}(D) \), where \([ -1 ]\) denotes the “dimension shift” one place to the right. Therefore:

**Remark 7.** We have \( \mathcal{C}(Z, RG)^{G}_t = [-1] \text{Tot}(D^{G}_1) \).

**Lemma 8.** Suppose that \( \mathcal{Z}_x \) and \( \mathcal{Z}_y \) are \( R \)-acyclic for all (nonempty) \( x \in \mathcal{S}(X) \) and \( y \in \mathcal{S}(Y) \). Then

\[
\tilde{H}(|X|/G, R) \cong \tilde{H}(|Y|/G, R) \cong \tilde{H}(|Z|/G, R).
\]

In particular, \( X/G \) is \( R \)-acyclic if and only if \( Y/G \) is \( R \)-acyclic.

**Proof.** Let \( E \) be the spectral sequence arising from the column-filtration of the double chain complex \( D^{G}_1 \). By the hypothesis on \( \mathcal{Z}_y \) and Proposition 6, \( E^{1}_{s,t} = \tilde{H}(|Y|/G, R) \) if \( s = -1 \), otherwise \( E^{1}_{s,t} = 0 \). Because the \( E^1 \)-page collapses to a single column,

\[
E^{1}_{s,t} \cong \tilde{H}_{s+1}(\text{Tot}(D^{G}_1)).
\]

But by Proposition 6 and Remark 7,

\[
\tilde{H}_{s+1}(\text{Tot}(D^{G}_1)) \cong \tilde{H}_{s+1+1}(|Z|/G, R).
\]

Therefore, \( \tilde{H}_{s}(|X|/G, R) = \tilde{H}_{s}(|Z|/G, R) \). To complete the argument, we interchange \( X \) and \( Y \) (in effect, switching to the row-filtration of \( D^{G}_1 \)).

**Proof of Theorem 2.** Because [2, Theorem 3] tells us that \( \mathcal{S}/G \) is simply connected, it suffices to show that \( \mathcal{S}/G \) is acyclic (over the rational integers). We may assume that \( \mathcal{S} \) contains non-Sylow \( p \)-subgroups of \( G \). Let \( X = \mathcal{S} \), and let \( Y \) be the \( G \)-subposet of \( \mathcal{S} \) obtained by deleting...
the $G$-conjugates of some minimal element of $\mathcal{S}$. Let $Z$ be such that $(X, Y, Z)$ is a double $G$-simplicial complex and, given (nonempty) $x \in \text{sd}(X)$ and $y \in \text{sd}(Y)$, then $xZy$ provides the maximal vertex $y$ of $y$ fixes $x$ under conjugation. Then $Zy$ is the $N_G(y)$-poset of $y$-fixed elements $X^y$, which is conically $N_G(y)$-contractible via the composite map $x \mapsto xy \mapsto y$. Meanwhile, $xZ$ consists of those $p$-subgroups of $N_G(x)$ which belong to $Y$, and by induction on the number of vertices of $X$, we may assume that $xZ/N_G(x)$ is acyclic. So Lemma 8 applies. By induction again, we may assume that $|Y|/G$ is acyclic, hence so is $|X|/G$, as required.

**Proof of Theorem 3.** Again, we shall apply Lemma 8. Because the maximal Brauer pairs containing $(1, b)$ are permuted transitively by $G$, we may assume that $\mathcal{S}$ contains a nonmaximal Brauer pair. Let $X = \mathcal{S}_{<e}$, and let $Y$ be the $G$-simplicial subcomplex of $X$ obtained by deleting the $G$-conjugates of some minimal vertex $(P^0, e^0)$ of $\mathcal{S}$. We form a double $G$-simplicial complex $(X, Y, Z)$ such that, given $xZ$ such that, given nonempty simplexes $(P, e) = ((P_0, e_0) < \cdots < (P_n, e_n))$ of $X$ and $(Q, f) = ((Q_0, f_0) < \cdots < (Q_m, f_m))$ of $Y$, then $(P, e)Z(Q, f)$ provided each $(P_i, e_i) \leq (Q_i, f_i)$. Fixing $(Q, f)$, then for each $(P, e) \in \text{sd}(Z(Q, f))$, let $(P, e)'$ be the element of $\text{sd}(Z(Q, f))$ obtained from $(P, e)$ by inserting $(Q_0, f_0)$ as the maximal term (if the maximal term is already $(Q_0, f_0)$, then $(P, e)' = (P, e)$). The barycentric subdivision $\text{sd}(Z(Q, f))$ of $Z(Q, f)$ is $N_G(Q, f)$-contractible via

$$(P, e) \mapsto (P, e)' \mapsto ((Q_0, f_0)).$$

Therefore, $|Z(Q, f)|/N_G(Q, f)$ is contractible, and perforce, acyclic.

By induction on the number of vertices of $X$, we may assume that $|Y|/G$ is acyclic. So, fixing a nonempty simplex $(P, e)$ as in the previous text, it suffices to show that $|(P, e)Z|/N_G(P, e)$ is acyclic. We need only worry about the case where $(P, e) = ((P^0, e^0))$, because if $(P, e)$ is not a $G$-conjugate of $(P^0, e^0)$, then we can consider the element $(Q, f)'$ of $\text{sd}(P, e)Z(Q, f)$ obtained from $(Q, f)$ by inserting $(P_n, e_n)$ as the minimal term, and the argument proceeds as before. Clearly, $(P^0, e^0)Z$ is nonempty. Also, $(P^0, e^0)Z$ is the $N_G(P^0, e^0)$-simplicial complex $\mathcal{S}_{>e}$, where $\mathcal{S}_{>e}$ consists of the Brauer pairs on $FN_G(P^0, e^0)$ strictly containing $(P^0, e^0)$. By induction, we may assume that $|\mathcal{S}_{>e}|/N_G(P^0, e^0)$ is acyclic, and now there is nothing left to prove.

**Sketch Proof of Theorem 4.** We indicate the modification to be made to the proof of Thévenaz–Webb [8, Theorem 2]. Given $(P, e) \in \mathcal{S}(b) - \mathcal{A}$, then the $N_G(P, e)$-posets,

$$\{(Q, f) : (1, b) < (Q, f) < (P, e)\} \quad \text{and} \quad \{Q : 1 < Q < P\}$$

are isomorphic, and we can apply [8, 1.7] to the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$, deducing the first asserted $G$-homotopy equivalence.
To demonstrate the second half of the assertion, we may assume that \( \mathcal{A} \) consists of precisely those Brauer pairs \((Q,f)\) such that \( Q \) is Abelian. Let \( \mathcal{P} := \text{sd}(\mathcal{A}(b)_q) \) as a \( G \)-poset. Let \( \phi \) be the surjective \( G \)-poset map \( \mathcal{P}^\text{op} \to \mathcal{A} \) such that, given \((P,e) = ((P_0,e_0) \subset \cdots \subset (P_n,e_n)) \in \mathcal{P} \), then \( \phi(P,e) := (A,f) \) where \( A \) is the intersection of the centres of the \( p \)-subgroups \( P_i \) and \( (A,f) \leq (P_0,e_0) \). Let us now fix \((A,f) \in \mathcal{A} \), and let \( \mathcal{Q} \) be the \( N(A,f) \)-subposet of \( \mathcal{P} \) consisting of the elements \((Q,f)\) such that \( \phi(Q,f) \geq (A,f) \). For such \((Q,f)\), let \((Q,f)'\) be the element of \( \mathcal{Q} \) obtained by inserting \((A,f)\) as the minimal term (leaving \((Q,f)\) unchanged if \((A,f)\) is already the minimal term). Then \( \mathcal{Q} \) is \( N(A,f) \)-contractible via \((Q,f) \mapsto (Q,f) \mapsto ((A,f)) \), and the assertion holds by \([8, \text{Theorem 1(ii)}]\).

Proof of Theorem 5. By Theorems 3–5, respectively, \(|\mathcal{A}(b)_q|/G\) is acyclic, \(|\mathcal{A}(b)/G| = |\mathcal{A}(b)_q|/G\), and \(|\mathcal{A}(b)|/G\) is simply connected.

Acknowledgments

The seed for this work was an unpublished theorem of Burkhard Kulshammer and Geoffrey R. Robinson. I also thank Klaus Haberland for some illuminating comments.

References