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ABSTRACT

Commutative d-torsion K-theory is a variant of topological K-theory constructed from commuting unitary matrices of order dividing d. Such matrices appear as solutions of linear constraint systems that play a role in the study of quantum contextuality and in applications to operator-theoretic problems motivated by quantum information theory. Using methods from stable homotopy theory, we modify commutative d-torsion K-theory into a cohomology theory that can be used for studying operator solutions of linear constraint systems. This provides an interesting connection between stable homotopy theory and quantum information theory.

I. INTRODUCTION

Commuting unitary matrices can be assembled into a generalized cohomology theory called commutative K-theory, a variant of topological K-theory first introduced in Ref. 1. This theory can be further modified by restricting to matrices whose order divides d. The resulting cohomology theory will be referred to as *commutative d-torsion K-theory*. Such matrices also play a significant role in quantum theory, especially in foundational areas concerning quantum contextuality[2,3] and linear constraint systems in the study of non-local games.[4] The goal of this paper is to make this connection precise. We construct another generalized cohomology theory, called the $C(d,m)$-cohomology, obtained from commutative d-torsion K-theory, which is tailored for studying operator solutions of linear constraint systems. We expect that stable homotopical methods introduced in this paper will provide further insight into operator-theoretic problems motivated by quantum information theory.

The cohomology theories studied in this paper are based on a classifying space construction introduced in Ref. 5. We write $B(\mathbb{Z}/d,G)$ to denote the classifying space of a topological group G constructed from tuples of pairwise commuting group elements where each group element has order dividing d, i.e., pairwise commuting d-torsion group elements. When G is the unitary group $U(m)$, this classifying space is constructed from tuples $(A_1,A_2,\dots,A_n)$ of matrices satisfying

$$A_iA_j = A_jA_i \quad \text{and} \quad (A_i)^d = I_m,$$

where I_m is the $m\times m$ identity matrix. Such matrices also appear as solutions to a linear constraint system specified by an equation $Mx=b$, where M is an $r\times c$ matrix over the additive group $\mathbb{Z}/d$ of integers modulo d. An operator solution consists of d-torsion $m\times m$-unitary matrices $A_1,A_2,\dots,A_c$ that satisfy

$$A_1^{M_{k1}}A_2^{M_{k2}}\dots A_c^{M_{kc}} = e^{2\pi i b_k/d}I_m \qquad \text{for all} \quad 1\le k\le r,$$

and $A_iA_j=A_jA_i$ whenever M_{ki} and M_{kj} are both non-zero. The data of a linear constraint system can be packaged as a pair $(\mathfrak{H},\tau)$, where $\mathfrak{H}$ is a hypergraph with a vertex set $V=\{v_1,v_2,\dots,v_c\}$ and an edge set $E=\{e_1,e_2,\dots,e_r\}$. Here, τ is the function $E\to\mathbb{Z}/d$ defined by $\tau(e_k)=b_k$. An operator solution $\{A_i\}$ can be regarded as a function $T:V\to U(m)$ where $T(v_i)=A_i$. The homotopical approach initiated in Refs. 6 and 7 associates a two-dimensional cell complex (i.e., CW complex) X, called a topological realization, to the hypergraph $\mathfrak{H}$ and the function τ represents a two-dimensional cohomology class $[\tau]\in H^2(X,\mathbb{Z}/d)$.

In this paper, we refine this approach by interpreting an operator solution as a map of topological spaces. For this, a quotient space $\bar{B}(\mathbb{Z}/d, G)$ of the classifying space $B(\mathbb{Z}/d, G)$ is introduced. An operator solution over $G = U(m)$ can be turned into a map, defined up to homotopy,

$$f_T : X \to \bar{B}(\mathbb{Z}/d, G).$$

Although our motivation comes from an urge to understand operator solutions of linear constraint systems, the classifying space $B(\mathbb{Z}/d, G)$ and its variants are of independent interest to algebraic topologists; see, for instance, Refs. 8–15.

A generalized cohomology theory is represented by a spectrum. Following Ref. 16, we show that $B(\mathbb{Z}/d, U)$, where U is the stable unitary group, is an infinite loop space and thus specifies a spectrum. This spectrum turns out to be stably equivalent to $ku \wedge B\mu_d$ (Proposition II.7), where $\mu_d = \{e^{2\pi i k/d} | 1 \le k \le d\}$ and ku is the connective complex K-theory spectrum. Commutative d-torsion K-theory is the generalized cohomology theory associated with this spectrum. Both the spectrum and the associated cohomology theory will be denoted by $k\mu_d$.

For applications to linear constraint systems, we introduce a stabilized version of the quotient space $\bar{B}(\mathbb{Z}/d, U(m))$. The usual stabilization process cannot be carried out in a straightforward manner. However, by working in the homotopy category of spectra, we construct a spectrum, denoted by $C(d, m)$, obtained from $k\mu_d$. It turns out that the resulting cohomology theory is represented by a product of two Eilenberg–MacLane spaces and thus can be constructed more directly. The infinite loop space $\bar{B}(d, m)$ associated with the spectrum $C(d, m)$ admits a map

$$\bar{\imath}_m : \bar{B}(\mathbb{Z}/d, U(m)) \to \bar{B}(d, m).$$

This space comes with a canonical cohomology class $\gamma_m^{\mathbb{S}}$ in $H^2(\bar{B}(d, m), \mathbb{Z}/d)$. By construction, homotopy groups of $C(d, m)$ are concentrated in dimensions $i = 1, 2$ and we show that there is an exact sequence

$$0 \to \pi_2 C(d, m) \to \mathbb{Z}/d \xrightarrow{\times m} \mathbb{Z}/d \to \pi_1 C(d, m) \to 0.$$

The kernel consists of the subgroup $(\mathbb{Z}/d)_m$ of m-torsion elements. Using the Atiyah–Hirzebruch spectral sequence, we describe $C(d, m)$-cohomology of a space.

Theorem III.5. *Let X be a connected CW complex. There is a commutative diagram*

$$\begin{tikzcd}
 & H^2(X, (\mathbb{Z}/d)_m) \arrow[r, "(i_m)_*"] \arrow[d, hook] & H^2(X, \mathbb{Z}/d) \arrow[d, equal] \\
k\mu_d(X) \arrow[r, "\zeta"] \arrow[d] & C(d, m)(X) \arrow[r, "cl"] \arrow[d, two heads] & H^2(X, \mathbb{Z}/d) \\
H^1(X, \mathbb{Z}/d) \arrow[r, "(\pi_m)_*"] & H^1(X, \frac{\mathbb{Z}/d}{m\mathbb{Z}/d}), &
\end{tikzcd} \tag{1.1}$$

where $\mathrm{cl}(f) = f^*(\gamma_m^{\mathbb{S}})$, *the image of* ζ *is contained in the kernel of* cl, *and the middle column is an exact sequence that (non-canonically) splits as follows:*

$$C(d, m)(X) \cong H^1\left(X, \frac{\mathbb{Z}/d}{m\mathbb{Z}/d}\right) \oplus H^2(X, (\mathbb{Z}/d)_m).$$

Going back to linear constraint systems, we show that the $C(d, m)$-cohomology informs us about the properties of operator solutions over $U(m)$. To an operator solution, we associate the class $[f]$ of the composite map

$$f : X \xrightarrow{f_T} \bar{B}(\mathbb{Z}/d, U(m)) \xrightarrow{\bar{\imath}_m} \bar{B}(d, m)$$

in the $C(d, m)$-cohomology of X. It turns out that $\mathrm{cl}(f) = 0$ if and only if the linear constraint system has a solution over $U(1)$, also known as a scalar solution.

Corollary IV.12. Let $(\mathfrak{H}, \tau)$ *be a linear constraint system over* $\mathbb{Z}/d$ *and X be a topological realization for* $\mathfrak{H}$.

1. *If* $(\mathfrak{H}, \tau)$ *has an operator solution and* $H^2(X, (\mathbb{Z}/d)_m) = 0$, *then* $(\mathfrak{H}, \tau)$ *has a scalar solution.*
2. *If d and m are coprime, then* $C(d, m)(X) = 0$. *In particular,* $(\mathfrak{H}, \tau)$ *has a scalar solution if it has an operator solution over* $U(m)$.
3. *If* $\pi_1(X)$ *is trivial and* $[\tau] \neq 0$, *then* $(\mathfrak{H}, \tau)$ *does not have an operator solution over* $U(m)$ *for any* $m \ge 2$.

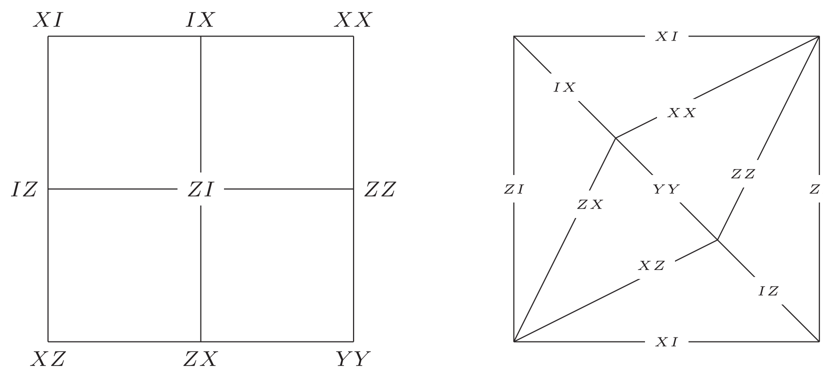

FIG. 1. (Left figure) $\mathfrak{H}_{\text{sq}}$ consists of nine vertices and six edges each consisting of three vertices in each row and column. All the incidence weights are equal to 1. The operator solution is given by the tensor product of two Pauli matrices, where the notation is simplified by omitting $\otimes$. The function τ_{sq} takes the value 0 for each hyperedge except the right-most column. (Right figure) A topological realization given by a torus together with a cell structure consisting of triangles. The operators are placed on the edges, and each triangle corresponds to a hyperedge. The cocycle τ_{sq} assigns 0 to each triangle except $\{XX, YY, ZZ\}$, which is assigned 1.

The most famous example of a linear constraint system, which does not admit a scalar solution, is the Mermin square construction.[17] This linear constraint system, defined over $\mathbb{Z}/2$, admits an operator solution in $U(2^n)$ for $n \geq 2$. A topological realization for the Mermin square linear constraint system can be chosen to be a torus $X = S^1 \times S^1$ with a certain cell structure (Fig. 1). Then, an operator solution specifies a class in the $C(2, 2^n)$-cohomology of the torus

$$M_n \in C(2, 2^n)(S^1 \times S^1).$$

We refer to this class as the Mermin class. In addition, we show that the Mermin class can also be identified with the generator of $\pi_2 C(2, 2^n) = \mathbb{Z}/2$. There is also a real version of these constructions that works for the orthogonal group $O(m)$. In this case, certain generalized cohomology classes can be realized as symmetry-protected topological (SPT) phases (Sec. IV F).

This paper is organized as follows: In Sec. II, we introduce the classifying space $B(\mathbb{Z}/d, G)$ and the type of principal bundles classified by this space. Γ-spaces are used to describe the spectrum $k\mu_d$, and Proposition II.7 informs us about its stable homotopy type. Low dimensional homotopy groups are described in Sec. II E. The quotient space $\bar{B}(\mathbb{Z}/d, G)$ and the spectrum $C(d, m)$ are introduced in Sec. III. In this section, we prove Theorem III.5, which describes the $C(d, m)$-cohomology of a space. Applications of $C(d, m)$-cohomology are discussed in Sec. IV, where we introduce linear constraint systems and a topological interpretation of operator solutions. Proposition IV.9 provides a computation of pointed homotopy classes of maps $X \to \bar{B}(\mathbb{Z}/d, G)$ when X is a two-dimensional CW complex. Applications to linear constraint systems are given in Corollary IV.12. The Mermin class is constructed in this section. In the Appendix, we introduce the real versions of these spectra obtained from the orthogonal group.

II. COMMUTATIVE *d*-TORSION *K*-THEORY

In this section, we introduce a new generalized cohomology theory obtained as a variant of commutative K-theory introduced in Refs. 1 and 9. Commutative K-theory has nice properties such as the spectrum ku_{com} representing the theory is stably equivalent to $ku \wedge \mathbb{C}P^\infty$ as proved in Ref. 18, where ku is the connective complex K-theory spectrum. For the d-torsion case, the spectrum representing the cohomology theory is denoted by $k\mu_d$. It is constructed from commuting unitary matrices whose eigenvalues belong to $\mu_d = \{e^{2\pi i k/d} | 1 \leq k \leq d\}$. To study this spectrum, we follow the Γ-space approach of Ref. 16. This description allows us to prove that $k\mu_d$ is stably equivalent to $ku \wedge B\mu_d$. There is also a real version ko_{sym} constructed from commuting symmetric orthogonal matrices. We describe low dimensional homotopy groups of these spectra.

A. Classifying spaces

Let G be a topological group (locally compact and Hausdorff with a non-degenerate base point $1_G \in G$). An element $g \in G$ is said to be *d-torsion* if g^d is the identity element 1_G. We are interested in a space constructed from pairwise commuting d-torsion group elements.

Definition II.1. We define $B(\mathbb{Z}/d, G)$ to be the geometric realization of the simplicial space

$$[n] \mapsto \operatorname{Hom}((\mathbb{Z}/d)^n, G),$$

where $\operatorname{Hom}((\mathbb{Z}/d)^n, G)$ is the subspace of G^n consisting of pairwise commuting n-tuples $(g_1, g_2, \ldots, g_n)$ such that $g_i^d = 1_G$ for all $1 \leq i \leq n$. The simplicial structure is given by

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$$d_i(g_1,g_2,\ldots,g_n)=\begin{cases}(g_2,\ldots,g_n), & i=0,\\ (g_1,\ldots,g_ig_{i+1},\ldots,g_n), & 0<i<n,\\ (g_1,g_2,\ldots,g_{n-1}), & i=n\end{cases}$$

and $s_j(g_1,g_2,\ldots,g_n)=(g_1,\ldots,g_j,1_G,g_{j+1},\ldots,g_n)$ for $0\le j\le n$.

Remark II.2. In general, for any cosimplicial group $\pi^{\cdot}$, there is a classifying space $B(\pi,G)$ obtained by a similar construction; see Ref. 15. When $\pi^{\cdot}$ is the level-wise free cosimplicial group $F^{\cdot}$, then this construction gives the usual classifying space BG. If the level-wise Abelianization $\mathbb{Z}^{\cdot}$ is used, then the resulting space is the *classifying space for commutativity* $B(\mathbb{Z},G)$.[9] Mod-d reduction in each level gives a cosimplicial group $(\mathbb{Z}/d)^{\cdot}$, and we recover the construction given in Definition II.1.

B. Stabilization

Let $\mathbb{C}^m$ denote the complex vector space of dimension m with a canonical basis $\{e_1,e_2,\ldots,e_m\}$. Inclusion of the canonical basis vectors induces a map $\mathbb{C}^m\to\mathbb{C}^{m+1}$, and the union (colimit) along these inclusions is denoted by $\mathbb{C}^\infty$. Let $U(m)$ denote the unitary group of $m\times m$ matrices. The stable unitary group U is the union along the inclusions

$$U(m)\to U(m+1),\quad A\mapsto\begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix}. \tag{2.1}$$

We write $B(\mathbb{Z}/d,U)$ for the union of $B(\mathbb{Z}/d,U(m))$ along the induced stabilization maps.

C. Spectra and Γ-spaces

For basic properties of spectra, we refer to Refs. 19 and 20. A more recent exposition with applications to topological field theories can be found in Ref. 21. A *spectrum* is a sequence $\{E_n\}_{n\ge0}$ of pointed topological spaces together with pointed maps $\sigma_n:\Sigma E_n\to E_{n+1}$. A morphism $f:E\to F$ between spectra consists of a sequence of pointed maps $\{f_n:E_n\to F_n\}_{n\ge0}$ that commute with the structure maps σ_n. Given a pointed topological space X, one can construct the suspension spectrum $\Sigma^\infty X$ consisting of the n-fold suspensions $\{\Sigma^nX\}_{n\ge0}$ where the structure maps σ_n are given by the identity maps. The spectrum $\Sigma^\infty(S^0)$ is called the sphere spectrum and is denoted by $\mathbb{S}$. There is a notion of homotopy for maps between spectra, and one can talk about the set $[E,F]$ of homotopy classes of maps. Let Σ^rE denote the shifted spectrum defined by $(\Sigma^rE)_n=E_{r+n}$, where $r\in\mathbb{Z}$ (by convention, E_{r+n} is a point if $r+n<0$). Homotopy groups of spectra are defined by

$$\pi_r(E)=[\Sigma^r\mathbb{S},E].$$

A cofiber sequence of spectra gives rise to a long exact sequence of homotopy groups. Given two spectra, one can construct the smash product $E\wedge F$. When $F=\Sigma^\infty X$, we will write $E\wedge X$ for the corresponding smash product. Spectra are used to define cohomology and homology theories. The E-cohomology and E-homology of X are defined by

$$E^r(X)=[\mathbb{S}\wedge X,\Sigma^rE]\quad\text{and}\quad E_r(X)=[\Sigma^r\mathbb{S},E\wedge X].$$

Given a spectrum E, one can define a space $\Omega^\infty E$ by taking the direct limit of the sequence of maps

$$E_0\xrightarrow{\omega_0}\Omega E_1\xrightarrow{\Omega\omega_1}\cdots\xrightarrow{\Omega^{n-1}\omega_{n-1}}\Omega^nE_n\xrightarrow{\Omega^n\omega_n}\cdots,$$

where $\omega_n:E_n\to\Omega E_{n+1}$ is the adjoint of σ_n. The space $\Omega^\infty E$ has the structure of an *infinite loop space* (Ref. 22, Sec. 1.7). An infinite loop space can be delooped indefinitely, that is, one can define the spaces $\Omega^{-r}(\Omega^\infty E)$. Let QX denote the space $\Omega^\infty(\Sigma^\infty X)$. The rth stable homotopy group of X is defined by

$$\pi_r^s(X)=\pi_r(QX),$$

which is also isomorphic to $\pi_r(\mathbb{S}\wedge X)$. The E-cohomology of X can also be defined as

$$E^r(X)=[X,\Omega^{-r}(\Omega^\infty E)].$$

We are interested in spectra that come from Γ-spaces. These objects are first introduced in Ref. 23. In this section, we will mostly follow the exposition given in Ref. 16, Sec. 2. For more details on Γ-spaces, see Ref. 24, Sec. 4, and Ref. 25, Appendix B. Let $\mathbf{Fin}_*$ denote the category whose objects are pointed finite sets $k_+ = \{1, 2, \ldots, k\} \sqcup \{+\}$, $k \geq 0$, and morphisms are pointed set maps $\alpha : k_+ \to l_+$. Let $\mathbf{Top}_*$ denote the category of pointed topological spaces. A Γ-*space* is a functor $F : \mathbf{Fin}_* \to \mathbf{Top}_*$. This can be extended to a functor $F : \mathbf{Top}_* \to \mathbf{Top}_*$ by a coend construction

$$F(X) = \int^{k_+} F(k_+) \times X^k. \tag{2.2}$$

More explicitly, the coend is the quotient of the disjoint union of $F(k_+) \times X^k$ over $k \geq 0$ under the equivalence relation generated by

$$(F(\alpha)(z); x_1, \ldots, x_l) \sim (z; x_{\alpha(1)}, \ldots, x_{\alpha(k)}),$$

where α runs over all pointed set maps $k_+ \to l_+$ and $(z; x_1, \ldots, x_l) \in F(k_+) \times X^l$. There is an assembly map $F(X) \wedge Y \to F(X \wedge Y)$ induced by $(z; x_1, \ldots, x_k) \wedge y \mapsto (z; x_1 \wedge y, \ldots, x_k \wedge y)$. The assembly map can be used to associate a spectrum to a Γ-space. The spectrum associated with F is denoted by $F(\mathbb{S})$ and consists of the spaces $\{F(S^n) | n \geq 0\}$ whose structure maps are induced by the assembly map $F(S^n) \wedge S^1 \to F(S^n \wedge S^1)$.

Example II.3. We will encounter the following examples:

1. Let $\mathbb{S} : \mathbf{Fin}_* \to \mathbf{Top}_*$ denote the inclusion functor. This means that we regard k_+ as a pointed topological space with discrete topology. We can think of this functor as $\mathrm{Hom}_{\mathbf{Fin}_*}(1_+, -)$. The associated spectrum is the sphere spectrum and is simply denoted by $\mathbb{S}$ (Ref. 26, Sec. 1).
2. Let $\mathbb{C}^d$ denote the complex vector space with the canonical basis $e_1, e_2, \ldots, e_d$. The Γ-space $ku : \mathbf{Fin}_* \to \mathbf{Top}_*$ is defined by

$$ku(k_+) = \coprod_{d_1, \ldots, d_k \in \mathbb{N}} \frac{L(\mathbb{C}^{d_1} \oplus \cdots \oplus \mathbb{C}^{d_k}, \mathbb{C}^\infty)}{U(d_1) \times \cdots \times U(d_k)},$$

where $L(-,-)$ denotes the space of complex linear isometric embeddings between two complex inner product spaces. A point in this space is specified by a tuple $(V_1, \ldots, V_k)$ of pairwise orthogonal subspaces of $\mathbb{C}^\infty$. Given $\alpha : k_+ \to l_+$, the map $ku(\alpha)$ is defined by

$$(V_1, \ldots, V_k) \mapsto (\oplus_{i \in \alpha^{-1}(1)} V_i, \ldots, \oplus_{i \in \alpha^{-1}(l)} V_i).$$

The spectrum $ku(\mathbb{S})$ we obtain is the connective complex K-theory spectrum, which will be denoted simply by ku. There is a canonical morphism $\mathbb{S} \to ku$ of Γ-spaces induced by the map $\mathbb{S}(1_+) \to ku(1_+) = \coprod_{q \geq 0} \mathrm{Gr}_q(\mathbb{C}^\infty)$ sending 1 to the subspace $\langle e_1 \rangle \subset \mathbb{C}^\infty$ and the base point $+$ to the point $\mathrm{Gr}_0(\mathbb{C}^\infty)$. There is a real version of this construction defined analogously but using $\mathbb{R}$-vector spaces. The resulting spectrum is the connective real K-theory spectrum ko. See Ref. 16 for an equivariant approach.
3. Let M be a commutative discrete monoid. There is an associated Γ-space denoted by $HM : \mathbf{Fin}_* \to \mathbf{Top}_*$, where $HM(k_+) = M^k$, and for $\alpha : k_+ \to l_+$, the map $HM(\alpha)$ is defined by sending $(x_1, \ldots, x_k)$ to $(\sum_{j \in \alpha^{-1}(1)} x_j, \ldots, \sum_{j \in \alpha^{-1}(l)} x_j)$. Applying Ω^∞ to the resulting spectrum $HM(\mathbb{S})$ gives a space homotopy equivalent to ΩBM, also known as the group completion of M. This can be seen from Segal's delooping construction for Γ-spaces (Ref. 23, Proposition 1.4). In particular, we can consider the monoid $\mathbb{N}$ and the associated spectrum $H\mathbb{N}(\mathbb{S})$. Since $\Omega^\infty H\mathbb{N}(\mathbb{S}) \simeq \mathbb{Z}$, we see that this spectrum is equivalent to the Eilenberg–Maclane spectrum $H\mathbb{Z}$. There is a map of Γ-spaces $\dim : ku \to H\mathbb{N}$ obtained by sending $(V_1, \ldots, V_k)$ to $(\dim(V_1), \ldots, \dim(V_k))$. A similar morphism of Γ-spaces $\dim : ku \to H\mathbb{Z}$ can be obtained by replacing $\mathbb{N}$ with the Abelian group of integers. The canonical morphisms from $\mathbb{S}$ fit into a commutative diagram of Γ-spaces

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[dr] & ku \arrow[d, "\dim"] \\ & H\mathbb{Z}. \end{tikzcd}$$

The diagonal arrow is induced by the inclusion map $1_+ \to \mathbb{Z}$ defined by $1 \mapsto 1$ and $+ \mapsto 0$.

D. The spectrum

Let $\mu_d \subset U(1)$ denote the subgroup generated by $e^{2\pi i/d}$.

Proposition II.4. Sending $(V_1, \ldots, V_k; \lambda_1, \ldots, \lambda_k)$, where V_i are pairwise orthogonal finite-dimensional subspaces of $\mathbb{C}^\infty$ and $\lambda_i \in (\mu_d)^n$, to the n-tuple $(A_1, \ldots, A_n)$ of pairwise commuting unitary matrices, where A_j acts on V_i by multiplication with $\lambda_i^{(j)}$ and trivially on the complement of $V_1 \oplus \cdots \oplus V_k$, induces a homeomorphism

$$ku((\mu_d)^n) \xrightarrow{\cong} \mathrm{Hom}((\mathbb{Z}/d)^n, U).$$

Moreover, this homeomorphism is compatible with the simplicial structures and induces a homeomorphism

$$ku(B\mu_d) \xrightarrow{\cong} B(\mathbb{Z}/d, U).$$

Proof. The statements are proved in Ref. 16 when $\lambda_i^{(j)} \in U(1)^n$. These arguments still go through when $U(1)$ is replaced by the subgroup μ_d. □

It is instructive to describe the inverse of the first homeomorphism. Let $(A_1, A_2, \ldots, A_n)$ be a tuple of pairwise commuting matrices in U such that $(A_j)^d = I$ for $1 \le j \le n$. These matrices are contained in $U(m)$ for some large enough m. We can simultaneously diagonalize the matrices

$$\begin{pmatrix} \lambda_1^{(1)} I_{d_1} & & & \\ & \lambda_2^{(1)} I_{d_2} & & \\ & & \ddots & \\ & & & \lambda_k^{(1)} I_{d_k} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_1^{(n)} I_{d_1} & & & \\ & \lambda_2^{(n)} I_{d_2} & & \\ & & \ddots & \\ & & & \lambda_k^{(n)} I_{d_k} \end{pmatrix}$$

such that $(\lambda_i^{(1)}, \lambda_i^{(2)}, \ldots, \lambda_i^{(n)})$ is distinct from $(\lambda_j^{(1)}, \lambda_j^{(2)}, \ldots, \lambda_j^{(n)})$ whenever $i \neq j$. Therefore, $(A_1, A_2, \ldots, A_n)$ amounts to specifying a tuple $(V_1, V_2, \ldots, V_k)$ of pairwise orthogonal finite-dimensional subspaces $V_i \subset \mathbb{C}^\infty$, $1 \le i \le k$, together with the eigenvalues $\lambda_i^{(j)} \in \mu_d$. Then, the inverse map sends $(A_1, \ldots, A_n)$ to the class of $(V_1, \ldots, V_k; \lambda_1, \ldots, \lambda_k)$ in the coend construction (2.2).

Given a pointed space X and a Γ-space F, we write F_X for the Γ-space defined by $F_X(k_+) = F(k_+ \wedge X)$. For $\alpha : k_+ \to l_+$, the map $F_X(\alpha)$ is obtained by the naturality of the coend construction. A Γ-space F is called *special* if the map $F((k+l)_+) \to F(k_+) \times F(l_+)$ induced by the projections $(k+l)_+ \to k_+$ and $(k+l)_+ \to l_+$ is a weak equivalence for all k_+, l_+. A special Γ-space is called *very special* if $\pi_0 F(1_+)$ is an Abelian group. [In general, $\pi_0 F(1_+)$ is a monoid since $F(1_+)$ is an H-space (Ref. 23, Sec. 1).]

Lemma II.5. Let X be a pointed space.

1. *If F is special, then F_X is also special.*
2. *The natural map $F(\mathbb{S}) \wedge X \to F_X(\mathbb{S})$ is a stable equivalence.*

Proof. Part (1) is implicitly mentioned in Ref. 24, Proof of Theorem 4.4, and part (2) is proved therein as Lemma 4.1. For a more recent exposition of the equivariant version of this statement, see Ref. 25 when X has finitely many cells and Ref. 16 for the general case. □

Definition II.6. The spectrum $ku_{B\mu_d}(\mathbb{S})$ will be called the *commutative d-torsion K-theory spectrum* and will be denoted by $k\mu_d$. The associated generalized cohomology theory will be referred to as the *commutative d-torsion K-theory*. We write $k\mu_d^n(X)$ to denote the nth $k\mu_d$-cohomology of X and $k\mu_d(X) = k\mu_d^0(X)$ for simplicity of notation (not to be confused with the Γ-space evaluated at X).

Proposition II.7. The spectrum $k\mu_d$ is stably equivalent to $ku \wedge B\mu_d$, and the space $\Omega^\infty k\mu_d$ is weakly equivalent to $B(\mathbb{Z}/d, U)$.

Proof. We modify the argument in Ref. 16 given for $B(\mathbb{Z}, U)$. Applying part (1) of the lemma to $F = ku$ and $X = B\mu_d$ and using the well-known fact that ku is special, we obtain that $ku_{B\mu_d}$ is special. Moreover, $ku_{B\mu_d}$ is very special since

$$ku_{B\mu_d}(1_+) = ku(1_+ \wedge B\mu_d) = B(\mathbb{Z}/d, U) \tag{2.3}$$

and thus $\pi_0(ku_{B\mu_d}(1_+)) = \pi_0 B(\mathbb{Z}/d, U) = 0$. It is a general fact that if F is very special, then $\Omega^\infty F(\mathbb{S}) \simeq F(1_+)$.[23] Therefore, $\Omega^\infty k\mu_d = \Omega^\infty ku_{B\mu_d}(\mathbb{S}) \simeq ku_{B\mu_d}(1_+) \cong B(\mathbb{Z}/d, U)$. The equivalence $k\mu_d \simeq ku \wedge B\mu_d$ follows from part (2) of Lemma II.5. □

As a consequence of this result, we have

$$k\mu_d^r(X) = [X, \Omega^{-r} B(\mathbb{Z}/d, U)].$$

Remark II.8. There is one important difference between $ku((\mu_d)^n)$ and $ku(U(1)^n)$ worth pointing out. The former is not an infinite loop space, whereas the latter is since $U(1)^n$ is path connected. Note that $\pi_0 ku((\mu_d)^n)$ can be identified with $\mathrm{Rep}((\mathbb{Z}/d)^n, U)$, the union of the quotient spaces $\mathrm{Hom}((\mathbb{Z}/d)^n, U(m))/U(m)$ under the conjugation action of $U(m)$.

Moreover, $\mathrm{Rep}((\mathbb{Z}/d)^n, U) \cong H\mathbb{N}((\mu_d)^n)$ and the quotient map

$$\mathrm{Hom}((\mathbb{Z}/d)^n, U) \to \mathrm{Rep}((\mathbb{Z}/d)^n, U)$$

can be described using the map of Γ-spaces $\dim : ku \to \mathbb{N}$; see Ref. 16. For example, when $d = 2$, we have that $H\mathbb{N}((\mu_2)^n) = \mathbb{N} \wedge (\mu_2)^n$, where $\mathbb{N}$ has 0 as its base point and $(\mu_2)^n$ is based at the identity element. The set of path components is not an Abelian group.

E. Low dimensional homotopy groups

As a consequence of Proposition II.7, the homotopy groups of $k\mu_d$ coincide with the ku-homology of $B\mu_d$. The groups $ku_*(B\mu_d)$ are computed in Ref. 27, Sec. 3.4; see also Ref. 28. In low degrees, we have

$$\pi_r B(\mathbb{Z}/d, U) \cong \pi_r(k\mu_d) \cong \pi_r(ku \wedge B\mu_d) = \begin{cases} 0, & r = 0, \\ \mathbb{Z}/d, & r = 1, \\ 0, & r = 2. \end{cases} \tag{2.4}$$

There is a commutative diagram

$$\begin{tikzcd} B\mu_d \arrow[r, hook] \arrow[dr, equal] & B(\mathbb{Z}/d, U) \arrow[d, "\det"] \\ & B\mu_d, \end{tikzcd} \tag{2.5}$$

which splits off the $\mathbb{Z}/d$ in $\pi_1 B(\mathbb{Z}/d, U)$. The determinant map factors through the geometric realization of the simplicial set of connected components, denoted by $|\pi_0\mathrm{Hom}((\mathbb{Z}/d)^\bullet, U)|$. Proposition II.4 implies that the connected components of $\mathrm{Hom}((\mathbb{Z}/d)^n, U)$ can be described as $\pi_0 ku((\mu_d)^n) = H\mathbb{N}((\mu_d)^n)$; see also Remark II.8. Therefore, we have

$$\pi_0\mathrm{Hom}((\mathbb{Z}/d)^\bullet, U) = H\mathbb{N}((\mu_d)^\bullet),$$

and the natural map $B(\mathbb{Z}/d, U) \to |\pi_0\mathrm{Hom}((\mathbb{Z}/d)^\bullet, U)|$ is given by the geometric realization of

$$ku((\mu_d)^\bullet) \to H\mathbb{N}((\mu_d)^\bullet)$$

induced by the Γ-space map $\dim : ku \to \mathbb{N}$, which sends a tuple of pairwise orthogonal subspaces $(V_1, V_2, \ldots, V_k)$ to their dimensions $(d_1, d_2, \ldots, d_k)$. Since $\mathbb{N}$ is a special Γ-space, we can apply Lemma II.5 to obtain an equivalence

$$|H\mathbb{N}((\mu_d)^\bullet)| \simeq \Omega^\infty(H\mathbb{N}(\mathbb{S}) \wedge B\mu_d). \tag{2.6}$$

Using the equivalence $H\mathbb{N}(\mathbb{S}) \simeq H\mathbb{Z}$, we obtain the following.

Proposition II.9. The determinant map factors as

$$\begin{tikzcd} B(\mathbb{Z}/d, U) \arrow[r] \arrow[dr, "\det"'] & {|\pi_0 Hom((\mathbb{Z}/d)^\bullet, U)|} \arrow[d] \\ & B\mu_d, \end{tikzcd}$$

where the homotopy groups of the simplicial set of connected components is given by

$$\pi_r|\pi_0\mathrm{Hom}((\mathbb{Z}/d)^\bullet, U)| \cong \tilde{H}_r(B\mu_d, \mathbb{Z}).$$

The determinant map induces a homomorphism

$$\det_* : k\mu_d(X) \to H^1(X, \mathbb{Z}/d),$$

which splits as a consequence of the diagram in (2.5). In general, since the homotopy groups of $k\mu_d$ are known, we can compute $k\mu_d$-cohomology using the Atiyah–Hirzebruch spectral sequence.[19] The E_2-page of the spectral sequence is given by

$$\tilde{H}^p(X, \pi_{-q}k\mu_d) \Rightarrow k\mu_d^{p+q}(X). \tag{2.7}$$

One special case, for which the computation is easy, is when X is a connected two-dimensional CW complex. In this case, the spectral sequence collapses in the E_2-page and $\det_*$ becomes an isomorphism

$$k\mu_d(X) \cong H^1(X, \mathbb{Z}/d). \tag{2.8}$$

III. $C(d,m)$-COHOMOLOGY

For each $m \geq 1$, we construct a spectrum, denoted by $C(d,m)$, obtained from the commutative d-torsion K-theory spectrum $k\mu_d$. In this section, we compute the homotopy groups of $C(d,m)$ and describe the $C(d,m)$-cohomology of a space. In Sec. IV, we will see that $C(d,m)$-cohomology informs us about operator solutions of linear constraint systems. These operator solutions play a significant role in quantum information theory.

A. A quotient space

Throughout this section, we assume that the topological group G contains a central subgroup isomorphic to μ_d. When $G = U(m)$, this will be the subgroup of $m \times m$ scalar matrices with entries in μ_d.

Definition III.1. Let $\bar{B}(\mathbb{Z}/d, G)$ denote the geometric realization of the simplicial space

$$[n] \mapsto \mathrm{Hom}((\mathbb{Z}/d)^n, G)/\sim,$$

where the quotient relation identifies $(A_1,\ldots,A_n)$ with $(\alpha_1 A_1,\ldots,\alpha_n A_n)$, where $\alpha_i \in \mu_d$. Simplicial structure maps are similar to the ones given in Definition II.1.

Let $\bar{G}$ denote the quotient group G/μ_d. The quotient space $\bar{B}(\mathbb{Z}/d, G)$ is a subspace of the classifying space $B\bar{G}$. Furthermore, there is a pull-back diagram

$$\begin{tikzcd} B(\mathbb{Z}/d,G) \arrow[r] \arrow[d] & BG \arrow[d] \\ \bar{B}(\mathbb{Z}/d,G) \arrow[r] & B\bar{G}, \end{tikzcd}$$

where the right-hand map is a fibration sequence with fiber $B\mu_d$. Therefore, we obtain a fibration sequence

$$B\mu_d \xrightarrow{\Delta_G} B(\mathbb{Z}/d, G) \to \bar{B}(\mathbb{Z}/d, G),$$

where the fiber inclusion is induced by $\mu_d \subset G$. By the classification of principal bundles, this fibration is determined by a cohomology class γ_G in $H^2(\bar{B}(\mathbb{Z}/d,G),\mathbb{Z}/d)$. When G is the unitary group $U(m)$, we simply write Δ_m for the fiber inclusion and γ_m for the cohomology class. The stabilization maps in (2.1) do not descend to $\bar{B}(\mathbb{Z}/d, U(m))$. However, we will construct a space that serves as a stabilization using methods from stable homotopy theory.

B. $C(d,m)$ spectrum

We begin with a spectrum level description of Δ_m. For $m \geq 1$, let us introduce a map of Γ-spaces,

$$\delta_m : \mathbb{S} \to ku, \tag{3.1}$$

induced by the map

$$1_+ \to \coprod_{q\geq 0} \mathrm{Gr}_q(\mathbb{C}^\infty)$$

that sends the element 1 to the subspace $\mathbb{C}^m = \langle e_1, e_2, \ldots, e_m\rangle$ and the base point + to $\mathrm{Gr}_0(\mathbb{C}^\infty)$. This assignment determines all the other maps $\mathbb{S}(k_+) \to ku(k_+)$ since $\mathbb{S}(-) \cong \mathrm{Hom}_{\mathbf{Fin}_*}(1_+, -)$.

Let $\delta_{d,m} : \mathbb{S}_{B\mu_d} \to ku_{B\mu_d}$ denote the Γ-space map induced by δ_m using the functoriality of the construction $F \mapsto F_X$. The associated spectrum maps will still be denoted by δ_m and $\delta_{d,m}$.

Consider the cofiber sequence

$$\mathbb{S}_{B\mu_d}(\mathbb{S}) \xrightarrow{\delta_{d,m}} k\mu_d \to C(\delta_{d,m}). \tag{3.2}$$

For applications, we are mainly interested in computing $[X, C(\delta_{d,m})]$ for a two-dimensional CW complex X. For a spectrum X, let $p_n : X \to P_n X$ denote the nth Postnikov section.[29]

Definition III.2. We define $C(d,m) = P_2 C(\delta_{d,m})$, i.e., the spectrum obtained from $C(\delta_{d,m})$ by killing the homotopy groups of degree greater than 2. We write $\bar{B}(d,m)$ for the associated infinite loop space $\Omega^\infty C(d,m)$.

To compute the homotopy groups of $C(d,m)$, we can use the cofiber sequence

$$\mathbb{S} \wedge B\mu_d \xrightarrow{\delta_m \wedge \mathrm{id}} ku \wedge B\mu_d \to C(\delta_{d,m}) \tag{3.3}$$

instead of (3.2) since we have a commutative diagram of spectra

$$\begin{tikzcd} \mathbb{S} \wedge B\mu_d \arrow[r, "\sim"] \arrow[d, "\delta_m \wedge \mathrm{id}"] & \mathbb{S}_{B\mu_d}(\mathbb{S}) \arrow[d, "\delta_{d,m}"] \\ ku \wedge B\mu_d \arrow[r, "\sim"] & ku_{B\mu_d}(\mathbb{S}) \end{tikzcd} \tag{3.4}$$

as a consequence of part (2) of Lemma II.5.

Lemma III.3. The homotopy groups of $C(d,m)$ fit into an exact sequence of Abelian groups,

$$0 \to \pi_2 C(d,m) \to \mathbb{Z}/d \xrightarrow{\phi} \mathbb{Z}/d \to \pi_1 C(d,m) \to 0, \tag{3.5}$$

where ϕ is given by multiplication with m.

Proof. The exact sequence in (3.5) is obtained from the homotopy exact sequence associated with (3.3) and using the homotopy groups of $ku \wedge B\mu_d$ given in (2.4) together with the isomorphism $\pi_1^s(B\mu_d) \cong \mathbb{Z}/d$. It remains to show that $\phi : \mathbb{Z}/d \to \mathbb{Z}/d$ is given by multiplication with m. The Γ-space map $\delta_m : \mathbb{S} \to ku$ fits into a commutative diagram of Γ-spaces

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[d, "\delta_m"'] & H\mathbb{Z} \arrow[d, "\times m"] \\ ku \arrow[r, "\dim"] & H\mathbb{Z} \end{tikzcd} ,$$

where the top map is the canonical map of Γ-spaces [Example II.3, part (3)]. The right-hand map is induced by $\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}$, the multiplication with m map. Replacing each Γ-space F in the diagram with $F_{B\mu_d}$, looking at the associated spectrum and using part (2) of Lemma II.5, we obtain a diagram of spectra

$$\begin{tikzcd} \mathbb{S} \wedge B\mu_d \arrow[r] \arrow[d, "\delta_m \wedge \mathrm{id}"'] & H\mathbb{Z} \wedge B\mu_d \arrow[d, "(\times m) \wedge \mathrm{id}"] \\ ku \wedge B\mu_d \arrow[r] & H\mathbb{Z} \wedge B\mu_d. \end{tikzcd} \tag{3.6}$$

Applying π_1 to this diagram gives a commutative diagram of Abelian groups,

$$\begin{tikzcd} \pi_1^s(B\mu_d) \arrow[r, "\cong"] \arrow[d, "\phi"] & H_1(B\mu_d, \mathbb{Z}) \arrow[d, "\times m"] \\ ku_1(B\mu_d) \arrow[r, "\cong"] & H_1(B\mu_d, \mathbb{Z}). \end{tikzcd}$$

Therefore, after identifying these groups with $\mathbb{Z}/d$, we see that ϕ is given by multiplication with m. □

We can extend diagram (3.6) by composing the horizontal maps with the first Postnikov section,

$$\begin{tikzcd} \mathbb{S} \wedge B\mu_d \arrow[r] \arrow[d, "\delta_m \wedge \mathrm{id}"'] & H\mathbb{Z} \wedge B\mu_d \arrow[r, "p_1"] \arrow[d, "(\times m) \wedge \mathrm{id}"] & \Sigma H\mathbb{Z}/d \arrow[d, "\times m"] \\ ku \wedge B\mu_d \arrow[r] & H\mathbb{Z} \wedge B\mu_d \arrow[r, "p_1"] & \Sigma H\mathbb{Z}/d \end{tikzcd} . \tag{3.7}$$

Corollary III.4. There is a cofiber sequence

$$\Sigma H\mathbb{Z}/d \xrightarrow{\times m} \Sigma H\mathbb{Z}/d \to C(d,m). \tag{3.8}$$

Moreover, we have an equivalence $\Sigma H\pi_1(C(d,m)) \vee \Sigma^2 H\pi_2(C(d,m)) \to C(d,m)$.

Proof. The cofiber sequence is a consequence of Lemma III.3. This sequence presents $C(d,m)$ as a $H\mathbb{Z}$-module spectrum since $\times m$ is a morphism of $H\mathbb{Z}$-modules. For a $H\mathbb{Z}$-module spectrum E, there is an equivalence $\vee_{n\in\mathbb{Z}}\Sigma^n H\pi_n(E) \to E$ of spectra (Ref. 30, Proposition 5.3), which gives the splitting. □

C. $C(d,m)$-cohomology

Let us introduce notation for the Abelian groups corresponding to the kernel and the cokernel of the exact sequence in (3.5),

$$0 \to (\mathbb{Z}/d)_m \xrightarrow{i_m} \mathbb{Z}/d \xrightarrow{\times m} \mathbb{Z}/d \xrightarrow{\pi_m} \frac{\mathbb{Z}/d}{m\mathbb{Z}/d} \to 0.$$

For a group homomorphism $h: A \to B$, we write $h_*: H^n(X,A) \to H^n(X,B)$ for the change of coefficients map. Note that both the kernel and the cokernel are isomorphic to $\mathbb{Z}/\gcd(d,m)$.

The $C(d,m)$-cohomology of a pointed space X is defined by the pointed homotopy classes of maps,

$$C(d,m)^r(X) = [X, \Omega^{-r}\bar{B}(d,m)],$$

and we simply write $C(d,m)(X) = [X,\bar{B}(d,m)]$ when $r = 0$.

Theorem III.5. *Let X be a connected CW complex. There is a commutative diagram*

$$\begin{tikzcd}
 & H^2(X,(\mathbb{Z}/d)_m) \arrow[r, "(i_m)_*"] \arrow[d, hook] & H^2(X,\mathbb{Z}/d) \arrow[d, equal] \\
k\mu_d(X) \arrow[r, "\zeta"] \arrow[d] & C(d,m)(X) \arrow[r, "cl"] \arrow[d, two heads] & H^2(X,\mathbb{Z}/d) \\
H^1(X,\mathbb{Z}/d) \arrow[r, "(\pi_m)_*"] & H^1(X,\frac{\mathbb{Z}/d}{m\mathbb{Z}/d}), &
\end{tikzcd} \tag{3.9}$$

where $\mathrm{cl}(f) = f^*(\gamma_m^{\mathbb{S}})$, *the image of* ζ *is contained in the kernel of* cl, *and the middle column is an exact sequence that (non-canonically) splits as follows:*

$$C(d,m)(X) \cong H^1\left(X, \frac{\mathbb{Z}/d}{m\mathbb{Z}/d}\right) \oplus H^2(X,(\mathbb{Z}/d)_m).$$

Proof. The diagram of spectra given in (3.7) extends below to the cofibers. Shifting the resulting diagram of cofibrations gives the following diagram of spectra:

$$\begin{tikzcd}
k\mu_d \arrow[r] \arrow[d, "c"] & \Sigma H\mathbb{Z}/d \arrow[d] \\
C(\delta_{d,m}) \arrow[r, "p_2"] \arrow[d] & C(d,m) \arrow[d, "\gamma"] \\
\Sigma(\mathbb{S}\wedge B\mu_d) \arrow[r] & \Sigma^2 H\mathbb{Z}/d\ .
\end{tikzcd}$$

The exactness claim about the middle row of (3.9) is obtained by evaluating the sequence $k\mu_d \xrightarrow{p_2 c} C(d,m) \xrightarrow{\gamma} \Sigma^2 H\mathbb{Z}/d$ at the space X. Applying to $\gamma: C(d,m) \to \Sigma^2 H\mathbb{Z}/d$ the functor Ω^∞ gives a map $\Omega^\infty\gamma: \bar{B}(d,m) \to B^2\mu_d$, which represents the cohomology class $\gamma_m^{\mathbb{S}} \in H^2(\bar{B}(d,m),\mathbb{Z}/d)$. Therefore, for a map $f: \mathbb{S}\wedge X \to C(d,m)$, the cohomology class $\mathrm{cl}(f)$, which is represented by the composition γf, coincides with $f^*(\gamma_m^{\mathbb{S}})$. The horizontal maps $(\pi_m)_*$ and $(i_m)_*$ are obtained by comparing the Atiyah–Hirzebruch spectral sequences and the commutativity of the squares that follow from the naturality of the spectral sequences. The homomorphism $k\mu_d(X) \to H^1(X,\mathbb{Z}/d)$ is the edge homomorphism in the Atiyah–Hirzebruch spectral sequence for the $k\mu_d$-cohomology of X. The splitting is given by the spectrum level splitting described in Corollary III.4. □

We end this section by considering the associated infinite loop space $\bar{B}(d,m) = \Omega^\infty C(d,m)$ and its relation to the unstable spaces $\bar{B}(\mathbb{Z}/d,U(m))$. We will identify $\mathbb{Z}/d$ with μ_d via the isomorphism $1 \mapsto \omega$. Recall that the fibration

$$B(\mathbb{Z}/d) \xrightarrow{\Delta_m} B(\mathbb{Z}/d,U(m)) \to \bar{B}(\mathbb{Z}/d,U(m))$$

is determined by a cohomology class $\gamma_m \in H^2(\bar{B}(\mathbb{Z}/d, U(m)), \mathbb{Z}/d)$. Applying the functor Ω^∞ to the cofiber sequence in (3.8) gives a (homotopy) fiber sequence

$$B(\mathbb{Z}/d) \xrightarrow{B(\times m)} B(\mathbb{Z}/d) \to \bar{B}(d,m), \tag{3.10}$$

which is determined by a cohomology class $\gamma_m^{\mathbb{S}} \in H^2(\bar{B}(d,m), \mathbb{Z}/d)$. Observe that the determinant map $\det : B(\mathbb{Z}/d, U(m)) \to B(\mathbb{Z}/d)$ descends to a map $\overline{\det} : \bar{B}(\mathbb{Z}/d, U(m)) \to B(\frac{\mathbb{Z}/d}{m\mathbb{Z}/d})$.

Lemma III.6.

1. *There are maps of (homotopy) fibrations*

$$\begin{tikzcd}
B(\mathbb{Z}/d) \arrow[r, equal] \arrow[d, "\Delta_m"] & B(\mathbb{Z}/d) \arrow[r, "B(\times m)"] \arrow[d, "B(\times m)"] & B(m\mathbb{Z}/d) \arrow[d] \\
B(\mathbb{Z}/d, U(m)) \arrow[r, "\det"] \arrow[d] & B(\mathbb{Z}/d) \arrow[r, equal] \arrow[d] & B(\mathbb{Z}/d) \arrow[d, "B\pi_m"] \\
\bar{B}(\mathbb{Z}/d, U(m)) \arrow[r, "\bar{\iota}_m"] \arrow[rr, bend right, "\overline{\det}"'] & \bar{B}(d,m) \arrow[r, "p_1"] & B\left(\frac{\mathbb{Z}/d}{m\mathbb{Z}/d}\right),
\end{tikzcd} \tag{3.11}$$

where the diagram commutes up to homotopy and $p_1 \bar{\iota}_m \simeq \overline{\det}$.

2. *The cohomology class* $\gamma_m^{\mathbb{S}}$ *maps to* γ_m *under the induced map* $(\bar{\iota}_m)^*$. *Moreover, the maps representing these cohomology classes fit into a homotopy commutative diagram*

$$\begin{tikzcd}
\bar{B}(\mathbb{Z}/d, U(m)) \arrow[r, "\bar{\iota}_m"] \arrow[d, "\bar{\gamma}_m"] \arrow[dd, bend right=60, "\gamma_m"'] & \bar{B}(d,m) \arrow[d, "\bar{\gamma}_m^{\mathbb{S}}"] \arrow[dd, bend left=60, "\gamma_m^{\mathbb{S}}"] \\
B^2(\mathbb{Z}/d)_m \arrow[r, equal] \arrow[d, "B^2 i_m"] & B^2(\mathbb{Z}/d)_m \arrow[d, "B^2 i_m"] \\
B^2(\mathbb{Z}/d) \arrow[r, equal] & B^2(\mathbb{Z}/d),
\end{tikzcd} \tag{3.12}$$

where $B^2(-) = B(B(-))$.

Proof. Part (1): Let G be a topological group and X be a G-space. The homotopy quotient of X, also known as the Borel construction (Ref. 31, Sec. 2.2), will be denoted by $X//G$. A G-map $X \to Y$ between two G-spaces induces a map $X//G \to Y//G$ between the homotopy quotients. We will apply this idea to construct the map $\bar{\iota}_m$ in diagram (3.11), which will make the lower part of the diagram commute up to homotopy. Since $\mathbb{Z}/d$ is an Abelian group, the classifying space $B(\mathbb{Z}/d) = |(\mathbb{Z}/d)^{\cdot}|$ is a topological Abelian group. The group structure is induced by the level-wise addition maps $\{m_n : (\mathbb{Z}/d)^n \times (\mathbb{Z}/d)^n \to (\mathbb{Z}/d)^n\}_{n\geq 0}$ defined by

$$m_n((\lambda_1, \lambda_2, \ldots, \lambda_n), (\lambda_1', \lambda_2', \ldots, \lambda_n')) = (\lambda_1 + \lambda_1', \lambda_2 + \lambda_2', \ldots, \lambda_n + \lambda_n').$$

More precisely, $(\mathbb{Z}/d)^{\cdot}$ is a simplicial Abelian group. Moreover, $B(\mathbb{Z}/d, U(m))$ and $B(\mathbb{Z}/d)$ are $B(\mathbb{Z}/d)$-spaces. Both spaces are geometric realizations of simplicial spaces, and the action is given by the geometric realization of a level-wise (simplicial) action. For the first one, the action on the space of n-simplices is given by

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \cdot (A_1, A_2, \ldots, A_n) \mapsto (\omega^{\lambda_1} A_1, \omega^{\lambda_2} A_2, \ldots, \omega^{\lambda_n} A_n),$$

and for the second space, the action is given by

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \cdot (\lambda_1', \lambda_2', \ldots, \lambda_n') \mapsto (m\lambda_1 + \lambda_1', m\lambda_2 + \lambda_2', \ldots, m\lambda_n + \lambda_n').$$

With these actions, $\det : B(\mathbb{Z}/d, U(m)) \to B(\mathbb{Z}/d)$ is $B(\mathbb{Z}/d)$-equivariant. Moreover, the base spaces in diagram (3.11) are homotopy equivalent to the corresponding homotopy quotients. Then, the map $\bar{\iota}_m$ is obtained as the composite

$$\bar{B}(\mathbb{Z}/d, U(m)) \simeq B(\mathbb{Z}/d, U(m))//B(\mathbb{Z}/d) \to B(\mathbb{Z}/d)//B(\mathbb{Z}/d) \simeq \bar{B}(d,m).$$

A similar argument applies to the right-hand square and produces the map p_1 between the homotopy quotients. The comparison maps between the homotopy quotients and the ordinary quotients give a commutative diagram

$$\begin{tikzcd} B(\mathbb{Z}/d, U(m))//B(\mathbb{Z}/d) \arrow[r, "p_1\bar{\iota}_m"] \arrow[d, "\sim"] & B(\mathbb{Z}/d)//B(m\mathbb{Z}/d) \arrow[d, "\sim"] \\ \bar{B}(\mathbb{Z}/d, U(m)) \arrow[r, "\overline{\det}"] & B\left(\frac{\mathbb{Z}/d}{m\mathbb{Z}/d}\right). \end{tikzcd}$$

Part (2): Delooping the left-hand square of diagram (3.11) shows that the maps $\gamma_m : \bar{B}(\mathbb{Z}/d, U(m)) \to B^2(\mathbb{Z}/d)$ and $\gamma_m^{\mathbb{S}} : \bar{B}(d,m) \to B^2(\mathbb{Z}/d)$ representing the corresponding cohomology classes fit into the outer diagram in (3.12). In particular, this implies that $\bar{\iota}_m^*(\gamma_m^{\mathbb{S}}) = \gamma_m$. Since up to homotopy $\gamma_m^{\mathbb{S}}$ is the homotopy fiber of the map $B^2(\mathbb{Z}/d) \xrightarrow{B^2\times m} B^2(\mathbb{Z}/d)$, it factors as $\bar{\gamma}_m^{\mathbb{S}} : \bar{B}(d,m) \to B^2(\mathbb{Z}/d)_m$. A similar factorization applies to γ_m since it is obtained as the pull-back of $\gamma_m^{\mathbb{S}}$ along $\bar{\iota}_m$, giving us the homotopy commutative diagram (3.12). □

Corollary III.7. The following diagram commutes up to homotopy:

$$\begin{tikzcd} \bar{B}(\mathbb{Z}/d, U(m)) \arrow[r, "\bar{\iota}_m"] \arrow[dr, "\overline{\det}\times\bar{\gamma}_m"'] & \bar{B}(d,m) \arrow[d, "p_1\times\bar{\gamma}_m^{\mathbb{S}}", "\sim"'] \\ & B\left(\frac{\mathbb{Z}/d}{m\mathbb{Z}/d}\right) \times B^2(\mathbb{Z}/d)_m. \end{tikzcd}$$

Proof. Both $(p_1 \times \bar{\gamma}_m^{\mathbb{S}})\bar{\iota}_m$ and $\overline{\det} \times \bar{\gamma}_m$ correspond to the same class in

$$H^1\left(\bar{B}(\mathbb{Z}/d, U(m)), \frac{\mathbb{Z}/d}{m\mathbb{Z}/d}\right) \times H^2(\bar{B}(\mathbb{Z}/d, U(m)), (\mathbb{Z}/d)_m)$$

since $p_1\bar{\iota}_m \simeq \overline{\det}$ and $\bar{\gamma}_m^{\mathbb{S}}\bar{\iota}_m \simeq \bar{\gamma}_m$ by Lemma III.6. Therefore, $(p_1 \times \bar{\gamma}_m^{\mathbb{S}})\bar{\iota}_m \simeq \overline{\det} \times \bar{\gamma}_m$. Note that $p_1 \times \bar{\gamma}_m^{\mathbb{S}}$ is a homotopy inverse of the homotopy equivalence obtained by applying Ω^∞ to the equivalence in Corollary III.4. □

IV. OPERATOR SOLUTIONS OF LINEAR CONSTRAINT SYSTEMS

Linear constraint systems arise in quantum information theory in the context of non-local games. Such games are played among a referee and two players where each player aims to win the game by satisfying a fixed set of rules. For some games, if the players use quantum resources, such as entangled quantum states and quantum measurements, then they can increase their likelihood of winning the game. Other than their applications in quantum information theory, linear constraint systems have found applications in resolving problems in the theory of operator algebras, such as Tsirelson problem[32] and Connes embedding conjecture.[33] In this section, we study operator solutions of linear constraint systems by using the generalized cohomology theory, $C(d,m)$-cohomology, introduced in Sec. III. We show that operator solutions of linear constraint systems correspond to classes in $C(d,m)$-cohomology. The paradigmatic example of a linear constraint system constructed by Mermin[17] (see also Refs. 34 and 35) gives rise to a non-trivial class in the $C(2,2^n)$-cohomology of a torus for $n \geq 2$. This connection to stable homotopy theory opens up a new direction in the study of linear constraint systems. In this respect, stable homotopy theory plays a similar role as it does in the classification of topological quantum phases;[36] see also Ref. 37 for applications of stable homotopical methods to quantum information theory.

A. Linear constraint systems

A *linear constraint system* is specified by a system of linear equations $Mx = b$ for some $r \times c$ matrix M with entries in $\mathbb{Z}/d$ and $b \in (\mathbb{Z}/d)^r$. We say that a linear constraint system (LCS) has an *operator solution* if there exists a collection of $m \times m$-unitary matrices A_i, $1 \leq i \leq c$, such that

1. $(A_i)^d$ is the identity matrix I_m for all $1 \leq i \leq c$,
2. $A_iA_j = A_jA_i$ whenever M_{ki} and M_{kj} are both non-zero for some $1 \leq k \leq r$, and
3. $A_1^{M_{k1}}A_2^{M_{k2}}\ldots A_c^{M_{kc}} = \omega^{b_k}I_m$, where $\omega = e^{2\pi i/d}$, for all $1 \leq k \leq r$.

When $m = 1$, we call such a solution a *scalar solution*. In the physics literature, an operator solution is usually called a *quantum solution* and a scalar solution is called a *classical solution*. A linear constraint system that admits no classical solutions is called *contextual*; otherwise, it is called *non-contextual*. Note that in this paper, we restrict our attention to operator solutions over finite-dimensional Hilbert spaces. The finiteness restriction can be removed for a more general discussion of the subject. For basic properties of linear constraint systems, we refer to Refs. 4, 7, 38, and 39.

B. Topological description

A linear constraint system can be formulated using hypergraphs. The data of a linear constraint system can be turned into a pair $(\mathfrak{H}, \tau)$, where $\mathfrak{H} = (V, E, \epsilon)$ is a hypergraph with a finite vertex set V, a finite edge set E, and an incidence weight ϵ and τ is a function $E \to \mathbb{Z}/d$. More concretely, let $\mathfrak{H}$ denote the hypergraph with $V = \{v_1, v_2, \ldots, v_c\}$, $E = \{e_1, e_2, \ldots, e_r\}$, where $e_k = \{v_i | M_{ki} \neq 0\}$, and $\epsilon_{e_k}(v_i) = M_{ki}$. The function τ is defined by $\tau(e_k) = b_k$. An operator solution can be regarded as a function $T : V \to U(m)$ where $T(v_i) = A_i$.

Remark IV.1. As in Sec. III A, let G be a group that contains a central subgroup isomorphic to μ_d. We can consider solutions over G instead of $U(m)$. Such an operator solution will be denoted by a function $T : V \to G$ where the group elements $\{T(v) | v \in V\}$ satisfy the d-torsion (1), commutativity (2), and linear constraint (3) conditions listed above.

We define a chain complex associated with the hypergraph

$$C_*(\mathfrak{H}) : C_2 \xrightarrow{\partial} C_1 \xrightarrow{0} C_0,$$

where

$$C_0 = \mathbb{Z}/d, \quad C_1 = \mathbb{Z}/d[V], \quad C_2 = \mathbb{Z}/d[E], \qquad \partial[e] = \sum_{v \in e} \epsilon_e(v)[v].$$

There is a corresponding cochain complex $C^*(\mathfrak{H})$. The function τ can be regarded as a 2-cocycle. We write $[\tau]$ for its cohomology class.

Let X be a CW complex. The set of n-cells will be denoted by X_n, and X^n will denote the n-skeleton. For each n-cell, there is an attaching map $\varphi^n : \partial D^n \to X^{n-1}$ and a characteristic map $\Phi^n : D^n \to X$. The exact sequence of homotopy groups associated with the pair (X^2, X^1) of complexes is given by

$$0 \to \pi_2(X^2) \to \pi_2(X^2, X^1) \xrightarrow{\partial} \pi_1(X^1) \to \pi_1(X^2) \to 0. \tag{4.1}$$

If X has a single 0-cell, then the fundamental group is generated by the homotopy classes $[\Phi^1]$ of the characteristic maps of 1-cells. The relative homotopy group $\pi_2(X, X^1)$ is generated, up to the action of $\pi_1(X^1)$, by the homotopy classes of the characteristic maps $[\Phi^2]$ of 2-cells (Ref. 40, Chap. II, Sec. 2.1). This observation applies to the quotient space $\bar{X} = X/X_0$ obtained by identifying the 0-cells in X. Let $q : X \to \bar{X}$ denote the quotient map. We will write $q^n : X^n \to \bar{X}^n$ to denote the induced map between the n-skeletons. The characteristic maps of $\bar{X}$ for $n \geq 1$ are given by the composites $\bar{\Phi}^n : D^n \xrightarrow{\Phi^n} X \xrightarrow{q} \bar{X}$.

Definition IV.2. A *topological realization* for the hypergraph $\mathfrak{H}$ is a connected two-dimensional CW complex $X = X(\mathfrak{H})$ that satisfies the following properties:

1. There are isomorphisms of sets: $X_1 \cong V$ and $X_2 \cong E$. The attaching (characteristic) maps corresponding to 1-cells and 2-cells will be labeled by V and E, respectively.
2. For each $e \in E$, the image of $\varphi_e^2 : \partial D^2 \to X^1$ is contained in the union of the images of the characteristic maps $\Phi_v^1 : D^1 \to X$, where $v \in e$.
3. For each $e \in E$, the boundary map $\partial : \pi_2(\bar{X}^2, \bar{X}^1) \to \pi_1(\bar{X}^1)$ in (4.1) satisfies

$$\partial[\bar{\Phi}_e^2] = \prod_{v \in e} [\bar{\Phi}_v^1]^{\epsilon_e(v)},$$

where the product is with respect to some ordering of the set e.

Remark IV.3. The notion of topological realization introduced above is more restricted than the one introduced in Ref. 7, Definition 6.1. Let $C_*(X)$ denote the chain complex X over $\mathbb{Z}/d$. Definition IV.2 induces a morphism of chain complexes $f_* : C_*(X) \to C_*(\mathfrak{H})$ such that f_1 and f_2 are isomorphisms, i.e.,

$$\begin{tikzcd}
C_2(X) \arrow[r, "\partial_2"] \arrow[d, "\cong"', "f_2"] & C_1(X) \arrow[r, "\partial_1"] \arrow[d, "\cong"', "f_1"] & C_0(X) \arrow[d, "f_0"] \\
\mathbb{Z}/d[E] \arrow[r, "\partial"] & \mathbb{Z}/d[V] \arrow[r, "0"] & \mathbb{Z}/d
\end{tikzcd} .$$

Construction IV.4. Let $(\mathfrak{H},\tau)$ be a loop control system (LCS) and $T : V \to G$ be an operator solution over G. For each edge $e \in E$ in the hypergraph, define the subgroup $A(e) \subset G$ (which is regarded as a discrete group) generated by $\mu_d \cup \{T(v)|v \in e\}$ and let $\bar{A}(e) = A(e)/\mu_d$. Let $X = X(\mathfrak{H})$ be a topological realization for $\mathfrak{H}$. We construct a map

$$f_T : X \to \bar{B}(\mathbb{Z}/d, G),$$

which is defined up to homotopy, as follows:

1. send each 0-cell in X_0 to the unique 0-cell of $\bar{B}(\mathbb{Z}/d, G)$,
2. send the 1-cell labeled by $v \in X_1$ to the 1-cell labeled by $[T(v)]$, the equivalence class of $T(v)$ under multiplication with elements in μ_d, and
3. by parts (2) and (3) of Definition IV.2, the boundary of a 2-cell labeled by $e \in X_2$ maps to a contractible loop in the subspace $B\bar{A}(e) \subset \bar{B}(\mathbb{Z}/d, G)$; extend this map to the interior of the disk by choosing a contracting homotopy that lies in $B\bar{A}(e)$.

Remark IV.5. Part (3) of Construction IV.4 requires some explanation. Let $Y = \bar{B}(\mathbb{Z}/d, G)$. Observe that f_T factors through the quotient map $q : X \to \bar{X}$ since all the 0-cells are identified. Let $\bar{f} : \bar{X} \to Y$ be such that $\bar{f}q = f_T$. This map induces a commutative diagram [see (4.1)]

$$\begin{tikzcd} \pi_2(\bar{X}, \bar{X}_2) \arrow[r, "\partial"] \arrow[d, "\bar{f}_2"] & \pi_1(\bar{X}^1) \arrow[d, "\bar{f}_1"] \\ \pi_2(Y, Y^1) \arrow[r, "\partial"] & \pi_1(Y^1). \end{tikzcd}$$

For a 2-cell labeled by $e \in X_2$, we have

$$\partial\bar{f}_2[\bar{\Phi}_e^2] = \bar{f}_1\partial[\bar{\Phi}_e^2] = \prod_{v\in e}\bar{f}_1[\bar{\Phi}_v^1]^{\epsilon_e(v)} = \prod_{v\in e}[T(v)]^{\epsilon_e(v)} = 1$$

as a consequence of the relation satisfied by the group elements $\{T(v)\}_{v\in e}$ (as part of the definition of a LCS). Therefore, the homotopy class of the image of the boundary of the 2-cell $\bar{\Phi}_e^2$ is contractible. Moreover, any two choices of a contracting homotopy extending the map on the boundary of a 2-cell of X are homotopic to each other since the image lies inside the subspace $B\bar{A}(e)$ [by part (2) of Definition IV.2], whose homotopy groups above degree 2 vanishes. Therefore, the map f_T is unique up to homotopy.

Let $[(X, x_0), (Y, y_0)]$ denote the set of pointed homotopy classes of maps between two based spaces. We will suppress the base points and simply write $[X, Y]$. This should not result in any confusion since in this paper, we do not consider the set of unpointed homotopy classes of maps.

Proposition IV.6. Let $(\mathfrak{H},\tau)$ be a linear constraint system.

1. *$(\mathfrak{H},\tau)$ has a scalar solution if and only if $[\tau] = 0$ in $H^2(C(\mathfrak{H}))$ and, thus, in the second cohomology group of any topological realization.*
2. *If T is an operator solution for $(\mathfrak{H},\tau)$, then $f_T^*(\gamma_G) = [\tau]$ for any map f_T constructed using the operator solution (Construction IV.4).*
3. *If $(\mathfrak{H},\tau)$ has an operator solution T and a topological realization X such that f_T induces the trivial map between the fundamental groups, then $(\mathfrak{H},\tau)$ has a scalar solution.*

Proof. Part (1) follows from the definition of the chain complex; see also Ref. 6. Part (2) follows from the observation that γ_G is the image of the identity map in $H^1(\mu_d, \mathbb{Z}/d) \cong \mathrm{Hom}(\mathbb{Z}_d, \mathbb{Z}_d)$ under the transgression map of the fibration $B\mu_d \to B(\mathbb{Z}/d, G) \to \bar{B}(\mathbb{Z}/d, G)$. In effect, the transgression map is computed using the connecting homomorphism $\delta : H^1(B\mu_d, \mathbb{Z}/d) \to H^2((B(\mathbb{Z}/d, G), B\mu_d), \mathbb{Z}/d)$ in the cohomology long exact sequence of the pair $(B(\mathbb{Z}/d, G), B\mu_d)$; see Ref. 41, p. 186. Let $\phi : \mu_d \to \mathbb{Z}/d$ be the 1-cocycle representing the identity map in cohomology. By definition, the connecting homomorphism $\delta[\phi]$ will be represented by the coboundary of a lift of ϕ to a 1-cochain on $B(\mathbb{Z}/d, G)$. To compute the value of the pull-back $f_T^*(\gamma)$ of the resulting 2-cocycle γ on a 2-cell of X labeled by $e \in E$, we can work with the fibration $B\mu_d \to BA(e) \to B\bar{A}(e)$ instead. Given the standard cell structure on $BA(e)$, it turns out that $f_T^*(\gamma) = -\tau$. To see this, observe that the description of the transgression map implies that the value of $f_T^*(\gamma)$ on the 2-cell Φ_e^2 can be computed by lifting the boundary of the composite $D^2 \xrightarrow{\Phi_e^2} X \xrightarrow{f_T} \bar{B}(\mathbb{Z}/d, G)$, which factors through the subspace $B\bar{A}(e)$, to $BA(e)$ and then identifying the homotopy class of the loop in $\pi_1(B\mu_d) \cong \mathbb{Z}/d$. The resulting element in μ_d is precisely $-\tau(e)$ by definition of f_T and the relation $\omega^{-\tau(e)}\prod_{v\in e}T(v)^{\epsilon_e(v)} = I_m$. A special case of part (3) is proved in Ref. 7, applicable to hypergraphs with $\epsilon_e(v) = \pm 1$, which has a simply connected topological realization. We sketch an alternative approach for the general case: the class γ_G comes from a class in $H^2(B\bar{G}, \mathbb{Z}/d)$, where $\bar{G} = G/\mu_d$, still denoted by the same symbol.

Let $H \subset G$ denote the discrete subgroup generated by $\{T(v)|v \in V\}$ together with μ_d. Let $\bar{H}$ denote the quotient H/μ_d. Since f_T induces the trivial map on π_1, we can reduce to the case where $\pi_1(X) = 1$ by collapsing the non-contractible loops in X. The composite

$$X \xrightarrow{f_T} \bar{B}(\mathbb{Z}/d, G) \subset B\bar{G}$$

factors through a map $X \to B\bar{H}$. Since $\pi_1(X) = 1$ and the homotopy groups of $B\bar{H}$ vanish above dimension 1, this map is null homotopic. Therefore, using part (2), we have $f_T^*(\gamma_G) = [\tau] = 0$. □

Example IV.7. Mermin square[17] is the prominent example of a contextual linear constraint system, i.e., it admits an operator solution but not a scalar solution. Let P_n denote the subgroup in $U(2^n)$ consisting of matrices of the form $i^a A_1 \otimes A_2 \otimes \cdots \otimes A_n$, where $a \in \mathbb{Z}/4$ and each A_i is one of the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The linear constraint system $(\mathfrak{H}_{\mathrm{sq}}, \tau_{\mathrm{sq}})$ and an operator solution $T_{\mathrm{sq}} : V \to P_2$ are depicted in Fig. 1 (left figure). As depicted in the right figure, $\mathfrak{H}_{\mathrm{sq}}$ has a topological realization given by a torus. The class $[\tau_{\mathrm{sq}}]$ is non-zero since the cocycle evaluates to 1 on the torus. Therefore, the linear constraint system does not admit a scalar solution.[6]

Another linear constraint system constructed in Ref. 17 is the Mermin star linear constraint system, which we denote by $(\mathfrak{H}_{\mathrm{st}}, \tau_{\mathrm{st}})$. An operator solution $T_{\mathrm{st}} : V \to P_3$ is displayed in Fig. 2 (left figure). The corresponding topological realization is again a torus but with a different cell structure (right figure); see Ref. 6.

C. Computing the homotopy classes

Definition IV.8. Let X be a pointed connected two-dimensional CW complex. Consider the collection of triples $(\mathfrak{H}, \tau, T)$ consisting of a linear constraint system $(\mathfrak{H}, \tau)$ over $\mathbb{Z}/d$ where $\mathfrak{H}$ admits a topological realization homotopy equivalent to X and an operator solution T over G. Two such triples $(\mathfrak{H}_0, \tau_0, T_0)$ and $(\mathfrak{H}_1, \tau_1, T_1)$ are said to be equivalent if f_{T_0} and f_{T_1} are homotopic as pointed maps. We write $\mathrm{Sol}(X; d, G)$ for the set of equivalence classes and refer to this set as the set of equivalence classes of operator solutions for (X, d) over G.

The equivalence classes of operator solutions map to the (pointed) homotopy classes of maps

$$\theta : \mathrm{Sol}(X; d, G) \hookrightarrow [X, \bar{B}(\mathbb{Z}/d, G)].$$

The target can be computed using an algebraic category (the category of *crossed modules*[42]) that captures the behavior of the homotopy category of two-dimensional CW complexes.

Let $\bar{\pi}_i$ denote the ith homotopy group of $\bar{B}(\mathbb{Z}/d, G)$.

Proposition IV.9. Let X be a connected two-dimensional CW complex. Sending a map to the homomorphism induced on π_1 gives a surjective map

$$\pi : [X, \bar{B}(\mathbb{Z}/d, G)] \to \mathrm{Hom}(\pi_1 X, \bar{\pi}_1)$$

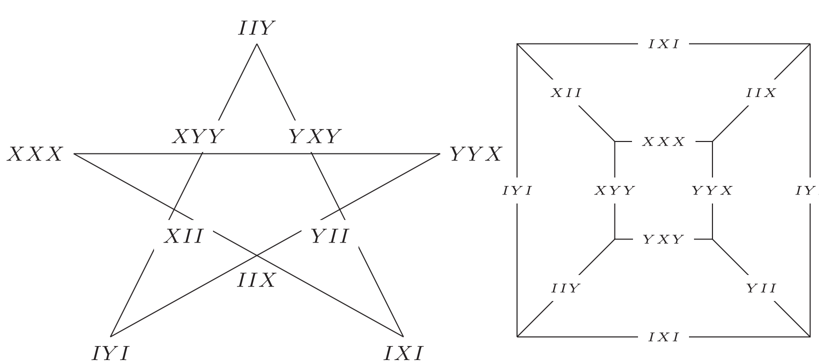

FIG. 2. (Left figure) $\mathfrak{H}_{\mathrm{st}}$ consists of ten vertices and five edges each consisting of four vertices in each line, and all the incidence weights are equal to 1. The function τ_{st} takes the value 0 for each hyperedge except the horizontal line. (Right figure) On the torus, τ_{st} specifies a 2-cocycle that assigns 0 to each cell except $\{XXX, YYX, YXY, XYY\}$, which is assigned 1.

such that for a fixed homomorphism α*, the preimage is given by*

$$\pi^{-1}(\alpha) \cong H^2(\tilde{X}, (\bar{\pi}_2)_\alpha),$$

where $(\bar{\pi}_2)_\alpha$ *is the* $\pi_1(X)$*-module determined by the homomorphism* α*.*

Proof. The statement holds for $[X, Y]$, where Y is an arbitrary CW complex. We will construct maps

$$Y \xrightarrow{r} \bar{Y} \xrightarrow{s} Y_{(2)},$$

where $Y_{(2)}$ is a two-dimensional CW complex, and the maps r and s are three-equivalences, i.e., each map induces an isomorphism on π_i for $0 \le i < 3$ and a surjection for $i = 3$. In this case, $r_* : [X, Y] \to [X, \bar{Y}]$, and similarly, s_* are bijections (Ref. 43, Corollary 23, p. 405). Before the construction, we first show how to finish the proof of the statement.

The set $[X, Y_{(2)}]$ can be computed algebraically; for details, we refer to Ref. 40, Chap. II, Sec. 4.2. Let us write $[X, Y_{(2)}]_\alpha$ for the set of homotopy classes of maps that induce the homomorphism α between the fundamental groups. The (cellular) chain complex for the universal cover $\tilde{X}$ consists of $\pi_1(X)$-modules, and we can talk about the cohomology groups $H^n(\tilde{X}, (\pi_2 Y_{(2)})_\alpha)$, where $\pi_2 Y_{(2)}$ is regarded as a $\pi_1(X)$-module via the homomorphism α. The cohomology group $H^2(\tilde{X}, (\pi_2 Y_{(2)})_\alpha)$ acts on $[X, Y_{(2)}]_\alpha$ in a transitive way (Ref. 40, Chap. II, Theorem 4.11), and this action determines a bijection

$$[X, Y_{(2)}]_\alpha \cong H^2(\tilde{X}, (\pi_2 Y_{(2)})_\alpha).$$

We turn to the construction of r and s. The first map is obtained by killing homotopy groups of Y above dimension 2. Construction of the second map uses the theory of crossed modules. The fundamental property we will use is that any free crossed module over a free base group is realizable by a two-dimensional CW complex, and maps between such crossed modules come from maps between the CW complexes that realize them (Ref. 40, Chap. II). Let us apply this to the crossed module given by the connecting homomorphism

$$\partial : \pi_2(\bar{Y}, \bar{Y}^1) \to \pi_1(\bar{Y}^1). \tag{4.2}$$

By the realization result, there is a two-dimensional CW complex $Y_{(2)}$ such that the crossed module $\partial : \pi_2(Y_{(2)}, Y^1_{(2)}) \to \pi_1(Y^1_{(2)})$ is isomorphic to the one given in (4.2). We will show that this isomorphism is realized by a map $s : Y_{(2)} \to \bar{Y}$. We start the construction of s from the 1-st skeleton. We can find a map $Y^1_{(2)} \to \bar{Y}^1$ that induces the desired isomorphism on π_1. Composing this map with the inclusion $\bar{Y}^1 \subset \bar{Y}$, we obtain $Y^1_{(2)} \to \bar{Y}$. This map lifts to a map $Y_{(2)} \to \bar{Y}$ since the set of 2-cells is a basis for the free group $\pi_2(Y_{(2)}, Y^1_{(2)})$ and the isomorphism between the crossed modules implies precisely the lifting condition in the algebraic language. □

Example IV.10. We will discuss an interesting example related to the Pauli group. For properties of this group, we refer to Ref. 44. The Pauli group P_n defined in Example IV.7 has a generalization for all primes p, which has a similar description as tensor products of $p \times p$ unitary matrices. As an abstract group, P_n is an almost extraspecial two-group for $p = 2$ and an extraspecial p-group of exponent p for odd primes. There is an irreducible complex representation of P_n, which allows us to regard it as a subgroup in $U(p^n)$.

The central quotient group of P_n, which will be denoted by E_n, is an elementary Abelian p-group of rank $2n$. There is a symplectic bilinear form $\mathfrak{b}$ on E_n induced by the commutator of P_n. We can choose a symplectic basis $\{x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n\}$ for E_n. Let $B(\mathfrak{b}, E_n)$ denote the geometric realization of the simplicial set

$$[k] \mapsto \{(a_1, a_2, \ldots, a_k) \in (E_n)^k \mid \mathfrak{b}(a_i, a_j) = 0 \;\forall i, j\}.$$

The simplicial structure is induced from the simplicial set $E_n^{\cdot}$ whose geometric realization is the classifying space BE_n. The space $\bar{B}(\mathbb{Z}/p, P_n)$ can be identified with $B(\mathfrak{b}, E_n)$. For $n \ge 2$, *it is known that*

$$\pi_1 \bar{B}(\mathbb{Z}/p, P_n) = \pi_1 B(\mathfrak{b}, E_n) \cong \begin{cases} \mathbb{Z}/2 \times E_n, & p = 2, \\ P_n, & p > 2, \end{cases}$$

and the higher homotopy groups are given by

$$\pi_i \bar{B}(\mathbb{Z}/p, P_n) = \pi_i B(\mathfrak{b}, E_n) \cong \pi_i\left(\bigvee^{N_{p,n}} S^n\right), \quad i \ge 2,$$

where $N_{p,n}$ has an explicit formula (Ref. 8, Sec. 6). Therefore, according to Proposition IV.9, the map

$$[X, \bar{B}(\mathbb{Z}/p, P_n)] \to \mathrm{Hom}(\pi_1 X, \bar{\pi}_1)$$

is an isomorphism when $n \geq 3$. However, for $n = 2$, it is only surjective and the kernel depends on the $\bar{\pi}_1$-module structure of $\bar{\pi}_2$, which is currently unknown.

The canonical class can be described as

$$\gamma_{P_n} = \begin{cases} x_0^2 + \sum_{i=1}^n x_i \cup z_i, & p = 2, \\ 0, & p > 2, \end{cases} \tag{4.3}$$

where $\{x_0, x_1, \ldots, x_n, z_1, \ldots, z_n\}$ is a basis for $\mathbb{Z}/2 \times E_n$; see Ref. 44 for details. Therefore, for odd p, every linear constraint system has a scalar solution if it has an operator solution over P_n. However, for $p = 2$, this depends on the map induced on π_1 as a result of the cup product decomposition in (4.3).

The operator solution T_{sq} of the Mermin square linear system $(\mathfrak{H}_{\mathrm{sq}}, \tau_{\mathrm{sq}})$ introduced in Example IV.7 gives a non-trivial class $[f_{T_{\mathrm{sq}}}]$ in $[S^1 \times S^1, \bar{B}(\mathbb{Z}/2, P_2)]$. For $n \geq 2$, let us write

$$T_n = T_{\mathrm{sq}} \otimes I_{2^{n-2}} \tag{4.4}$$

for the operator solution obtained by tensoring with the identity matrix $A \mapsto A \otimes I_{2^{n-2}}$. Then, $[f_{T_n}]$ gives a non-trivial class in $[S^1 \times S^1, \bar{B}(\mathbb{Z}/2, P_n)]$ for all $n \geq 2$. Similarly, the Mermin star example $(\mathfrak{H}_{\mathrm{st}}, \tau_{\mathrm{st}})$ specifies a class in $[S^1 \times S^1, \bar{B}(\mathbb{Z}/2, P_3)]$. It turns out that this class coincides with $[f_{T_3}]$ since there is a refined cell structure (Ref. 6) on the torus, as depicted in Fig. 3. More precisely, there is a commutative diagram

$$\begin{tikzcd} X_{sq} \arrow[r, hook] \arrow[d] & X_{ref} \arrow[d] & X_{st} \arrow[l, hook'] \arrow[d] \\ \bar{B}(\mathbb{Z}/2, P_2) \arrow[r, "\otimes I_2"] & \bar{B}(\mathbb{Z}/2, P_3) \arrow[r, equal] & \bar{B}(\mathbb{Z}/2, P_3) \end{tikzcd}$$

relating the topological realizations $X = S^1 \times S^1$ with different cell structures as indicated by the subscripts.

D. Application of $C(d, m)$-cohomology

Now, we focus on operator solutions in unitary groups. For notational simplicity, let us write $\mathrm{Sol}(X; d, m)$ for the set of equivalence classes of operator solutions over $U(m)$. Recall the map

$$\bar{\imath}_m : \bar{B}(\mathbb{Z}/d, U(m)) \to \bar{B}(d, m)$$

introduced in (3.11). Composing with $\bar{\imath}_m$ gives a map

$$\hat{\theta} : \mathrm{Sol}(X; d, m) \hookrightarrow [X, \bar{B}(\mathbb{Z}/d, U(m))] \xrightarrow{(\bar{\imath}_m)_*} C(d, m)(X),$$

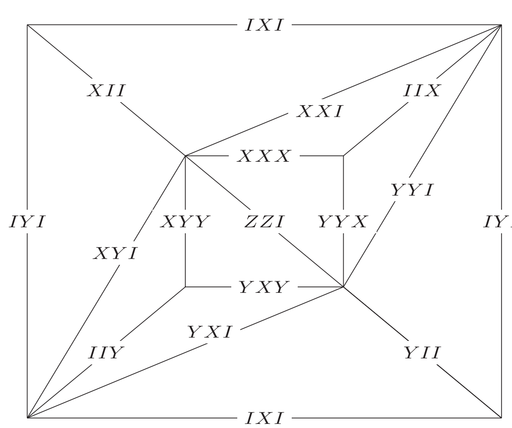

FIG. 3. Refined topological realization.

where we have identified $[X, \bar{B}(d,m)]$ with the 0-th $C(d,m)$-cohomology of X since the target space is the infinite loop space associated with the spectrum representing the cohomology theory. Given an operator solution T, the image of f_T under $(\bar{\imath}_m)_*$ will be denoted by $\hat{f}_T$.

Remark IV.11. By Lemma III.6, the pull-back $\hat{f}_T^*(\gamma_m^{\mathbb{S}})$ coincides with $f_T^*(\gamma_m)$. Therefore, for a linear constraint system, the existence of a scalar solution is determined in a stable manner, i.e., $\hat{f}_T^*(\gamma_m^{\mathbb{S}}) = 0$ if and only if a scalar solution exists.

Corollary IV.12. Let $(\mathfrak{H}, \tau)$ be a linear constraint system over $\mathbb{Z}/d$ and X be a topological realization for $\mathfrak{H}$.

1. *If $(\mathfrak{H}, \tau)$ has an operator solution and $H^2(X, (\mathbb{Z}/d)_m) = 0$, then $(\mathfrak{H}, \tau)$ has a scalar solution.*
2. *If d and m are coprime, then $C(d,m)(X) = 0$. In particular, $(\mathfrak{H}, \tau)$ has a scalar solution if it has an operator solution over $U(m)$.*
3. *If $\pi_1(X)$ is trivial and $[\tau] \neq 0$, then $(\mathfrak{H}, \tau)$ does not have an operator solution over $U(m)$ for any $m \geq 2$.*

Proof. Let T be an operator solution for $(\mathfrak{H}, \tau)$ and $f : \mathbb{S} \wedge X \to C(d,m)$ represent the class $\hat{\theta}(T)$. If $H^2(X, (\mathbb{Z}/d)_m) = 0$, then by Lemma III.6, part (2), $f_T^*(\gamma_m) = 0$ and the fibration $B\mu_d \to \tilde{X} \to X$ classified by $f_T^*(\gamma_m)$ splits, i.e., $\tilde{X} \simeq X \times B\mu_d$. Choosing a splitting $X \to \tilde{X}$ and composing with the map $\tilde{X} \to B(\mathbb{Z}/d, U) \xrightarrow{\det} B\mu_d$ shows that there is a class α in $k\mu_d(X)$ such that $[f] = (\zeta(\alpha), \mathrm{cl}(f))$, where $\mathrm{cl}(f)$ is regarded as an element of $H^2(X, (\mathbb{Z}/d)_m)$, under the decomposition given in Theorem III.5. Since $H^2(X, (\mathbb{Z}/d)_m) = 0$, we have $\mathrm{cl}(f) = 0$ and (1) follows from Proposition IV.6, parts (1) and (2). Part (2) follows from part (1) since if $(d,m) = 1$, then $H^2(X, (\mathbb{Z}/d)_m) = 0$. Part (3) follows from Proposition IV.6, part (3). The existence of an operator solution implies that $[\tau] = 0$ since X is simply connected. □

E. The Mermin class

In the physics literature, a quantum system with Hilbert space $(\mathbb{C}^2)^{\otimes n}$ is called an *n-qubit*. Such systems play a significant role in quantum information theory. Operator solutions in $U(2^n)$ of linear constraint systems over $\mathbb{Z}/2$ produce classes in $C(d,m)$-cohomology, where $d = 2$ and $m = 2^n$. Theorem III.5 gives an isomorphism

$$C(2, 2^n)(X) \cong H^1(X, \mathbb{Z}/2) \oplus H^2(X, \mathbb{Z}/2). \tag{4.5}$$

We will construct non-trivial classes that come from the operator solutions of the Mermin square linear constraint system described in Example IV.7. Our topological realization is a torus $X = S^1 \times S^1$. An operator solution for $n = 2$ is given in Fig. 1. Let T_1 denote this solution. We define an operator solution in $U(2^n)$ by tensoring with the identity as in (4.4), i.e., by constructing an operator solution T_n defined by $T_n(v) = T_1(v) \otimes I_{2^{n-1}}$ for $v \in V$. Let $[T_n]$ denote the class of this solution in $\mathrm{Sol}(S^1 \times S^1; 2, 2^n)$. Let M_n denote the class $\hat{\theta}(T_n)$ in $C(2, 2^n)(S^1 \times S^1)$. This class will be called the *Mermin class*. As a consequence of the isomorphism (4.5), M_n is represented by a pair of cohomology classes

$$(\varphi_1; \varphi_2) \in H^1(X, \mathbb{Z}/2) \oplus H^2(X, \mathbb{Z}/2).$$

For each $n \geq 2$, the cohomology class $[\tau] \neq 0$ since, as we have seen in Example IV.7, the Mermin square linear constraint system does not admit a scalar solution. Therefore, φ_2 is the non-trivial class in $H^2(S^1 \times S^1, \mathbb{Z}/2) = \mathbb{Z}/2$. To determine φ_1, we will construct a homotopy commutative diagram

$$\begin{tikzcd} X_{\varphi_2} \arrow[r] \arrow[d] & B(\mathbb{Z}/2, U) \arrow[d] \\ X \arrow[r] & \bar{B}(2, 2^n), \end{tikzcd} \tag{4.6}$$

where X_{φ_2} is the $B\mu_2$-bundle determined by the non-trivial class φ_2. The bottom map in the diagram is given by the composite

$$X \xrightarrow{f_{T_n}} \bar{B}(\mathbb{Z}/2, P_n) \to \bar{B}(\mathbb{Z}/2, U(2^n)) \xrightarrow{\bar{\imath}_{2^n}} \bar{B}(2, 2^n), \tag{4.7}$$

where the middle map is induced by the inclusion $P_n \subset U(2^n)$. Note that the map obtained using Construction IV.4 factors through $\bar{B}(\mathbb{Z}/2, P_n)$ since the operator solution T_n is over the Pauli group P_n. Each map in (4.7) can be extended to a map between homotopy fibrations over $B^2\mu_2$ corresponding to the cohomology classes φ_2, γ_{P_n} [see (4.3)], γ_{2^n}, and $\gamma_{2^n}^{\mathbb{S}}$. The top map in (4.6) is given by the composite map induced between the homotopy fibers. Now, we observe that the class φ_1 is determined by the map induced on $\pi_1 X \to \pi_1 \bar{B}(2, 2^n)$. Applying π_1 to the diagram in (4.6), we obtain

$$\begin{tikzcd} \pi_1 X_{\varphi_2} \arrow[r] \arrow[d, two heads] & \pi_1 B(\mathbb{Z}/2, U) \arrow[d, "\cong"] \\ \pi_1 X \arrow[r] & \pi_1 \bar{B}(2, 2^n) . \end{tikzcd}$$

Let $\tilde{x}$ and $\tilde{z}$ denote the elements lifting the generators $x = (1,0)$ and $z = (0,1)$ of the quotient group $\pi_1 X = \mathbb{Z}^2$. It suffices to determine the images of $\tilde{x}$ and $\tilde{z}$ under the top horizontal map. Figure 1 tells us that $\tilde{x}$ maps to the loop determined by $X \otimes I_{2^{n-1}}$ and $\tilde{z}$ maps to $Z \otimes I_{2^{n-1}}$. We can understand the induced map on π_1 by composing with the determinant map $\det : B(\mathbb{Z}/2, U) \to B\mu_2$. This amounts to taking the determinant of the matrices representing the loops, which gives 1 in both cases. Thus, both of the loops map to the trivial loop in $B\mu_2$. Therefore, $\varphi_1 = (0,0) \in (\mathbb{Z}/2)^2$ (also follows from Corollary III.7). In summary, the Mermin class M_n is represented by $(0,0;1)$. Since f_{T_n} induces the trivial map on π_1, it factors as

$$\begin{tikzcd} S^1 \times S^1 \arrow[r, "f_{T_n}"] \arrow[d] & \bar{B}(2,2^n), \\ S^2 \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd} \tag{4.8}$$

where the vertical map collapses the non-contractible loops corresponding to x and z. The homotopy class of $\bar{f}$ is the generator of $\pi_2 C(2,2^n) = \mathbb{Z}/2$. By slight abuse of notation, we will also write M_n for this class and refer to it as the Mermin class as well.

Let us compare to the unstable situation. By looking at the homotopy fibers of the maps in (4.7) regarded as homotopy fibrations over $B\mu_2$ as before, we see that the diagram (4.6) factors as

$$\begin{tikzcd} X_{\varphi_2} \arrow[r] \arrow[d] & B(\mathbb{Z}/2, P_n) \arrow[r, "\tilde{g}"] \arrow[d] & B(\mathbb{Z}/2, U) \arrow[d] \\ X \arrow[r, "f"] & \bar{B}(\mathbb{Z}/2, P_n) \arrow[r, "g"] & \bar{B}(2,2^n), \end{tikzcd}$$

where $g^*(\gamma^{\mathbb{S}}_{2^n}) = \gamma_{P_n}$. The homotopy class $[f]$ is non-trivial in $[X, \bar{B}(\mathbb{Z}/2, P_n)]$, which surjects onto $\mathrm{Hom}(\pi_1 X, \bar{\pi}_1)$ as we have seen in Example IV.10. However, the composite $g\,f$ induces the trivial map on π_1. This is not in conflict with part (3) of Proposition IV.6 if we take $G = U$. This is because the subgroup $\mu_2 \hookrightarrow U$ is not a central, or even not a normal, subgroup. Part (3) of Proposition IV.6 also implies that the diagonal map $\bar{f}$ in (4.8) does not factor through $\bar{B}(\mathbb{Z}/2, U(2^n))$.

F. Relation to SPT phases

Replacing $U(m)$ with the orthogonal group $O(m)$ gives the real versions of the spectra considered in this paper. As explained in the Appendix, we can define the real symmetric K-theory spectrum ko_{sym} and the spectrum $C_{\mathbb{R}}(m)$, the real version of the $C(d,m)$ spectrum. The Mermin square construction and its n-qubit version T_n introduced in Sec. IV E can be regarded as an operator solution over $O(2^n)$ since the matrices involved have real entries. Let $M_n^{\mathbb{R}} \in C_{\mathbb{R}}(2^n)(S^1 \times S^1)$ denote the corresponding class. As in the complex case, we find that $M_n^{\mathbb{R}}$ can be identified with the generator of the quotient in the exact sequence [see (A6)]

$$0 \to \pi_2 ko_{\mathrm{sym}} \to \pi_2 C_{\mathbb{R}}(2^n) \to H^2(S^2, \mathbb{Z}/2) \to 0.$$

The generator of $\pi_2 ko_{\mathrm{sym}} = \mathbb{Z}/2$ has also a physical interpretation. It can be realized as a non-trivial SPT phase: The ko-orientation $M\mathrm{Spin} \to ko$ of the spin cobordism spectrum $M\mathrm{Spin}$ is highly connected (this follows from Ref. 45, Theorem 2.2; see also Ref. 46, Remark 6.1). In particular, it induces an isomorphism on π_2. Therefore, smashing this map with $B\mu_2$ induces an isomorphism $\pi_2(M\mathrm{Spin} \wedge B\mu_2) \to \pi_2(ko \wedge B\mu_2)$. The generator of $\pi_2(M\mathrm{Spin} \wedge B\mu_2)$ is identified as the Gu–Wen phase, a fermionic SPT phase constructed in Ref. 47; see also Ref. 48, Sec. 5. This class hits the generator of $\pi_2 ko_{\mathrm{sym}}$ under the identification $\pi_2(ko \wedge B\mu_2) \cong \pi_2(ko_{\mathrm{sym}})$.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: REAL VERSION

In this section, we describe the real versions of the constructions introduced in Secs. II D and III B. The main idea is to replace $U(m)$ with the orthogonal group $O(m)$. As in the complex case, we obtain a commutative version of the topological real K-theory and a cohomology theory, denoted by $C_{\mathbb{R}}(m)$, which can be used to study operator solutions of LCSs over $O(m)$. Every Abelian subgroup of $O(m)$ can be conjugated into $\mathrm{SO}(2)^j \times O(1)^{m-2j}$ for some $j \leq \lfloor m/2 \rfloor$ (Ref. 49, Appendix A). Thus, a homomorphism $f : \mathbb{Z}^m \to O(m)$, when regarded as a representation, is isomorphic to a direct sum

$$f \cong \eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_j \oplus \ell_1 \oplus \ell_2 \oplus \cdots \oplus \ell_{2m-j},$$

where $\eta_i : \mathbb{Z}^m \to SO(2)$ and $\ell_i : \mathbb{Z}^m \to O(1)$. In particular, a matrix is diagonalizable in $O(m)$ if and only if it is *symmetric*, i.e., $A^T = A$. Thus, in the real case, we will consider 2-torsion orthogonal matrices. The resulting space $B(\mathbb{Z}/2, O(m))$ is constructed from pairwise commuting symmetric orthogonal matrices. We can stabilize over m, similar to the complex case, to obtain $B(\mathbb{Z}/2, O)$. The real version of Proposition II.4 gives a homeomorphism

$$ko((\mu_2)^n) \xrightarrow{\cong} \mathrm{Hom}((\mathbb{Z}/2)^n, O),$$

where ko is the corresponding Γ-space of the connective real K-theory spectrum. Similar to the complex case, this homeomorphism is compatible with the simplicial structures and induces a homeomorphism

$$ko(B\mu_2) \xrightarrow{\cong} B(\mathbb{Z}/2, O).$$

From ko, we can construct the Γ-space $ko_{B\mu_2}$ and consider the associated spectrum

$$ko_{\mathrm{sym}} = ko_{B\mu_2}(\mathbb{S}).$$

We will refer to ko_{sym} as the *real symmetric K-theory*. There is a similar stable equivalence $ko_{\mathrm{sym}} \simeq ko \wedge B\mu_2$ and a weak equivalence $B(\mathbb{Z}/2, O) \simeq \Omega^\infty ko_{\mathrm{sym}}$ by the real versions of Propositions II.4 and II.7 (see also Ref. 16, Remark 2.9). The homotopy groups of $B(\mathbb{Z}/2, O)$ are isomorphic to the (reduced) ko-homology of $B\mu_2$,

$$\pi_{8k+\epsilon}(ko_{\mathrm{sym}}) = \begin{cases} \mathbb{Z}/2, & \epsilon = 1, 2, \\ \mathbb{Z}/2^{4k+3}, & \epsilon = 3, \\ \mathbb{Z}/2^{4k+4}, & \epsilon = 7, \\ 0, & \text{otherwise} \end{cases} \tag{A1}$$

(taken from the unreduced version in Ref. 50, Sec. 12.2.D).

Similar to the complex case, $\pi_1(ko_{\mathrm{sym}})$ can be understood by considering the composition of $B\mu_2 \subset B(\mathbb{Z}/2, O)$ with the determinant map $\det : B(\mathbb{Z}/2, O) \to B\mu_2$. This composition is the identity map and splits off the $\mathbb{Z}/2$ in the first homotopy group. Moreover, the unit map $\mathbb{S} \to ko$ is 3-connected, i.e., induces an isomorphism on π_i for $0 \leq i < 3$ and a surjection on $i = 3$ [mainly because $\pi_1(ko)$ is generated by the image of the Hopf map $\eta \in \pi_1(\mathbb{S})$ (Ref. 51, Theorem 3.1.26)]. From the Atiyah–Hirzebruch spectral sequence, we see that $\mathbb{S} \wedge B\mu_2 \to ko \wedge B\mu_2$ is also 3-connected. In addition, since$\pi_3(QB\mu_2) \cong \mathbb{Z}/8$,[52] the map $Q(B\mu_2) \to B(\mathbb{Z}/2, O)$ extending the inclusion $B\mu_2 \subset B(\mathbb{Z}/2, O)$ induces an isomorphism on π_r for $0 \leq r \leq 3$. Therefore, we have

$$\pi_r^s(B\mu_2) = \pi_r(QB\mu_2) \cong \begin{cases} 0, & r = 0, \\ \mathbb{Z}/2, & r = 1, 2, \\ \mathbb{Z}/8, & r = 3. \end{cases} \tag{A2}$$

Let $\delta_m : \mathbb{S} \to ko$ denote the morphism of Γ-spaces corresponding to the real version of (3.1). There is an induced map of spectra $\delta_m \wedge \mathrm{id} : \mathbb{S} \wedge B\mu_2 \to ko \wedge B\mu_2$.

Definition A.1. The spectrum $C_{\mathbb{R}}(m)$ is obtained by killing the homotopy groups above degree 2 of the cofiber of $\delta_m \wedge \mathrm{id}$.

Theorem III.5 has also a real version where $C(d, m)$ is replaced by $C_{\mathbb{R}}(m)$.

Lemma A.2. Let $(\delta_m \wedge \mathrm{id})_* : \pi_n(\mathbb{S} \wedge B\mu_2) \to \pi_n(ko \wedge B\mu_2)$ *denote the homomorphism between the homotopy groups induced by the spectrum map* $\delta_m \wedge \mathrm{id} : \mathbb{S} \wedge B\mu_2 \to ko \wedge B\mu_2$*. Then,*

$$(\delta_m \wedge \mathrm{id})_* = \underbrace{(\delta_1 \wedge \mathrm{id})_* + \cdots + (\delta_1 \wedge \mathrm{id})_*}_{m}.$$

Proof. We first show that $\delta_m : \mathbb{S} \to ko$ is the m-fold sum $\delta_1 + \cdots + \delta_1$. By definition, the Γ-space morphism δ_1 is completely determined by its value on $\mathbb{S}(1_+) = 1_+$, which sends 1 to the subspace $\langle e_1 \rangle \subset \mathbb{R}^\infty$. We have an H-space structure on $ko(1_+)$, which comes from being a special Γ-space, which is induced by

$$ko(1_+) \times ko(1_+) \to ko(1_+) \tag{A3}$$

that sends (V, W) to the direct sum $V \oplus W$. This H-space structure is responsible for the Abelian group structure on the set of homotopy classes of maps $[\mathbb{S}, ko]$. Thus, $\delta_1 + \delta_1$ is computed by using (A3). In effect, we obtain a map $\mathbb{S}(1_+) \to ko(1_+)$ that sends 1 to the direct sum $\langle e_1 \rangle \oplus \langle e_1 \rangle \cong \langle e_1, e_2 \rangle$. This is precisely δ_2. In a similar way, we can proceed to show that δ_m is the m-fold sum of δ_1 as claimed. To finish the proof, we rely on the following basic properties of the homotopy category of spectra: Let K, L, M be spectra, X be a space, and $f, f' : L \to M$ be maps of spectra.

1. $\wedge\mathrm{id} : [K, L] \to [K \wedge X, L \wedge X]$, defined by $f \mapsto f \wedge \mathrm{id}$, is a homomorphism of Abelian groups, i.e., $(f + f') \wedge \mathrm{id} = f \wedge \mathrm{id} + f' \wedge \mathrm{id}$.
2. Consider the induced map $f_* : [K, L] \to [K, M]$, defined by $f_*(g) = fg$, and f'_* similarly defined. Then, $(f + f')_* = f_* + f'_*$.

Both of these results follow from the basic properties of addition of spectrum maps. We apply (1) to $[\mathbb{S}, ko] \to [\mathbb{S} \wedge B\mu_2, ko \wedge B\mu_2]$ and obtain

$$\delta_m \wedge \mathrm{id} = (\delta_1 + \cdots + \delta_1) \wedge \mathrm{id} = (\delta_1 \wedge \mathrm{id}) + \cdots + (\delta_1 \wedge \mathrm{id}). \tag{A4}$$

Note that the map induced on π_n can be thought of as a map

$$(\delta_m \wedge \mathrm{id})_* : [\Sigma^n \mathbb{S}, \mathbb{S} \wedge B\mu_2] \to [\Sigma^n \mathbb{S}, ko \wedge B\mu_2]. \tag{A5}$$

Now, we apply (2) to the decomposition given in (A4) and obtain that

$$(\delta_m \wedge \mathrm{id})_* = (\delta_1 \wedge \mathrm{id})_* + \cdots + (\delta_1 \wedge \mathrm{id})_*.$$

□

A similar result applies to the complex version and can be used to give an alternative Proof of Lemma III.3. In the real case, $\mathbb{S} \wedge B\mu_2 \to ko \wedge B\mu_2$ induces an isomorphism on π_r for $0 \le r \le 3$ as observed above. Then, using Lemma A.2 together with the homotopy groups of ko_{sym} given in (A1) and of $Q(B\mu_2)$ given in (A2), we obtain an exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{\times m} \mathbb{Z}/2 \xrightarrow{\alpha} \pi_2 C_{\mathbb{R}}(m) \xrightarrow{\beta} \mathbb{Z}/2 \xrightarrow{\times m} \mathbb{Z}/2 \to \pi_1 C_{\mathbb{R}}(m) \to 0.$$

We see that if m is odd, then $\pi_i C_{\mathbb{R}}(m) = 0$ for $i = 1, 2$. Thus, the interesting case is when m is even. In this case, the homotopy groups fit into the exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{\alpha} \pi_2 C_{\mathbb{R}}(m) \xrightarrow{\beta} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\cong} \pi_1 C_{\mathbb{R}}(m) \to 0. \tag{A6}$$

Let X be a connected two-dimensional CW complex. For m, even there is a commutative diagram

$$\begin{tikzcd}
H^1(X,\mathbb{Z}/2) \arrow[r, "\delta"] & H^2(X,\mathbb{Z}/2) \arrow[r, "\alpha_*"] \arrow[d, hook] & H^2(X,\pi_2 C_{\mathbb{R}}(m)) \arrow[r, two heads, "\beta_*"] \arrow[d, hook] & H^2(X,\mathbb{Z}/2) \arrow[d, equal] \\
 & ko_{\mathrm{sym}}(X) \arrow[r, "\zeta"] \arrow[d, two heads] & C_{\mathbb{R}}(m)(X) \arrow[r, "\mathrm{cl}"] \arrow[d, two heads] & H^2(X,\mathbb{Z}/2) \\
 & H^1(X,\mathbb{Z}/2) \arrow[r, "\cong"] & H^1(X,\mathbb{Z}/2).
\end{tikzcd}$$

where δ is the connecting homomorphism of the exact sequence associated with $0 \to \mathbb{Z}/2 \to \pi_2 C_{\mathbb{R}}(m) \to \mathbb{Z}/2 \to 0$. The image of ζ is contained in the kernel of cl.

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