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Bounded unit root processes with non-stationary volatility

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ABSTRACT

This article concerns the unit root testing under nonstandard conditions for a time series process, such as having an innovation process with non-stationary variance and being limited inside an interval. These conditions are investigated separately in the unit root literature and shown to cause problems, such as size distortions. In this article, we consider the presence of both conditions in the unit root tests simultaneously. The simulation results indicate that the previous methods fail to provide satisfactory inference performance under the simultaneous presence of these conditions. To alleviate this issue, we propose a robust unit root testing mechanism and derive this procedure's asymptotic properties.

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1. Introduction

In time series analysis, modeling non-stationarity for bounded (limited) processes is an overlooked avenue of research. Granger (2010) defined a bounded process as 'a process possessing bounds either below (at zero, say) or above (full capacity) or both'. We can categorize series such as expenditure shares, unemployment rates, nominal interest rates as inherently limited processes based on this definition. Also, series like minimum wages and target zone exchange rates are bounded by policy control. Granger (2010) asserts that the existence of bounds compels integrated series to behave like stationary ones as they approach the bounds. Neglecting this feature makes the inference of the standard unit root tests theoretically invalid. As a remedy, Cavaliere (2005) extends the asymptotic theory of the standard unit root tests to the limited time series domain by introducing the limited unit root distribution to the literature. Cavaliere and Xu (2014) improve the finite sample properties of the unit root tests defined in Cavaliere (2005) by accounting for potential autocorrelation. Although these studies briefly consider the possible permanent volatility shifts in the innovations, they do not provide a detailed theoretical investigation of the issue and an extensive finite sample performance of the widely used unit root tests under this setup. In this paper, we close this gap in the literature and propose a new approach to unit root testing robust to non-stationary volatility and the range constraints.

Some crucial examples of non-stationary volatility observed in bounded time series can be listed as the U.S. short-term nominal interest rate, EMS target exchange rate and U.S. unemployment rate (see Romer 1986; Bollen, Gray, and Whaley 2000; Rodrigues and Rubia 2005). The authors Cavaliere and Taylor (2007, 2008, 2009) examine the impact of significant volatility

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breaks in such series on the standard unit root tests. These studies develop new unit root tests robust to a broad class of volatility breaks and smooth volatility change. However, the tests that they propose are not reliable under the limited-time series assumption.

We first improve the asymptotic theory for integrated time series under permanent volatility shifts. We analyze the variance shifted bounded Brownian motion and present the convergence of the partial sum processes obtained from the observed series. Second, we develop generalizations of the standard unit root tests to the limited time series with a variety of volatility changes. Therefore, we provide a general class of unit root tests robust to boundedness and non-stationary volatility. The tests in this class are asymptotically valid. Third, by consistently estimating bound parameters and variance profile, we use a direct simulation method to obtain the critical values of our proposed tests.

The rest of the article is organized as follows. [Section 2](#) introduces the bounded integrated processes with non-stationary volatility and details the necessary assumptions. The limited unit root distribution under permanent variance changes is demonstrated, and the convergence results are presented in [Sec. 3](#). [Section 4](#) presents the asymptotic results for ADF and M type unit root tests in the presence of range constraints and non-stationary volatility. [Section 5](#) shows the unit root testing mechanism. The finite sample properties of new unit root tests are presented in [Sec. 6](#). [Section 7](#) concludes. The tables for the simulation results and proofs of the theorems and lemmas are placed in the [Appendix B](#). $\mathcal{C}$ is the space of continuous processes on $[0, 1]$, $\mathcal{D}$ is the space of right continuous with left limit (càdlàg) processes on $[0, 1]$ interval, $\xrightarrow{w}$ denotes weak convergence, $\xrightarrow{p}$ denotes the convergence in probability and $\lfloor x \rfloor$ is the closest integer to x . All limits in the article are computed as the sample size goes to positive infinity.

2. The model and assumptions

In this section, we introduce the family of the bounded non-stationary processes where the innovations display a general class of permanent volatility changes. Let $\{X_t\}_{t=0}^T$ be a bounded time series with the fixed and known bounds $\underline{b}$ and $\bar{b}$ ($\underline{b} < \bar{b}$). We define this time series with the drift parameter θ as

$$X_t = \theta + Y_t, \quad (1)$$

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \rho = 1, \quad (2)$$

initialized at $Y_0 = O_p(1)$. The term ε_t is decomposed as follows:

$$\varepsilon_t = \psi(L)v_t + \underline{\xi}_t - \bar{\xi}_t, \quad (3)$$

where $\underline{\xi}_t$ and $\bar{\xi}_t$ are non-negative processes such that $\underline{\xi}_t > 0$ if and only if $Y_{t-1} + \psi(L)v_t < \underline{b} - \theta$ and $\bar{\xi}_t > 0$ if and only if $Y_{t-1} + \psi(L)v_t > \bar{b} - \theta$. Since we consider non-stationary volatility in the innovations, we consider v_t as a process in the form $v_t = \sigma_t z_t$ where $z_t \sim iid(0, 1)$ and σ_t is the time-varying standard deviation of v_t .

Remark 1. Notice that the innovation term v_t is characterized by σ_t and z_t . Since conditional on σ_t , z_t has independent and identical distribution (i.i.d.), the innovation v_t has mean zero and time-varying variance σ_t^2 .

Remark 2. In this paper, we only consider a time-invariant specification of the deterministic term θ and the case without deterministic term follows analogously by allowing $\theta = 0$ in [Eq. \(1\)](#). Although time-varying deterministic component is possible in this setup, it will complicate the notation. For instance, we need to consider time-varying bounds, such as $\underline{b}_t$ and $\bar{b}_t$. Moreover, time-varying bounds case is not a realistic scenario for economics and finance. For further discussion of time-varying bounds, see Cavaliere and Xu (2014) and Carrion-I Silvestre and Gadea (2013).

Throughout this chapter, we will utilize the following assumptions, which will be taken to hold on (1) – (3) :

Assumption A.

1. The lag operator $\psi(L) = \sum_{j=0}^{\infty} \psi_j L^j$ satisfies $\psi(z) \neq 0$ for all $|z| \leq 1$ and $\sum_{j=0}^{\infty} j^s |\psi_j| < \infty$ for some $s \geq 1$.
2. $\{z_t, \mathcal{F}_t\}$ is a MDS (martingale difference sequence) with respect to some filtration $\mathcal{F}_t$, such that (a) $E(z_t^2) = 1$, (b) $T^{-1} \sum_{t=1}^T z_t^2 \xrightarrow{p} 1$, and (c) $E|z_t|^r < \infty$ for some $r \geq 4$.
3. $\sup_{t=1, \dots, T} E|\xi_t|^r < \infty$ and $\sup_{t=1, \dots, T} E|\bar{\xi}_t|^r < \infty$ or $\max_{t=1, \dots, T} |\xi_t|$ and $\max_{t=1, \dots, T} |\bar{\xi}_t|$ are of $\mathcal{O}_p(T^{1/2})$.
4. $\frac{\bar{b}-\theta}{\lambda T^{1/2}} = \underline{c} + o(1)$ and $\frac{\bar{b}-\theta}{\lambda T^{1/2}} = \bar{c} + o(1)$ where $\underline{c} \leq \bar{c}$ and $\lambda^2 = \psi(1)^2 \bar{\omega}^2$ denotes the long run variance.

Assumption A1 implies that u_t is stable and has an invertible representation in terms of the v_t 's. A2 sets the sequence of the error terms z_t to be martingale differences. These assumptions are quite standard in the unit root testing literature. Assumption A3 presents a restrictive condition that prevents X_t from doing excessive leaps at the bounds. Under Assumption A4, the upper and lower bound locations are taken as nuisance parameters. Moreover, Assumption A4 states that, under the presence of bounds, the $\sim T^{1/2}$ order of the random walk part of the series is preserved. For more details, we refer Cavaliere and Xu (2014).

The following assumption is given to capture the heteroscedasticity in the innovations:

Assumption B. The volatility term σ_t satisfies $\sigma_{[sT]} := \omega(s)$ for all $s \in [0, 1]$, where $\omega(\cdot) \in \mathcal{D}$ is non-stochastic and strictly positive. For $t < 0$, $\sigma_t \leq \sigma < \infty$.

Assumption B is the critical assumption that permits the innovation variance dynamics to be considered in a general framework. Under this assumption, the variance of the error term needs to be bounded and exhibit a countable amount of jumps. Therefore, a broad family of a volatility process can be allowed. The conventional homoscedasticity assumption made in the unit root testing literature that is $\sigma_t = \sigma$ for all t , also satisfies Assumption B.

To conduct the asymptotic analysis, we need the variance profile function, which is a fundamental object to capture non-stationary volatility in the asymptotic analysis. The variance profile function is defined in C as,

$$\eta(s) = \left[\int_0^1 \omega(r)^2 dr \right]^{-1} \int_0^s \omega(r)^2 dr \quad \text{for all } s \in [0, 1]. \quad (4)$$

The object defined in (4) reflects the time series pattern of the volatility of the observed series. Furthermore, Cavaliere and Taylor (2007) show that $\int_0^1 \omega(r)^2 dr = \bar{\omega}^2$ is the limit of $T^{-1} \sum_{t=1}^T \sigma_t^2$, where $\bar{\omega}^2$ represents the asymptotic average variance of the error term.

Remark 3. The assumption that the volatility function $\omega(\cdot)$ is deterministic is given to simplify the theoretical setup. However, this assumption can be relaxed by allowing for the cases where the error terms z_t and $\omega(\cdot)$ are stochastically independent. In fact, in such cases, if the sample paths of the volatility process $\omega(\cdot)$ satisfy Assumption B, the results obtained in this chapter can be considered as conditional on the realization of $\omega(\cdot)$.

The proposed time series model possesses a similar structure with the models in Cavaliere and Xu (2014) and Cavaliere and Taylor (2007). However, we extend both models by considering joint occurrence of bounds and nonstationary volatility. One important consequence of such mixture is that the presence nonstationary volatility may locally decrease or increase the speed of the time series within the bounds relative to the constant variance case. In other words, the

probability of the time series process approaches to the bounds now also depends on the volatility component, which is assumed to be constant in Cavaliere and Xu (2014). For instance, a high volatility period may cause the series hit faster if its direction is also toward the bound, or vice versa. In the next section, we present such impact of nonstationary volatility on the asymptotic objects of regulated time series processes.

3. The variance transformed bounded unit root distribution

This section presents how the presence of both range constraints and permanent volatility shifts alters the asymptotic framework. First, we focus on the effect of the volatility pattern on the asymptotic theory. Throughout this chapter, one of the key roles on the limiting distributions is constructed on $B_\eta(s) = B(\eta(s))$, which is first defined in Clark (1973) as subordinate stochastic process (see also, Cavaliere and Taylor 2007; Davidson 1994; Revuz and Yor 2013). $B(\cdot)$ is a standard Brownian motion and $\eta(\cdot)$ is an increasing homomorphism with $\eta(0) = 0$ and $\eta(1) = 1$. Consequently, $B_\eta(\cdot)$ is referred to as the ‘variance-transformed’ Brownian motion.

The variance transformed Brownian motion is a continuous process, i.e., $B_\eta(\cdot) \in \mathcal{C}$; see Davidson (1994) for the proof. This property coupled with Proposition 6 in Chapter 2 of Harrison (1985) tells us that we can find a pair of functions $L(\cdot)$ and $Q(\cdot)$ such that $B_\eta(\cdot) + L(\cdot) - Q(\cdot)$ is regulated process. Now, we introduce the following key definition:

Definition 1. Let $B_\eta(t)$ be variance shifted Brownian motion defined on $\mathcal{C}$ (see Cavaliere and Taylor 2007; Davidson 1994; Revuz and Yor 2013; Clark 1973). Suppose that we have fix bounds $\underline{c}$ and $\bar{c}$. If $B_\eta(0) \in [\underline{c}, \bar{c}]$, then there exists continuous functions $L(\cdot)$ and $Q(\cdot)$ satisfying the conditions of Lemma 1 such that $B_\eta^\varepsilon, \bar{c}(t) = B_\eta(t) + L(t) - Q(t) \in [\underline{c}, \bar{c}]$ for all $t \in [0, 1]$. The functions $L(\cdot)$ and $Q(\cdot)$ are called regulators and the function $B_\eta^\varepsilon, \bar{c}(\cdot)$ is called ‘regulated variance transformed Brownian motion’ with bounds at $[\underline{c}, \bar{c}]$.

Next, consider the continuous-time approximation of a process $\{Y_t\}$ on the Cadlg space $\mathcal{D}$ to transform the original series through the broken line process.

$$Y_T(s) = (\lambda^2 T)^{-1/2} (Y_{\lfloor Ts \rfloor} - Y_0), s \in [0, 1].$$

The main result of this section is now summarized in the following theorem:

Theorem 1. Let $\{Y_t\}$ be the process defined in (2)–(3) and the fix bounds are given as $[\underline{b}, \bar{b}]$. If $\mathcal{D}$ is endowed with uniform topology and $Y_0 \in [\underline{c}, \bar{c}]$ then $Y_T(s) \xrightarrow{w} B_\eta^\varepsilon, \bar{c}(s)$.

3.1. Extension for the deterministic components

In the previous analysis, we only consider the pure random walk process Y_t , but in our initial characterization we observe the series X_t with a drift term. To remove this drift term, we regress X_t on a constant term and obtain the residuals $\hat{X}_t = Y_t - (\hat{\theta} - \theta)$ where $\hat{\theta}$ is OLS estimate of θ . In Lemma 1 below, we now provide the representations for the OLS de-meant time transformed Brownian motions:

Lemma 1. Under assumption $\mathcal{A}$ and X_t is generated by (1)–(3), we define the partial sum process for de-meant process as $\hat{X}_T(s) = (\lambda^2 T)^{-1/2} (\hat{X}_{\lfloor Ts \rfloor} - \hat{X}_0)$, $s \in [0, 1]$. Then we have $\hat{X}_T(s) \xrightarrow{w} B_{j,\eta}^\varepsilon, \bar{c}(s)$ for $j=0, 1$. $j=0$ when there is no deterministic component and $j=1$ when there is only a drift term, where

$$B_{j,\eta}^\varepsilon, \bar{c}(s) = B_\eta^\varepsilon, \bar{c}(s) \quad \text{for } j = 0,$$

$$B_{j,\eta}^\varepsilon, \bar{c}(s) = B_\eta^\varepsilon, \bar{c}(s) - \left[\int_0^1 B_\eta^\varepsilon, \bar{c}(r) dr \right] \quad \text{for } j = 1.$$

4. Unit root tests and their large sample properties

In this section, we present the implementations of the limited time transformed unit root distribution for the well-known unit root tests. Accordingly, we discuss the effect of the time-varying variances of the shocks on the asymptotic distributions of augmented Dickey-Fuller and M unit root tests when the series is regulated.

For given sample $\{X_t\}_{t=0}^T$, the ADF statistics are based on the OLS regression:

$$\hat{X}_t = \alpha \hat{X}_{t-1} + \sum_{i=1}^k \alpha_i \Delta \hat{X}_{t-i} + \varepsilon_{t,k}, \quad (5)$$

and these tests are defined as

$$ADF_\alpha = \frac{T(\hat{\alpha} - 1)}{\hat{\alpha}(1)}, \quad ADF_t = \frac{\hat{\alpha} - 1}{s(\hat{\alpha})},$$

where $\hat{\alpha}(1) = 1 - \sum_{i=1}^k \hat{\alpha}_i$, with $\hat{\alpha}_i$ denoting OLS estimator of α_i in (5) and $s(\hat{\alpha})$ the OLS standard error of $\hat{\alpha}$. Here $\hat{X}_t$ is the de-meaned defined in [Sec. 3.1](#).

We can define the M statistics as

$$MZ_\alpha = \frac{T^{-1} \hat{X}_T^2 - T^{-1} \hat{X}_0^2 - s_{AR}^2(k)}{2T^{-2} \sum_{t=1}^T \hat{X}_{t-1}^2}, \quad MSB = \left[T^{-2} \sum_{t=1}^T \hat{X}_{t-1}^2 / s_{AR}^2(k) \right]^{1/2}, \quad MZ_t = MZ_\alpha \times MTB,$$

where $s_{AR}^2(k)$ is an autoregressive estimator of the spectral density frequency zero of $\{\varepsilon_t\}$. Specifically, $s_{AR}^2(k) = \hat{\sigma}^2 / \hat{\alpha}(1)^2$ where $\hat{\sigma}^2$ is the OLS variance estimator from the regression 5. By following Lewis and Reinsel (1985), for the aforementioned tests, the lag truncation parameter is assumed to satisfy the assumption below:

Assumption C. As $T \rightarrow \infty$, $1/k + k^2/T \rightarrow 0$.

Under this assumption, one can consistently estimate the length truncation parameter k by using an information criterion such as Bayesian Information Criterion. In the theorem below, we introduce the representations for the limiting null distributions of the aforementioned test statistics in the presence of range constraints and non-stationary volatility.

Theorem 2. Let $\{X_t\}_{t=0}^T$ be generated as in (1)–(3) with $\rho = 1$. Under Assumptions A, B and C,

- i. $s_{AR}^2(k) \xrightarrow{P} \bar{\omega}^2 \psi(1)^2$, where $\bar{\omega}^2 \psi(1)^2$ is the long run variance of the innovation term;
- ii. $ADF_\alpha, MZ_\alpha \xrightarrow{w} 0.5 \left[B_{\eta,j}^\varepsilon, \bar{c}(1)^2 - B_{\eta,j}^\varepsilon, \bar{c}(0)^2 - 1 \right] \left[\int_0^1 B_{\eta,j}^\varepsilon, \bar{c}(s)^2 ds \right]^{-1}$;
- iii. $MSB \xrightarrow{w} \left[\int_0^1 B_{\eta,j}^\varepsilon, \bar{c}(s)^2 ds \right]^{1/2}$;
- iv. $ADF_t, MZ_t \xrightarrow{w} 0.5 \left[\hat{B}_{\eta,j}^\varepsilon, \hat{c}(1)^2 - B_{\eta,j}^\varepsilon, \bar{c}(0)^2 - 1 \right] \left[\int_0^1 B_{\eta,j}^\varepsilon, \bar{c}(s)^2 ds \right]^{-1/2}$,

where $B_{\eta,j}^\varepsilon, \bar{c}(s)$ is defined as in [Lemma 1](#).

Remark 4. The limiting distributions used in ADF and M unit root tests in [Theorem 2](#) differ from the standard and regulated integrated of order one -I(1)- asymptotic distributions. The sample paths of limiting process in our case are bounded between $\underline{c}$ and $\bar{c}$ and time-shifted by the variance profile $\eta(\cdot)$. The well known case of $I(1)$ process has no bounds and $\eta(s) = s$.

5. Testing for unit root tests under the presence of bounds and non-stationary volatility

As we have discussed in the previous sections, the conventional unit root inference is influenced by the presence of bounds and permanent volatility shifts. Therefore the limiting null distributions of the frequently used test statistics become inapplicable. Moreover, the new asymptotic distributions contain additional nuisance parameters, which need to be estimated before the inference. To alleviate this issue, we propose a three steps procedure to generate the proposed asymptotic distributions of unit root tests for bounded processes with non-stationary volatility. First, [Sec. 5.1](#) presents the consistent estimation of the variance profile $\eta(s)$. Second, [Sec. 5.2](#) introduces the consistent estimators of the nuisance parameters $\underline{c}$ and $\bar{c}$. Last, in [Sec. 5.3](#), we develop the simulation-based tests which can be performed to acquire asymptotically valid unit root tests in the presence of the bounds and non-stationary volatility.

5.1. Consistent estimation of the variance profile

In this section, we present the variance profile estimator, which is defined in Cavaliere and Taylor (2007). Let $\hat{X}_t$ be de-trended or demeaned version of observed series X_t , then we have

$$\hat{\eta}(s) = \frac{\sum_{t=1}^{\lfloor sT \rfloor} (\Delta \hat{X}_t)^2 + (sT - \lfloor sT \rfloor)(\Delta \hat{X}_{\lfloor sT \rfloor + 1})^2}{\sum_{t=1}^T (\Delta \hat{X}_t)^2}. \quad (6)$$

In the following Lemma, we now demonstrate that $\hat{\eta}(\cdot)$ is a uniformly consistent estimator for the variance profile $\eta(\cdot)$ in the presence of bounds by generalizing [Theorem 2](#) of Cavaliere and Taylor (2007) to the present framework.

Lemma 2. *Under the assumptions $\mathcal{A}$ and $\mathcal{B}$, we have $\hat{\eta}(s) \xrightarrow{P} \eta(s)$ uniformly for all $s \in [0, 1]$.*

5.2. Consistent Estimation of the range parameters

In the previous parts, we assume that the bounds $\underline{b}, \bar{b}$ are known. Therefore, there is a feasible way to estimate the nuisance parameters $\underline{c}, \bar{c}$ consistently. To this aim, in our setup, consider the estimators which are easy to implement:

$$\hat{\underline{c}} = \frac{\underline{b} - \hat{\theta}}{s_{AR}(k)T^{1/2}}, \quad \hat{\bar{c}} = \frac{\bar{b} - \hat{\theta}}{s_{AR}(k)T^{1/2}}, \quad (7)$$

where $s_{AR}^2(k)$ is the spectral AR estimator of the long run variance as defined in [Sec. 2.4](#). The result on the consistency of $\hat{\underline{c}}$ and $\hat{\bar{c}}$ follows from Cavaliere (2005) and Cavaliere and Xu (2014), thus we skip the proofs for brevity.

5.3. Simulation based tests

Now, given the estimate $\hat{\eta}(\cdot)$ and $\hat{\underline{c}}$ and $\hat{\bar{c}}$, we are able to approximate the quantiles from the asymptotic null distributions of the conventional M and ADF statistics given in [Theorem 2](#), by utilizing direct simulation methods.

Let $\{e_t\}$ be i.i.d. (0,1) random variable. From [Theorem 3](#) of Cavaliere and Taylor (2007) we know that

$$B_{\hat{\eta}, T}(s) := T^{-1/2} \sum_{t=1}^{\lfloor \hat{\eta}(\lfloor sT \rfloor / T) T \rfloor} e_t \xrightarrow{w} B_{\eta}(s) \text{ for all } s \in [0, 1]. \quad (8)$$

Therefore, define $f_t = \Delta B_{\hat{\eta}, T}(t/T) = B_{\hat{\eta}, T}(t/T) - B_{\hat{\eta}, T}((t-1)/T)$ for $t = 1, 2, \dots, T$. Consider the recursive setup as follows:

$$\tilde{X}_t = \begin{cases} \hat{\underline{c}} & \text{if } \tilde{X}_{t-1} + f_t < \hat{\underline{c}}T^{1/2}, \\ \hat{\bar{c}} & \text{if } \tilde{X}_{t-1} + f_t > \hat{\bar{c}}T^{1/2}, \\ \tilde{X}_{t-1} + f_t & \text{otherwise.} \end{cases} \quad (9)$$

Now, we define the following theorem as sample T approaches to infinity:

Theorem 3. Under the assumptions $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, we have $\tilde{X}_T(s) \xrightarrow{w} B_{\tilde{\eta}}^{\underline{c}, \bar{c}}(s)$ for all $s \in [0, 1]$ where $\tilde{X}_T(s) = T^{-1/2}X_{[sT]}$.

Remark 5. Theorem 3 shows that as T approaches to infinity, the simulated process $\tilde{X}_t$ is distributed as the variance transformed bounded Brownian motion $B_{\tilde{\eta}}^{\underline{c}, \bar{c}}(\cdot)$. We, therefore, approximate the quantiles from the non-pivotal limiting null distributions in Theorem 2 by utilizing numerical simulations based on estimating $\underline{c}$ and $\bar{c}$ and using the Cadlg processes $\tilde{X}_t$ and f_t . Using the conventional ADF and M statistics in Sec. 4, we can now obtain the simulation based versions of the ADF and M tests. Accordingly, testing for unit roots can be performed by the simulated critical values for a given significance level.

In practice, our proposed tests, which are robust to presence of both non-stationary volatility and bounds can be applied as follows. First, by creating a sequence of $iid(0, 1)$ random numbers, $B_{\tilde{\eta}}(s)$ is generated. Then, with the recursive procedure defined in (9), $B_{\tilde{\eta}}^{\underline{c}, \bar{c}}(\cdot)$ is constructed. Next, corresponding demeaned process $B_{\tilde{\eta}, 1}^{\underline{c}, \bar{c}}(\cdot)$ is obtained by standard OLS projection. For illustration, let us take the test of ADF_t . The quantity $0.5[B_{\tilde{\eta}, j}^{\underline{c}, \bar{c}}(j)^2 - B_{\tilde{\eta}, j}^{\underline{c}, \bar{c}}(0)^2 - 1][T^{-1}B_{\tilde{\eta}, j}^{\underline{c}, \bar{c}}((t-1)/T)^2]^{-1/2}$ corresponding to discretized version of the limiting distribution is computed. This operation is needed to be employed for MC independent replications, where MC is the number of Monte-Carlo simulation. Next step is to construct order statistic of the MC outcomes as $ADF_t^{(1)} \leq ADF_t^{(2)} \leq \dots \leq ADF_t^{(MC)}$. For a significance level ζ , an estimate of the desired quantile is calculated as $ADF_t^{(\lfloor \zeta \times MC \rfloor)}$. The rejection decision of the unit root null at significance level ζ is given where ADF_t is lower than $ADF_t^{(\lfloor \zeta \times MC \rfloor)}$. Note that, MZ_α , MZ_t and ADF_α tests can be performed with the same procedure.

Remark 6. It is noteworthy to state that, by construction, power functions of our tests closely approximate the size-adjusted powers of the corresponding standard unit root tests, heteroscedasticity robust unit root tests of Cavaliere and Taylor (2007, 2008) and bounded unit root tests of Cavaliere and Xu (2014).

6. Finite sample simulations

This section presents Monte Carlo simulation results for ADF_a and MZ_a unit root tests under a variety of volatility patterns and bound parameters. We compare the performance of our proposed unit root tests ADF_a^{bnv} and MZ_a^{bnv} , which are robust to the presence of both bounds and heteroscedasticity with (i) standard tests ADF_a^s and MZ_a^s , (ii) heteroscedasticity robust tests ADF_a^{nv} and MZ_a^{nv} of Cavaliere and Taylor (2007, 2008), and (iii) bounded unit root tests ADF_a^b and MZ_t^b of Cavaliere and Xu (2014).¹

We generate data as in Eqs. (1)–(3) with the sample sizes $T = \{250, 500\}$ by setting $Y_0 = \theta = 0$. We consider the case of symmetric bounds ($\bar{c} = -\underline{c} =: c > 0$). Additionally, to investigate the performance of the tests under heteroscedasticity, we consider the following error term variance specifications²:

¹The performance of ADF_t , MZ_t and MSB tests are omitted since they exhibit similar performance to MZ_a and ADF_a tests and qualitatively their contribution is negligible to the reported results. They are available upon request.

²In the error term variance specifications, we follow Cavaliere and Taylor (2007).

1. Constant Volatility (CV): $\omega_1(s) = 1$, for $s \in [0, 1]$.
2. Single Break in Volatility (SBV): $\omega_2(s) = 1 + (\frac{1}{3} - 1)^* I(s > 0.2T)$ for $s \in [0, 1]$.
3. Trending Volatility (TV): $\omega_3(s) = 1 + (\frac{1}{3} - 1)^* s$ for $s \in [0, 1]$.
4. Exponential Integrated Stochastic Volatility (EISV): $\omega_4(s) = \exp(9B(s))$ for $s \in [0, 1]$ and $B(s)$ is standard Brownian motion process.

All experiments are performed under 10.000 replications. All unit root tests are employed at a nominal 5% significance level. Following Cavaliere and Taylor (2009) and Cavaliere and Xu (2014), we draw the innovations z_t from i.i.d. $N(0, 1)$. We obtain the distribution of $\varepsilon_t = \Delta Y_t$ by reflecting the distribution of $u_t = \psi(L)v_t$ at $\underline{b} - Y_{t-1}$ and $\bar{b} - Y_{t-1}$.³ Additionally, we set $X_t = Y_t$ since the asymptotic distributions are invariant to the value of the deterministic terms. The critical values of our tests are obtained by the direct simulation method described in Sec. 5.3 by fixing the discretization step size to T . The variance profile and bound parameters are estimated as described in Secs. 5.1 and 5.2. All of the test results are based on OLS demeaned data.

We report the performance of the aforementioned unit root tests for serially uncorrelated and correlated innovations. For the case where the innovations are serially uncorrelated, we set $\psi(L) = 1$ and correspondingly $k=0$ in (5). Then, we allow for weak dependence in u_t . Under this scenario, we choose the optimal lags in equation (5) by using MAIC lag length selection criterion of Ng and Perron (2001) with $k \leq \lfloor 12(T/100)^{0.25} \rfloor$. Finally, under weak dependence, by following Ferretti and Romo (1996), Chang and Park (2002) and Cavaliere and Xu (2014), to improve the finite sample performance of the tests, we use the re-colored simulation-based tests. The procedure in these tests re-builds stationary serial correlation into the MC innovations without changing the limiting distributions presented in Theorem 2.⁴

There are four serial correlation scenarios considered for the innovation sequence $\{u_t\}$. In the first scenario, innovations do not contain any serial correlation. In the second one, u_t follows MA(1) model, i.e., $u_t = v_t + 0.5v_{t-1}$. In the third one, u_t is a stationary AR(1) process, i.e., $u_t = 0.5u_{t-1} + v_t$. Finally, the last one follows an ARMA(1,1) process, i.e., $u_t = -0.5u_{t-1} + v_t + 0.5v_{t-1}$.⁵ We set $\rho \in \{1, 0.93, 0.86\}$, where $\rho=1$ is for the size exercise, and other values are for the power evaluation of the tests.

6.1. Size performance

In this section, we report the finite sample size performance of the tests mentioned above. Table 1 demonstrates the size performance under the uncorrelated shocks. In Tables 2, 3 and 4, we report the size performance for the weakly dependent shocks with the models of MA(1), AR(1) and ARMA(1, 1), respectively. Additionally, in these simulations, the ideal size is equal to the nominal significance level, which is 5%. On the one hand, if the size of the test is greater than the nominal size, we say that the test is oversized and rejecting the null hypothesis of unit root more than it should. On the other hand, if the size of the test is less than the nominal size, we say that the test is undersized and rejecting the null hypothesis less than it should.

³We set $\underline{\xi}_t := 2[\underline{b} - (Y_{t-1} + \varepsilon_t)]I(Y_{t-1} + \varepsilon_t < \underline{b})$ and $\bar{\xi}_t := 2[\bar{b} - (Y_{t-1} + \varepsilon_t)]I(Y_{t-1} + \varepsilon_t < \bar{b})$. We also set various types of truncation mechanisms and obtain very similar results. They are also available upon request.

⁴We refer the reader to Chang and Park (2002) for the detailed information about re-colored simulation-based tests.

⁵The results for other serial correlation scenarios do not differ qualitatively from those reported here.

6.1.1. Uncorrelated errors

First, consider the constant volatility case, where the errors are homoscedastic. Under the case of no bounds, all of the tests have the same performance and are close to the nominal significance level. This situation is an expected result since it is a benchmark case for all of the tests. However, in the presence of bounds, standard tests ADF_a^s - MZ_a^s and heteroscedasticity robust tests ADF_a^{nv} - MZ_a^{nv} become significantly over-sized, since they are not robust to the presence of bounds. For instance, under $T=250$, for $c=0.4$, the empirical rejection frequency (ERF) of ADF_a^s is 32% and ERF of ADF_a^{nv} is 27%. We do not see an over-sizing problem in ADF_a^b and MZ_a^b , since by nature, they are robust to homoscedastic data with bounds. Our tests ADF_a^{bmv} and MZ_a^{bmv} exhibit satisfactory performance under this case.

Next, consider the single break in volatility case. Under the case of no bounds, ADF_a^s - MZ_a^s and ADF_a^b - MZ_a^b are significantly oversized. We observe that under $T=250$, with no bounds, the ERFs of MZ_a^s and MZ_a^b are 16%. We expect these results since these tests are not robust to non-stationary volatility. Under no bounds, as expected ADF_a^{nv} - MZ_a^{nv} and our tests ADF_a^{bmv} - MZ_a^{bmv} have the correct size. However, as the bounds become tight, ADF_a^{nv} , MZ_a^{nv} becomes size distorted since, as we mention before, these tests are not robust to the presence of the bounds. For instance, under $T=500$ where $c=0.6$, the ERFs of ADF_a^s , ADF_a^{nv} and ADF_a^b are 33%, 15% and 14% respectively. However, the ERF of our proposed tests ADF_a^{bmv} and MZ_a^{bmv} are so close to the nominal significance level under this case, for all bound specifications.

Now, consider the trending volatility case. In this case, we observe a similar performance to the single break volatility case. Under this scenario, ADF_a^s - MZ_a^s and ADF_a^b - MZ_a^b are still oversized, but the size distortion is not as severe as the single break volatility case. This result is consistent with the findings of Cavaliere and Taylor (2007). Again, under $T=250$, with no bounds, the ERFs of MZ_a^s and MZ_a^b are 7%. We observe that as the bounds become tighter and tighter, ADF_a^{nv} , MZ_a^{nv} have nearly the same size distortion as in the single break in volatility. It is important to note that, under this heteroscedasticity scenario with all bound parameters, our tests ADF_a^{bmv} and MZ_a^{bmv} are largely satisfactory and avoid the over-sizing problems associated with the other tests.

Under the uncorrelated shocks, finally, we consider the case of exponential integrated stochastic volatility. ADF_a^s - MZ_a^s and ADF_a^b - MZ_a^b have the most severe size distortion under this scenario. For instance, under $T=500$ with $c=0.4$, the ERFs of ADF_a^s and MZ_a^b are 64% and 26%. Also, we observe that under the same case, the ERF of ADF_a^{nv} is 23%. However, our tests ADF_a^{bmv} and MZ_a^{bmv} are always successful in controlling size under the exponential integrated stochastic volatility.

Overall, we observe that the size distortions observed in ADF_a^s - MZ_a^s , ADF_a^b - MZ_a^b and ADF_a^{nv} - MZ_a^{nv} tests do not disappear as the sample size T increases, because the nuisance parameter problems arising from the non-stationary volatility and bounds do not disappear in the long run.

6.1.2. Autocorrelated errors

The size performance of the aforementioned tests is now investigated for u_t following a linear stationary process. As it can be seen from Tables B2-B4, our tests ADF_a^{bmv} and MZ_a^{bmv} are successful in controlling size under the weakly dependent shocks regardless of the bound parameter, volatility type and sample size.

Under the weakly dependent errors, the standard MZ_a^s and ADF_t^s have a similar over-sizing problem with uncorrelated errors. For example, consider the results in Table B3, where the innovations follow AR(1) process. We observe that in the single break volatility case with $c=0.6$ and $T=250$, the ERF of MZ_a^s is 23%. Under the autocorrelated errors, the tests MZ_a^{nv} and ADF_a^{nv} are still oversized under the tight bounds. For an evidence, let us take a closer look at Table B2, where the innovations follow MA(1) process. For instance, under $T=250$ and the data with

single break volatility and bound parameter $c = 0.4$, the ERF of ADF_a^{bnv} is 11%. Finally, the tests MZ_a^b and ADF_a^b still exhibit a significant over-sizing in the presence of the non-stationary volatility under the weakly dependent shocks. Moreover, let us consider Table B4, where the shocks follow $ARMA(1,1)$ process. We can observe that in the exponential integrated stochastic volatility case with no bounds and $T = 500$, the ERF of ADF_a^b is 0.28. Overall, we observe in Tables B2–B4 that, in addition to being robust to non-stationary volatility and bounds, our tests ADF_a^{bnv} and MZ_a^{bnv} are also successful in controlling size in the presence of the serial correlation problem. Therefore, the ERFs of our tests are still so close to the nominal significance level except a mild under-sizing problem in some cases.⁶

6.2. Power performance

In this section, we investigate the empirical power against particular alternatives $\rho \in \{0.93, 0.86\}$. In the previous section, we observe that other unit root tests do not exhibit a satisfactory size performance. Therefore, their empirical power results are not reliable. So, we need the size-adjusted power of these tests to make a meaningful comparison. Moreover, as stated in Remark 10, by construction, our tests possess the power functions that closely approximate the size-adjusted power functions of the aforementioned tests in the literature. Therefore, following Cavaliere and Taylor (2007), we only report the power performance for our ADF_a^{bnv} and MZ_a^{bnv} tests. These results are similar to the size-adjusted power performance of the other tests.

We start by looking through Table B5, where the innovations are uncorrelated. First, under the constant volatility, we can conclude that for both ADF_a^{bnv} and MZ_a^{bnv} , as the bounds get tighter and tighter, the finite sample power decrease, but we still have satisfactory performance. This result is important since Cavaliere and Xu (2014) do not analyze the power performance of their unit root tests. Second, under the single break volatility, power is less, compared to the constant volatility case. Here, again, as the bounds become tighter, the power performance declines. Under the trending volatility, we observe the higher power compared to the single break volatility case, but the lower than the constant volatility case. In this case, we see a pattern of decrease in the power as the bounds become tighter and tighter. Finally, we see a similar performance in the trending volatility case under the exponential integrated stochastic volatility. Overall, we observe that ADF_a^{bnv} is more powerful compared to MZ_a^{bnv} .

Next, we consider Tables B6–B8, where the weak dependence models are $MA(1)$, $AR(1)$ and $ARMA(1,1)$ respectively. Overall the pattern is similar to the uncorrelated case under the various heteroscedasticity scenarios and range specifications. However, the empirical powers compared to the uncorrelated case are lower. The power performance of the tests under the single break volatility case is especially much lower than the uncorrelated case. We now observe that under the serially correlated errors, the power of ADF_a^{bnv} test is outperformed by MZ_a^{bnv} test.

7. Conclusion

In this paper, we have shown that the presence of both non-stationary volatility and range constraints has severe implications on the validity of the unit root tests in the literature. These tests suffer from significant size distortions. As a remedy, we have proposed a unit root testing mechanism that provides reliable inference under a very general class of volatility dynamics and bound specifications. For the construction of the testing procedure, we introduce the ‘variance shifted bounded Brownian motion’ process and provide the convergence of the observed series to this process.

⁶We observe in all tables the size distortions created by the EISV are different. We explain these high variations by the stochastic feature of the volatility process and the simulation uncertainty.

This new testing mechanism is based on the direct simulation methods that approximate the asymptotic null distributions of the unit root statistics. We consistently estimate the variance profile and bound parameters since the asymptotic distributions depend on the nuisance parameters arising from heteroscedasticity and bounds. The finite sample simulations show that the testing approach outlined in this study is successful in eliminating the size distortions observed in the other unit root tests.

Appendix A: proofs of the lemmas and theorems

Proof of Theorem 1. This proof closely follows the strategy outlined in Cavaliere (2005); thus, we will not give the full proof, but provide insight for brevity. In this regard, first let $v_t = \sigma_t \epsilon_t$ be the innovation process where ϵ_t is an i.i.d. white noise process and σ_t is assumed to be known. This situation is the only difference between our and Cavaliere's 2005 models. Cavaliere (2005) utilizes a homoscedastic innovation term with $\sigma_t = 1$. Accordingly, we can apply the findings of Cavaliere (2005). Although we can utilize the construction of Cavaliere (2005), we need to replace the standard Brownian motion $B(\cdot)$ with the variance-shifted Brownian motion $B_\eta(\cdot)$. Since $B_\eta(\cdot)$ is a continuous process as $B(\cdot)$, we can invoke the results in the proof of Theorem 1 in Cavaliere (2005). For space concerns, we skip the rest of the proof.

Proof of Lemma 1. The proof of this lemma is directly follows Theorem 1 and application of the continues mapping theorem.

Proof of Theorem 2. Throughout, to simplify the proof, it is assumed without loss of generality that no deterministic terms are included in the model and the estimation, hence $X_t = Y_t$.

Following Cavaliere and Xu (2014), we can decompose X_t as

$$X_t = \psi(1) \sum_{i=1}^t v_i + \sum_{i=1}^t (\xi_t - \bar{\xi}_t) + \tilde{u}_0 - \tilde{u}_t. \quad (A1)$$

In Eq. (A1), it is seen that non-stationary part of the $BI(1)$ process can be broken down into a random walk, $\sum_{i=1}^t v_i$, and the accumulated regulators, $\sum_{i=1}^t (\xi_t - \bar{\xi}_t)$. Assumption A4 implies that the order of these two terms is same. As a result, regulators are allowed to affect both short run and long run dynamics of X_t .

Under Assumption A, let us define $\psi(z)^{-1} = \alpha(z) = 1 - \sum_{j=1}^{\infty} \alpha_j z^j$. By letting $\xi_t^* = \alpha(L)\xi_t$ and $\bar{\xi}_t^* = \alpha(L)\bar{\xi}_t$, define $u_t^* = u_t + \xi_t^* - \bar{\xi}_t^*$, where $e_t = \psi(L)u_t + \xi_t - \bar{\xi}_t = \psi(L)u_t^* + \xi_t^* - \bar{\xi}_t^*$, $w_t^* = \sum_{i=1}^t u_i^*$ and $r_t^* = \xi_t^* - \bar{\xi}_t^*$.

We will present the proof for ADF_t and ADF_{η} statistics. The proof for other statistics follows similarly.

Let $Z_{t,k} = (\Delta X_{t-1}, \dots, \Delta X_{t-k})^\top$, $\beta := (\alpha_1, \dots, \alpha_k)^\top$ and ADF regression is

$$X_t = \alpha X_{t-1} + \beta^\top Z_{t,k} + u_{t,k},$$

with $u_{t,k} = u_t^* + \sum_{i=k+1}^{\infty} \alpha_i \Delta X_{t-i} = u_t^* + \sum_{i=k+1}^{\infty} \alpha_i e_{t-i}$.

The proof for ADF_t is as follows. Under the null hypothesis $\rho = 1$, by using Lemma A4 in Cavaliere and Xu (2014), we can conclude that (a) $T^{-1} \sum_{t=1}^T X_{t-1} u_{t,k} = \psi(1) T^{-1} \sum_{t=1}^T w_{t-1}^* u_t^* o_p(1)$, (b) $T^{-2} \sum_{t=1}^T X_{t-1}^2 = \psi(1)^2 T^{-2} \sum_{t=1}^T (w_{t-1}^*)^2 + o_p(1)$ and (c) $T^{-1} \sum_{t=1}^T u_{t,k}^2 = T^{-1} \sum_{t=1}^T (u_t^*)^2 + o_p(1)$.

By using Lemma A5 in Cavaliere and Xu (2014) and Lemma 3 in Cavaliere and Taylor (2007), we can also conclude that as $T \rightarrow \infty$ (a) $\|(T^{-1} \sum_{t=1}^T Z_{t,k} Z_{t,k}^\top)^{-1}\| = O_p(1)$; (b) $\|\sum_{t=1}^T Z_{t,k} X_{t-1}\| = O_p(Tk^{1/2})$ and (c) $\|\sum_{t=1}^T Z_{t,k} u_{t,k}\| = O_p(1)$.

Furthermore, by Lemma 2 in Cavaliere and Taylor (2007), we can conclude that $T^{-1} \sum_{t=1}^T (u_t^*)^2 \xrightarrow{p} \bar{\omega}^2$. Then we can proceed as in Chang and Park (2002) to prove that

$$ADF_t = \bar{\omega}^{-1} \left(T^{-2} \sum_{t=1}^T (w_{t-1}^*)^2 \right)^{-1/2} \left(T^{-1} \sum_{t=1}^T w_{t-1}^* v_t^* \right) + o_p(1)$$

Then Theorem 1 and the continuous mapping theorem (CMT) implies

$$T^{-1} \sum_{t=1}^T w_{t-1}^* v_t^* = \frac{1}{2T} w_T^{*2} - \frac{1}{2T} v_T^{*2} \xrightarrow{w} \frac{\bar{\omega}^2}{2} \quad \text{and} \quad T^{-2} \sum_{t=1}^T w_{t-1}^{*2} \xrightarrow{w} \bar{\omega}^2 \int_0^1 B_\eta^\epsilon \bar{c}(s)^2 ds, (B_\eta^\epsilon, \bar{c}(1)^2 - 1),$$

which completes the proof for ADF_t . Following a similar manner,

$$ADF_\alpha = \frac{T(\hat{\alpha} - 1)}{\hat{\alpha}(1)} = \left(T^{-2} \sum_{t=1}^T w_{t-1}^{*2} \right)^{-1} \left(T^{-1} \sum_{t=1}^T w_{t-1}^* v_t^* \right) + o_p(1).$$

Under Lemma A5 of Cavaliere and Xu (2014) and Theorem 1 of Cavaliere and Taylor (2007), we can conclude that $\hat{\alpha}(1) \xrightarrow{p} 1/\psi(1)$. This completes the proof of Part (i), consistency of s_{AR}^2 , and ADF_α . $\square$

Proof of Lemma 2. $\hat{\eta}(s)$ and $\eta(s)$ are both continuous, bounded between $[0, 1]$ and monotone for all $s \in [0, 1]$, we only need to show the pointwise convergence as in Cavaliere and Taylor (2007). Consider

$$\hat{\eta}(s) = \frac{\sum_{t=1}^{\lfloor sT \rfloor} \epsilon_t^2 + (sT - \lfloor sT \rfloor) \epsilon_{\lfloor sT \rfloor}^2}{\sum_{t=1}^T \epsilon_t^2},$$

since $\frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} (\Delta \hat{x}_t)^2 = \sum_{t=1}^{\lfloor sT \rfloor} \epsilon_t^2 + o_p(1)$. Then we can write,

$$\sup_{s \in [0, 1]} \left| \hat{\eta}(s) - \frac{\sum_{t=1}^{\lfloor sT \rfloor} \epsilon_t^2}{\sum_{t=1}^T \epsilon_t^2} \right| \leq \left(\frac{1}{T} \sum_{t=1}^T \epsilon_t^2 \right)^{-1} \max_{1 \leq t \leq T} \frac{\epsilon_t^2}{T}.$$

But $\max_{1 \leq t \leq T} \frac{\epsilon_t^2}{T} = \left(\max_{1 \leq t \leq T} \frac{\epsilon_t^2}{T^{1/2}} \right)^2 \xrightarrow{p} 0$ from Theorem 1. For the first factor, notice that $\sum_{t=1}^{\lfloor sT \rfloor} \epsilon_t^2 = \sum_{t=1}^{\lfloor sT \rfloor} (\psi(L)u_t + \xi_t - \bar{\xi}_t)^2$. Next, we define $r_t = \xi_t - \bar{\xi}_t$ and $u_t^* = \psi(L)u_t$ then,

$$\frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} \epsilon_t^2 = \frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} (u_t^* + r_t)^2 = \frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} (u_t^*)^2 + \frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} (r_t)^2 + \frac{2}{T} \sum_{t=1}^{\lfloor sT \rfloor} u_t^* r_t.$$

From Cavaliere and Taylor (2007) $\frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} (u_t^*)^2 \xrightarrow{w} \kappa \eta(s)$ where $\kappa = \omega^2 \psi(1)$. However, $\frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} r_t^2 + u_t^* r_t = o_p(1)$ since:

$$\begin{aligned} \left| \frac{1}{T} \left(\sum_{t=1}^{\lfloor sT \rfloor} r_t^2 + u_t^* r_t \right) \right| &\leq \frac{1}{T} \sum_{t=1}^{\lfloor sT \rfloor} |r_t^2 + u_t^* r_t| \\ &\leq \frac{1}{T} [\max |u_t^*| + \max |r_t|] \sum_{t=1}^T 2|r_t| = o_p(1), \end{aligned}$$

from Assumption 1 we have $\max |u_t|, \max |r_t|$ are $o_p(T^{-1/2})$ and $\sum_{t=1}^T 2|r_t| = o_p(T^{1/2})$. Since $\frac{1}{T} \sum_{t=1}^T (u_t^*)^2 \xrightarrow{w} \kappa \eta(1) = \kappa$, the $\hat{\eta}(s) \xrightarrow{w} \frac{\kappa \eta(s)}{\kappa} = \eta(s)$. $\square$

Proof of Theorem 3. We need to first show the difference process $\Delta B_{\hat{\eta}, T}(t/T)$ is identical to $dB_{\eta}(s)$ in the limit. In order to show this, we will use a similar structure as in Cavaliere and Taylor (2007). First, we define the usual partial sum process $S_T(s) = \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor Ts \rfloor} e_t$. Then $B_{\hat{\eta}, T}(s) = S_T(\hat{\eta}(\lfloor sT \rfloor/T))$. Cavaliere and Taylor (2007) already showed that $S_T(\hat{\eta}(\lfloor sT \rfloor/T)) \xrightarrow{w} B(\eta(s)) = B_{\eta}(s)$. Now we can write $\Delta B_{\hat{\eta}, T}(t/T) = S_T(\hat{\eta}(t/T)) - S_T(\hat{\eta}(t/T - 1/T))$. Next, we define $h = 1/T$

$$\begin{aligned} \lim_{T \rightarrow \infty, h \rightarrow 0} \Delta B_{\hat{\eta}, T}(t/T) &= S_T(\hat{\eta}(s)) - S_T(\hat{\eta}(s - h)) = dS_T(\hat{\eta}(s)), \\ &\xrightarrow{w} dB(\eta(s)) = dB_{\eta}(s). \end{aligned}$$

The last line follows from the joint convergence of $(S_T(\cdot), \hat{\eta}(\cdot))$ to $(B(\cdot), \eta(t))$ which is shown in Cavaliere and Taylor (2007). The rest of the proof is the application of the CMT and the conjuncture which is described in Theorem 1. Simply replacing the increments of $S_T^*(s)$ in the proof of Theorem 1 of Cavaliere (2005) with $\Delta B_{\hat{\eta}, T}(t)$, we obtain

$$B_{\hat{\eta}}^{\bar{c}}, \bar{c}(s) = B_{\eta}(s) + L(s) - Q(s),$$

where $L(s)$ and $Q(s)$ are the continuous regulators. $\square$

Appendix B: tables for the simulation results

Table B1. Finite sample size performance of ADF_a and MZ_a tests for i.i.d. shocks.

		ADF_a^s	ADF_a^{nv}	ADF_a^b	ADF_a^{bnv}	MZ_a^s	MZ_a^{nv}	MZ_a^b	MZ_a^{bnv}
Volatility type	c	T = 250							
CV	∞	0.05	0.05	0.05	0.05	0.04	0.04	0.05	0.05
	0.8	0.11	0.10	0.05	0.05	0.10	0.09	0.05	0.05
	0.6	0.17	0.16	0.05	0.05	0.16	0.15	0.04	0.04
	0.4	0.32	0.27	0.04	0.04	0.30	0.25	0.03	0.04
SBV	∞	0.16	0.05	0.16	0.05	0.15	0.04	0.15	0.05
	0.8	0.29	0.12	0.19	0.04	0.28	0.10	0.18	0.04
	0.6	0.32	0.14	0.14	0.04	0.30	0.12	0.12	0.04
	0.4	0.48	0.22	0.10	0.04	0.46	0.20	0.09	0.04
TV	∞	0.08	0.05	0.08	0.05	0.07	0.04	0.07	0.05
	0.8	0.19	0.12	0.10	0.05	0.17	0.11	0.08	0.04
	0.6	0.23	0.16	0.07	0.04	0.22	0.15	0.08	0.04
	0.4	0.35	0.23	0.07	0.04	0.32	0.21	0.07	0.04
EISV	∞	0.17	0.04	0.17	0.04	0.16	0.03	0.16	0.04
	0.8	0.28	0.06	0.21	0.04	0.27	0.05	0.19	0.04
	0.6	0.41	0.10	0.22	0.04	0.40	0.08	0.20	0.04
	0.4	0.60	0.21	0.26	0.04	0.58	0.18	0.24	0.03
		T = 500							
CV	∞	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.8	0.11	0.11	0.05	0.05	0.10	0.11	0.05	0.05
	0.6	0.19	0.18	0.04	0.04	0.18	0.18	0.04	0.04
	0.4	0.32	0.30	0.04	0.05	0.31	0.29	0.04	0.04
SBV	∞	0.16	0.05	0.16	0.05	0.15	0.05	0.15	0.05
	0.8	0.30	0.12	0.18	0.05	0.29	0.11	0.17	0.04
	0.6	0.33	0.15	0.14	0.04	0.32	0.14	0.13	0.04
	0.4	0.53	0.23	0.10	0.04	0.52	0.22	0.09	0.04
TV	∞	0.08	0.05	0.07	0.05	0.07	0.04	0.07	0.04
	0.8	0.18	0.12	0.09	0.05	0.17	0.11	0.08	0.04
	0.6	0.23	0.17	0.08	0.04	0.22	0.16	0.07	0.04
	0.4	0.36	0.27	0.08	0.04	0.35	0.26	0.07	0.04
EISV	∞	0.21	0.04	0.20	0.04	0.17	0.03	0.17	0.04
	0.8	0.28	0.08	0.22	0.04	0.31	0.05	0.23	0.04
	0.6	0.46	0.11	0.26	0.04	0.43	0.08	0.23	0.04
	0.4	0.64	0.23	0.29	0.04	0.62	0.18	0.26	0.04

Notes: (i) Nominal 5% asymptotic level. (ii) Tests based on OLS de-meaned data. (iii) Tests with superscripts s, nv, b, bnv are for standard, non-stationary volatility robust, bounds robust, and both bounds and non-stationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table B2. Finite sample size performance of ADF_a and MZ_a tests for MA(1) shocks.

		ADF_a^s	ADF_a^{nv}	ADF_a^b	ADF_a^{bnv}	MZ_a^s	MZ_a^{nv}	MZ_a^b	MZ_a^{bnv}
Volatility type		T = 250							
CV	∞	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.05
	0.8	0.09	0.08	0.04	0.04	0.08	0.09	0.05	0.05
	0.6	0.14	0.13	0.05	0.05	0.13	0.13	0.05	0.05
	0.4	0.21	0.20	0.05	0.05	0.19	0.21	0.05	0.05
SBV	∞	0.17	0.04	0.13	0.04	0.12	0.04	0.10	0.05
	0.8	0.23	0.06	0.13	0.04	0.18	0.08	0.11	0.04
	0.6	0.23	0.09	0.10	0.04	0.18	0.11	0.08	0.05
	0.4	0.25	0.11	0.06	0.04	0.19	0.13	0.06	0.04
TV	∞	0.08	0.04	0.07	0.04	0.07	0.04	0.07	0.05
	0.8	0.15	0.08	0.08	0.04	0.13	0.09	0.09	0.05
	0.6	0.18	0.11	0.09	0.05	0.15	0.11	0.10	0.05
	0.4	0.23	0.14	0.10	0.05	0.20	0.15	0.10	0.05
EISV	∞	0.09	0.07	0.08	0.04	0.06	0.04	0.07	0.05
	0.8	0.12	0.07	0.12	0.03	0.09	0.07	0.11	0.04
	0.6	0.14	0.09	0.13	0.03	0.11	0.08	0.13	0.04
	0.4	0.15	0.11	0.24	0.03	0.11	0.12	0.22	0.04
		T = 500							
CV	∞	0.05	0.05	0.05	0.04	0.05	0.05	0.05	0.05
	0.8	0.10	0.09	0.05	0.05	0.09	0.09	0.05	0.05
	0.6	0.15	0.15	0.06	0.05	0.14	0.15	0.05	0.05
	0.4	0.21	0.22	0.05	0.05	0.20	0.23	0.06	0.05
SBV	∞	0.17	0.04	0.14	0.04	0.15	0.04	0.13	0.04
	0.8	0.26	0.07	0.13	0.04	0.23	0.07	0.12	0.04
	0.6	0.25	0.09	0.12	0.04	0.22	0.08	0.10	0.04
	0.4	0.31	0.12	0.08	0.04	0.26	0.09	0.07	0.05
TV	∞	0.07	0.04	0.07	0.04	0.06	0.04	0.07	0.04
	0.8	0.15	0.10	0.09	0.05	0.14	0.10	0.09	0.05
	0.6	0.19	0.13	0.08	0.05	0.18	0.12	0.07	0.05
	0.4	0.26	0.19	0.06	0.05	0.24	0.18	0.07	0.05
EISV	∞	0.09	0.04	0.08	0.04	0.08	0.04	0.08	0.04
	0.8	0.13	0.06	0.09	0.04	0.11	0.07	0.11	0.05
	0.6	0.16	0.08	0.13	0.04	0.15	0.09	0.12	0.05
	0.4	0.39	0.14	0.26	0.04	0.32	0.15	0.24	0.03

Notes: (i) Nominal 5% asymptotic level. (ii) Tests based on OLS de-meaned data. (iii) Tests with superscripts s, nv, b, bnv are for standard, non-stationary volatility robust, bounds robust, and both bounds and non-stationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table B3. Finite sample size performance of ADF_a and MZ_a tests for AR(1) shocks.

		ADF_a^s	ADF_a^{nv}	ADF_a^b	ADF_a^{bnv}	MZ_a^s	MZ_a^{nv}	MZ_a^b	MZ_a^{bnv}
Volatility type		T = 250							
CV	∞	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
	0.8	0.08	0.08	0.04	0.05	0.07	0.08	0.04	0.05
	0.6	0.12	0.12	0.05	0.05	0.11	0.12	0.05	0.05
	0.4	0.19	0.17	0.05	0.05	0.18	0.18	0.05	0.05
SBV	∞	0.16	0.04	0.13	0.04	0.12	0.05	0.10	0.05
	0.8	0.23	0.08	0.15	0.04	0.19	0.08	0.13	0.05
	0.6	0.23	0.10	0.11	0.04	0.19	0.08	0.09	0.04
	0.4	0.24	0.12	0.08	0.03	0.19	0.09	0.06	0.04
TV	∞	0.07	0.04	0.07	0.05	0.06	0.04	0.06	0.05
	0.8	0.13	0.09	0.09	0.05	0.11	0.10	0.09	0.05
	0.6	0.15	0.11	0.07	0.05	0.14	0.12	0.07	0.05
	0.4	0.19	0.14	0.08	0.05	0.18	0.15	0.07	0.05
EISV	∞	0.39	0.04	0.26	0.03	0.24	0.04	0.16	0.04
	0.8	0.49	0.08	0.30	0.04	0.33	0.09	0.20	0.04
	0.6	0.50	0.10	0.29	0.03	0.35	0.12	0.20	0.04
	0.4	0.48	0.12	0.24	0.03	0.34	0.14	0.16	0.03
		T = 500							
CV	∞	0.05	0.05	0.05	0.05	0.04	0.05	0.05	0.05
	0.8	0.09	0.09	0.05	0.05	0.09	0.09	0.05	0.05
	0.6	0.14	0.14	0.05	0.05	0.13	0.14	0.05	0.05
	0.4	0.20	0.21	0.05	0.05	0.19	0.21	0.05	0.05
SBV	∞	0.16	0.04	0.13	0.03	0.14	0.04	0.12	0.04
	0.8	0.26	0.07	0.15	0.04	0.23	0.08	0.14	0.04
	0.6	0.26	0.08	0.12	0.04	0.24	0.09	0.11	0.05
	0.4	0.29	0.09	0.08	0.04	0.25	0.10	0.07	0.04
TV	∞	0.08	0.05	0.07	0.05	0.07	0.05	0.07	0.05
	0.8	0.14	0.09	0.09	0.05	0.14	0.09	0.09	0.05
	0.6	0.18	0.12	0.08	0.05	0.18	0.13	0.08	0.05
	0.4	0.22	0.16	0.06	0.05	0.22	0.17	0.06	0.05
EISV	∞	0.38	0.04	0.28	0.04	0.24	0.04	0.18	0.05
	0.8	0.47	0.08	0.32	0.05	0.32	0.09	0.24	0.05
	0.6	0.51	0.10	0.26	0.05	0.37	0.12	0.25	0.05
	0.4	0.47	0.12	0.23	0.05	0.33	0.14	0.18	0.05

Notes: (i) Nominal 5% asymptotic level. (ii) Tests based on OLS de-meaned data. (iii) Tests with superscripts s, nv, b, bnv are for standard, non-stationary volatility robust, bounds robust, and both bounds and non-stationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table B4. Finite sample size performance of ADF_a and MZ_a tests for ARMA(1,1) shocks.

		ADF_a^s	ADF_a^{nv}	ADF_a^b	$ADF_a^{b,nv}$	MZ_a^s	MZ_a^{nv}	MZ_a^b	$MZ_a^{b,nv}$
Volatility type		T = 250							
CV	∞	0.04	0.05	0.05	0.05	0.03	0.05	0.05	0.05
	0.8	0.09	0.09	0.05	0.05	0.08	0.10	0.05	0.05
	0.6	0.14	0.16	0.06	0.05	0.12	0.16	0.06	0.05
	0.4	0.24	0.26	0.06	0.05	0.22	0.27	0.06	0.05
SBV	∞	0.16	0.03	0.13	0.03	0.12	0.04	0.11	0.04
	0.8	0.23	0.06	0.13	0.03	0.18	0.07	0.11	0.04
	0.6	0.24	0.08	0.11	0.04	0.18	0.07	0.09	0.04
	0.4	0.26	0.10	0.07	0.04	0.20	0.09	0.07	0.05
TV	∞	0.07	0.05	0.07	0.05	0.06	0.05	0.07	0.05
	0.8	0.14	0.10	0.09	0.05	0.12	0.11	0.09	0.05
	0.6	0.18	0.14	0.08	0.05	0.16	0.14	0.08	0.05
	0.4	0.26	0.18	0.06	0.05	0.23	0.18	0.06	0.06
EISV	∞	0.29	0.03	0.21	0.04	0.22	0.03	0.16	0.04
	0.8	0.39	0.08	0.25	0.04	0.30	0.08	0.20	0.04
	0.6	0.42	0.09	0.22	0.04	0.32	0.11	0.17	0.03
	0.4	0.47	0.10	0.22	0.04	0.34	0.12	0.17	0.03
		T = 500							
CV	∞	0.04	0.05	0.05	0.05	0.04	0.05	0.05	0.05
	0.8	0.10	0.11	0.06	0.05	0.09	0.11	0.06	0.05
	0.6	0.17	0.17	0.06	0.05	0.16	0.17	0.06	0.05
	0.4	0.26	0.26	0.05	0.05	0.25	0.27	0.05	0.05
SBV	∞	0.16	0.03	0.14	0.03	0.14	0.04	0.12	0.04
	0.8	0.24	0.06	0.14	0.04	0.22	0.07	0.13	0.04
	0.6	0.26	0.07	0.10	0.04	0.23	0.08	0.10	0.04
	0.4	0.32	0.09	0.07	0.04	0.26	0.11	0.06	0.05
TV	∞	0.07	0.05	0.08	0.05	0.06	0.05	0.07	0.05
	0.8	0.15	0.10	0.09	0.05	0.14	0.11	0.10	0.05
	0.6	0.19	0.15	0.08	0.05	0.18	0.15	0.08	0.05
	0.4	0.29	0.20	0.06	0.05	0.27	0.20	0.06	0.05
EISV	∞	0.17	0.03	0.13	0.03	0.18	0.03	0.14	0.03
	0.8	0.25	0.06	0.13	0.03	0.25	0.07	0.15	0.03
	0.6	0.35	0.08	0.15	0.03	0.34	0.09	0.17	0.04
	0.4	0.61	0.14	0.22	0.04	0.56	0.14	0.23	0.03

Notes: (i) Nominal 5% asymptotic level. (ii) Tests based on OLS de-meanned data. (iii) Tests with superscripts s, nv, b, b, nv are for standard, non-stationary volatility robust, bounds robust, and both bounds and non-stationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table B5. Finite sample power performance of ADF_a and MZ_a tests for i.i.d. shocks.

		$ADF_a^{b,nv}$				$MZ_a^{b,nv}$			
		T = 250		T = 500		T = 250		T = 500	
Volatility type	$c \backslash \rho$	0.93	0.86	0.93	0.86	0.93	0.86	0.93	0.86
CV	∞	0.87	1.00	1.00	1.00	0.86	1.00	1.00	1.00
	0.8	0.71	1.00	1.00	1.00	0.68	1.00	1.00	1.00
	0.6	0.49	0.99	0.99	1.00	0.43	0.99	0.99	1.00
	0.4	0.19	0.91	0.90	1.00	0.14	0.86	0.88	1.00
SBV	∞	0.55	0.97	0.97	1.00	0.52	0.96	0.96	1.00
	0.8	0.31	0.87	0.86	1.00	0.26	0.83	0.84	1.00
	0.6	0.20	0.75	0.77	1.00	0.15	0.67	0.74	1.00
	0.4	0.16	0.73	0.75	1.00	0.09	0.60	0.70	1.00
TV	∞	0.77	1.00	1.00	1.00	0.74	1.00	1.00	1.00
	0.8	0.49	0.99	0.99	1.00	0.44	0.99	0.98	1.00
	0.6	0.31	0.95	0.95	1.00	0.26	0.93	0.94	1.00
	0.4	0.16	0.84	0.83	1.00	0.11	0.76	0.79	1.00
EISV	∞	0.33	0.78	0.31	0.61	0.29	0.74	0.27	0.56
	0.8	0.30	0.75	0.34	0.65	0.26	0.70	0.29	0.60
	0.6	0.27	0.66	0.38	0.70	0.21	0.57	0.32	0.64
	0.4	0.17	0.56	0.62	0.81	0.10	0.40	0.52	0.72

Notes: (i) Tests based on OLS de-meanned data. (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter. (iv) Tests with superscript b, nv are for both bounds and non-stationary volatility robust versions, respectively.

Table B6. Finite sample power performance of ADF_a and MZ_a tests for MA(1) shocks.

		ADF_a^{bmv}				MZ_a^{bmv}			
		T = 250		T = 500		T = 250		T = 500	
Volatility type	$c \backslash \rho$	0.93	0.86	0.93	0.86	0.93	0.86	0.93	0.86
CV	∞	0.65	0.94	0.99	1.00	0.65	0.90	0.98	1.00
	0.8	0.50	0.91	0.97	1.00	0.52	0.87	0.96	0.99
	0.6	0.39	0.85	0.93	1.00	0.41	0.81	0.91	0.98
	0.4	0.26	0.72	0.84	0.99	0.26	0.69	0.82	0.96
SBV	∞	0.24	0.58	0.71	0.95	0.28	0.57	0.70	0.89
	0.8	0.18	0.50	0.59	0.92	0.21	0.49	0.58	0.85
	0.6	0.18	0.50	0.56	0.91	0.22	0.48	0.55	0.84
	0.4	0.20	0.52	0.60	0.90	0.22	0.48	0.57	0.81
TV	∞	0.50	0.88	0.96	1.00	0.51	0.83	0.94	0.99
	0.8	0.33	0.79	0.89	0.99	0.37	0.76	0.87	0.97
	0.6	0.27	0.74	0.84	0.99	0.29	0.71	0.82	0.96
	0.4	0.22	0.64	0.78	0.98	0.24	0.61	0.76	0.93
EISV	∞	0.33	0.52	0.82	0.97	0.44	0.59	0.80	0.94
	0.8	0.24	0.44	0.78	0.96	0.29	0.46	0.76	0.92
	0.6	0.22	0.38	0.72	0.94	0.28	0.40	0.71	0.88
	0.4	0.15	0.30	0.68	0.91	0.17	0.29	0.65	0.85

Notes: (i) Tests based on OLS de-meanded data. (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter. (iv) Tests with superscript *bmv* are for both bounds and non-stationary volatility robust versions, respectively.

Table B7. Finite sample power performance of ADF_a and MZ_a tests for AR(1) shocks.

		ADF_a^{bmv}				MZ_a^{bmv}			
		T = 250		T = 500		T = 250		T = 500	
Volatility type	$c \backslash \rho$	0.93	0.86	0.93	0.86	0.93	0.86	0.93	0.86
CV	∞	0.72	0.94	0.99	1.00	0.72	0.91	0.98	0.99
	0.8	0.59	0.92	0.97	1.00	0.62	0.89	0.96	0.99
	0.6	0.45	0.87	0.95	1.00	0.48	0.84	0.94	0.98
	0.4	0.33	0.78	0.87	0.99	0.35	0.77	0.86	0.96
SBV	∞	0.22	0.57	0.71	0.94	0.30	0.58	0.69	0.88
	0.8	0.17	0.49	0.58	0.90	0.22	0.50	0.58	0.83
	0.6	0.15	0.46	0.54	0.88	0.20	0.48	0.54	0.82
	0.4	0.17	0.46	0.54	0.87	0.20	0.46	0.55	0.79
TV	∞	0.56	0.88	0.96	1.00	0.57	0.84	0.94	0.99
	0.8	0.41	0.81	0.91	0.99	0.45	0.79	0.89	0.97
	0.6	0.32	0.77	0.84	0.99	0.35	0.74	0.83	0.96
	0.4	0.26	0.68	0.77	0.98	0.29	0.66	0.76	0.94
EISV	∞	0.37	0.54	0.84	0.97	0.46	0.60	0.82	0.95
	0.8	0.26	0.47	0.79	0.96	0.30	0.48	0.77	0.92
	0.6	0.23	0.39	0.74	0.92	0.29	0.42	0.73	0.88
	0.4	0.17	0.32	0.68	0.90	0.19	0.31	0.67	0.86

Notes: (i) Tests based on OLS de-meanded data. (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter. (iv) Tests with superscript *bmv* are for both bounds and non-stationary volatility robust versions, respectively.

Table B8. Finite sample power performance of ADF_a and MZ_a tests for ARMA(1,1) shocks.

Volatility type	$c \backslash \rho$	ADF_a^{bmv}				MZ_a^{bmv}			
		T = 250		T = 500		T = 250		T = 500	
		0.93	0.86	0.93	0.86	0.93	0.86	0.93	0.86
CV	∞	0.76	0.96	0.99	1.00	0.77	0.92	0.99	1.00
	0.8	0.63	0.93	0.98	1.00	0.64	0.89	0.96	0.99
	0.6	0.46	0.90	0.95	1.00	0.48	0.86	0.93	0.98
	0.4	0.38	0.82	0.89	0.99	0.39	0.78	0.87	0.96
SBV	∞	0.25	0.64	0.73	0.96	0.32	0.61	0.70	0.89
	0.8	0.18	0.57	0.61	0.93	0.22	0.55	0.61	0.85
	0.6	0.17	0.55	0.61	0.93	0.23	0.53	0.59	0.84
	0.4	0.26	0.62	0.66	0.92	0.30	0.57	0.63	0.82
TV	∞	0.63	0.91	0.97	1.00	0.64	0.86	0.96	0.99
	0.8	0.40	0.84	0.92	0.99	0.43	0.79	0.89	0.97
	0.6	0.33	0.81	0.87	0.99	0.37	0.76	0.84	0.96
	0.4	0.30	0.75	0.82	0.98	0.33	0.71	0.79	0.93
EISV	∞	0.33	0.76	0.29	0.59	0.27	0.72	0.26	0.53
	0.8	0.30	0.73	0.31	0.62	0.24	0.68	0.27	0.58
	0.6	0.26	0.64	0.35	0.68	0.20	0.56	0.30	0.63
	0.4	0.15	0.53	0.60	0.78	0.09	0.36	0.50	0.71

Notes: (i) Tests based on OLS de-measured data. (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter. (iv) Tests with superscript *bmv* are for both bounds and non-stationary volatility robust versions, respectively.

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