Rings in which elements are a sum of a central and a unit element

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Abstract

In this paper we introduce a new class of rings whose elements are a sum of a central and a unit element, namely a ring R is called CU if each element $a \in R$ has a decomposition $a = c + u$ where c is central and u is unit. One of the main results in this paper is that if F is a field which is not isomorphic to $\mathbb{Z}_2$, then $M_2(F)$ is a CU ring. This implies, in particular, that any square matrix over a field which is not isomorphic to $\mathbb{Z}_2$ is the sum of a central matrix and a unit matrix.

1 Introduction

Throughout this paper, all rings considered are associative with an identity unless otherwise stated. In what follows, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the ring of integers, the ring of rational numbers and the ring of complex numbers, respectively. For a ring R, $U(R)$, $J(R)$, $N(R)$ and $C(R)$ denote the group of units, the Jacobson radical, the set of nilpotent elements and the center of R, respectively. For a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n and $M_n(R)$ denotes the full matrix ring over R, $U_n(R)$ is the subring of $M_n(R)$ consisting of all $n \times n$ upper triangular matrices and $GL_n(R)$ is the general linear group of $M_n(R)$.

If $A \in M_n(\mathbb{C})$, then there exists $c \in \mathbb{C}$ such that c is not an eigenvalue of A. Hence, $A - cI_n$ is invertible in $M_n(\mathbb{C})$. Thus, $A = cI_n + (A - cI_n)$ is a

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CU-decomposition of A. Therefore $M_n(\mathbb{C})$ is a CU ring. For more general case, let F be an algebraically closed field. Then every element in $M_n(F)$ is a sum of a central and a unit element in $M_n(F)$. Motivated by these facts, we investigate elementary properties of rings in which the elements are the sum of a central and a unit element.

2 Properties of CU Rings

In this section we introduce a CU ring and investigate its elementary properties. We give relations between CU rings and some related rings.

We now give our main definition.

Definition 2.1. Let R be a ring. An element $a \in R$ has a CU-decomposition if $a = c + u$ for some $c \in C(R)$ and $u \in U(R)$. Then R is called a CU ring if every element of R has a CU-decomposition.

Examples 2.2. (1) Every commutative ring is CU.
(2) Every local ring is CU.
(3) Every nilpotent element in any ring has a CU-decomposition.
(4) CU property for rings is preserved under homomorphic images.

Proof. (1) If $a \in R$, then $a = 1 + (a - 1)$.
(2) Let $a \in R$. Since R is local, a is invertible or $1 - a$ is invertible. If a is invertible, then $a = 0 + a$. On the other hand, we have $a = 1 + (a - 1)$, if $1 - a$ is invertible.
(3) Let a be any nilpotent element in a ring R. Then a has a CU-decomposition such that $a = 1 + (a - 1)$.
(4) Clear since invertible elements and central elements are preserved under epimorphisms of rings. ∎

Proposition 2.3. *Let R be a ring and $a \in R$. Then a has a CU-decomposition if and only if for each $p \in U(R)$, pap^{-1} has a CU-decomposition.*

Proof. Assume that a has a CU-decomposition. Then $a = c + u$ where c is central and u is invertible in R. For an invertible $p \in R$, $pap^{-1} = pcp^{-1} + pup^{-1}$ where $pcp^{-1} = c$ is central and pup^{-1} is invertible. Conversely, suppose that $pap^{-1} \in R$ has a CU-decomposition. So $pap^{-1} = t + x$ where t is central and x is invertible. Hence $a = p^{-1}tp + p^{-1}xp$ is a CU-decomposition of a. ∎

Lemma 2.4. *Let R be a commutative ring. For any positive integer n, $A \in M_n(R)$ has a CU-decomposition if and only if $A - cI_n \in GL_n(R)$ for some $c \in R$.*

Proof. Assume that $A \in M_n(R)$ has a CU-decomposition. By assumption there exists $c \in R$ such that $A - cI_n \in GL_n(R)$. Note that for any $B \in M_n(R)$, B is central in $M_n(R)$ if and only if there exists $c \in R$ such that $B = cI_n$. Conversely, suppose that for any $A \in M_n(R)$, there exists $c \in R$ such that $A - cI_n \in GL_n(R)$. As cI_n is central in $M_n(R)$, $A \in M_n(R)$ has a CU-decomposition. ∎

For a commutative ring R, the following result is important to find out whether $A \in M_n(R)$ has a CU-decomposition.

Corollary 2.5. *Let R be a commutative ring. For any positive integer n, $A \in M_n(R)$ has a CU-decomposition if and only if $f(c)$ is invertible in R for some $c \in R$, where $f(x) \in R[x]$ is the characteristic polynomial of A.*

Proof. Note that in a commutative ring R, for any $A \in M_n(R)$ and $c \in R$, $A - cI_n \in GL_n(R)$ if and only if $f(c) = \det(A - cI_n)$ is invertible in R. The result is clear by Lemma 2.4. ∎

For a positive integer n, one may suspect that if R is a CU ring, then the full matrix ring $M_n(R)$ is CU. The following example shows that this is not true in general.

Example 2.6. Since $\mathbb{Z}$ is commutative, it is a CU ring. But $M_2(\mathbb{Z})$ is not a CU ring. Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \in M_2(\mathbb{Z})$ which is not invertible. Let $f(\lambda) = \det(A - \lambda I_2) = \lambda(\lambda - 7)$ denote the characteristic polynomial of A. Then there is no $c \in \mathbb{Z}$ such that $f(c)$ is invertible in $\mathbb{Z}$. Hence A has not a CU-decomposition by Corollary 2.5.

Theorem 2.7. *Let R be a division ring. For any positive integer n, $M_n(R)$ is a CU ring if and only if $|C(R)| > n$.*

Proof. Assume that $|C(R)| \leq n$. Consider A as a diagonal matrix which has the property that each element of $C(R)$ is one of the diagonal entries of A. For such a matrix A there is no $c \in C(R)$ for which $A - cI$ is a unit. Hence $M_n(R)$ is not a CU ring. For the reverse implication, we will prove something a little stronger. We show that if $|C(R)| > n$, then for every $A \in M_n(R)$, there are at most n elements $c \in C(R)$ for which $A - cI$ is not a unit. We complete the proof by induction on n. If $n = 1$, the result is clear. Suppose that for every $0 < m < n$ and $A \in M_m(R)$, there exist at most m elements $c \in C(R)$ for which $A - cI$ is not a unit. Let $A \in M_n(R)$ and $c \in C(R)$. If $A - cI$ is a unit or nilpotent, we are done. Otherwise, $A - cI$ is neither a unit nor nilpotent. By Fitting's Lemma, applied to $A - cI$, we know that $A - cI$ is similar to a block diagonal matrix $\operatorname{diag}(A_1, A_2)$, where A_1 is a unit and A_2 is nilpotent and $A_1 \in M_m(R)$ and $A_2 \in M_{n-m}(R)$, where $0 < m < n$, since $A - cI$ is neither a unit nor is nilpotent. Since c is central, A is similar to $\operatorname{diag}(B_1, B_2)$ where $B_1 = A_1 + cI$ and $B_2 = A_2 + cI$. By induction, there are at most m elements $c \in C(R)$ for which $B_1 - cI$ is not a unit, and at most $n - m$ elements $c \in C(R)$ for which $B_2 - cI$ is not a unit. It follows that there are at most $n = m + (n - m)$ elements $c \in C(R)$ for which $A - cI$ is not a unit. ∎

The following is more a corollary of the above proof, rather than the statement, the center of R need not map onto the center of $R/J(R)$, even for local rings, namely, skew power series rings.

Corollary 2.8. *Let R be local ring. For a positive integer n, $M_n(R)$ is a CU ring if and only if the image of $C(R)$ in $R/J(R)$ has strictly more than n elements.*

Proof. Note that $R/J(R)$ is a division ring, and $a \in R$ is invertible in R if and only the image of a in $R/J(R)$ is invertible in $R/J(R)$. So the forward implication is clear. For the reverse implication, let $A \in M_n(R)$. We know that there are at most n elements in $C(R/J(R))$ for which the image of $A - cI$ is not a unit in $M_n(R/J(R))$. Since $M_n(R/J(R)) \cong M_n(R)/(M_n(J(R)) \cong M_n(R)/J(M_n(R))$, $A - cI$ is not a unit in $M_n(R)$. Since the image of $C(R)$ in $C(R/J(R))$ has strictly more than n elements, there exists at least one $c \in C(R)$ for which $A - cI$ is unit. ∎

Theorem 2.9. *Let R be a commutative local ring. For any positive integer n, $M_n(R)$ is a CU ring if and only if $R/J(R)$ is not isomorphic to $\mathbb{Z}_2$.*

Proof. Assume that $M_n(R)$ is a CU ring. By contradiction, suppose that $R/J(R)$ is isomorphic to $\mathbb{Z}_2$. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(R)$ and $B = A \oplus 0_{n-2} \in M_n(R)$ where 0_{n-2} is the $(n-2) \times (n-2)$ zero matrix and $f(\lambda) = \det(B - \lambda I_n)$ is the characteristic polynomial of B. By Corollary 2.5, there exists $c \in R$ such that $f(c) = c^{n-1}(c-1)$ is invertible in R. Hence c is nonzero, therefore $c \in J(R)$ or $c - 1 \in J(R)$. This is a contradiction. Conversely, since $R/J(R)$ is a field and $M_n(R)/J(M_n(R)) \cong M_n(R/J(R))$, we may assume that R is a field and R is not isomorphic to $\mathbb{Z}_2$. Let $A \in M_n(R)$. If $A \in GL_n(R)$, then there is nothing to do. Assume that $A \notin GL_n(R)$. We complete the proof by induction on n. Assume that $n = 2$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is not invertible, $ad = bc$. Being $R \ncong \mathbb{Z}_2$, there exists $0 \neq u \in R$ such that $a + d - u \neq 0$. Let $C = uI_2$. Then C is central and $A - C \in GL_2(R)$ since $\det(A - C) = -u(a + d - u) \neq 0$.
Assume that the claim holds for all $k < n$ and $A \in M_n(R)$. Since R is a local ring, it follows by [1, Corollary 7.3.2] that there exist $P, Q \in GL_n(R)$ such that $PAQ = diag(I_r, 0_{n-r})$. We have

$$PAP^{-1} = diag(I_r, 0_{n-r})Q^{-1}P^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}.$$

By induction hypothesis, there exist nonzero $c \in R$, $U \in GL_n(R)$ such that $A_1 = diag(c, c, \ldots, c) + U$. Then

$$PAP^{-1} = diag(c, c, \ldots, c) + \begin{bmatrix} U & A_2 \\ 0 & diag(-c, -c, \ldots, -c) \end{bmatrix}.$$

By Proposition 2.3, A has a CU-decomposition. ∎

Corollary 2.10. *Let F be a field. For any positive integer n, $M_n(F)$ is CU if and only if F is not isomorphic to $\mathbb{Z}_2$.*

Let R be a ring and $U_n(R)$ the subring of $M_n(R)$ consisting of all $n \times n$ upper triangular matrices. Then we have the following.

Corollary 2.11. *Let R be a commutative ring. For a positive integer n, $A \in U_n(R)$ has a CU-decomposition if and only if there exists $c \in R$ such that $A - cI_n$ is invertible in $U_n(R)$.*

Proof. Same as the proof of Lemma 2.4. ■

The following result can be useful to determine under what conditions the ring of 2×2 upper triangular matrices $U_2(R)$ is CU.

Proposition 2.12. *Let R be a commutative ring. Then $U_2(R)$ is a CU ring if and only if for any $a, b \in R$, there exists $c \in R$ such that $a - c, b - c \in U(R)$.*

Proof. Assume that $U_2(R)$ is CU. Let $a, b \in R$. Consider $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in U_2(R)$. By Corollary 2.11, there exists $c \in R$ such that $A - cI_2$ is invertible in $U_2(R)$. Hence $a - c$ and $b - c$ are invertible. Conversely, let $A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in U_2(R)$. There exists $c \in C(R)$ such that $x - c$, $z - c \in U(R)$. Let $U = \begin{bmatrix} x - c & y \\ 0 & z - c \end{bmatrix}$ and $C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$. Then $C \in C(U_2(R))$ and $U \in U(U_2(R))$. Hence $A = C + U$ is a CU-decomposition of A. ■

For a positive integer n, the next example shows that if R is a CU ring, then $U_n(R)$ need not be a CU ring.

Example 2.13. Consider the ring $\mathbb{Z}$, let $a = 2$ and $b = 3$. Then there is no $c \in \mathbb{Z}$ such that $2 - c$ and $3 - c$ are invertible. By Proposition 2.12, $U_2(\mathbb{Z})$ is not CU.

In spite of the fact that $U_n(R)$ need not be a CU ring for any positive integer n and for some rings R, we now show that CU subrings of $U_n(R)$ are rich. If $D_n(R) = \{(a_{ij}) \in U_n(R) \mid \text{ all diagonal entries of } (a_{ij}) \text{ are equal}\}$, then we have the following result.

Proposition 2.14. *Let R be a ring. For any positive integer n, R is a CU ring if and only if $D_n(R)$ is a CU ring.*

Proof. We assume that R is a CU ring and $a \in R$. Let $A = diag(a, a, \ldots, a) + (a_{ij}) \in D_n(R)$ with $i < j$. Then a has CU-decomposition such as $a = c + u$ where c is central and u is invertible. Consider $C = diag(c, c, \ldots, c)$ and $U = diag(u, u, \ldots, u) + (a_{ij})$ with $i < j$. Then C is central and U is invertible in $D_n(R)$. Hence $A = C + U$ is a CU-decomposition of A. Conversely, suppose that $D_n(R)$ is a CU ring for some positive integer n. Let $a \in R$ and consider $A = diag(a, a, \ldots, a) \in D_n(R)$. Then there exist central $C = diag(c, c, \ldots, c) \in D_n(R)$ and invertible $U = diag(u, u, \ldots, u) \in D_n(R)$ such that $A = C + U$. Hence $a = c + u$ is the CU-decomposition of a. ■

Let $V_n(R)$ the subrings of $U_n(R)$ where n is a positive integer.

$$V_n(R) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{bmatrix} \mid a_i \in R, 1 \leq i \leq n \right\}.$$

Let (x^n) denote the ideal generated by x^n in $R[x]$. Then we have $R[x]/(x^n) \cong V_n(R)$ in a natural way.

Theorem 2.15. *Let R be a ring. For any positive integer n, the following statements are equivalent.*

(1) R is a CU ring.

(2) $V_n(R)$ is a CU ring.

Proof. (2) $\Rightarrow$ (1) Let $r \in R$ and $A = \{(a_i) \mid a_1 = r \text{ and } a_i = 0 \text{ if } i \neq 1\} \in V_n(R)$. By (2) A has a CU-decomposition $A = C + U$ where $U = (u_i)$ is invertible and $C = (c_i)$ central in $V_n(R)$. Then u_1 is invertible in R and c_1 is central in R. Hence $r = c_1 + u_1$ is a CU-decomposition of r in R.

(1) $\Rightarrow$ (2) Let $A = (a_i) = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{bmatrix} \in V_n(R)$. There are invertible elements u_i in R and central elements c_i of R with $a_i = c_i + u_i$. Let

$$U = (u_i) = \begin{bmatrix} u_1 & u_2 & u_3 & \dots & u_{n-1} & u_n \\ 0 & u_1 & u_2 & \dots & u_{n-2} & u_{n-1} \\ 0 & 0 & u_1 & \dots & u_{n-3} & u_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_1 & u_2 \\ 0 & 0 & 0 & \dots & 0 & u_1 \end{bmatrix} \in V_n(R),$$

$$C = (c_i) = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_{n-1} & c_n \\ 0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ 0 & 0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_1 & c_2 \\ 0 & 0 & 0 & \dots & 0 & c_1 \end{bmatrix} \in V_n(R).$$

Then $A = C + U$ is a CU-decomposition of A. ■

Lemma 2.16. *Let R be a ring and e an idempotent in R. Then we have the following.*

(1) If $u \in U(R)$ and $eu = ue$, then eue is invertible in eRe.

(2) If $c \in C(R)$, then ec is central in eRe.

Proof. (1) Let $uu^{-1} = 1 = u^{-1}u$ and $eu = ue$. Then $eu^{-1} = u^{-1}e$. Hence $e = euu^{-1} = (eue)(eu^{-1}e)$. Similarly, $e = eu^{-1}u = (eu^{-1}e)(eue)$. So $eu^{-1}e$ is the inverse of eue in eRe.
(2) Let $exe \in eRe$. Note that $ec = ece = ce$. Then $(exe)(ec) = (exe)c = c(exe) = (ce)(exe)$. Hence ec is central in eRe. ∎

Theorem 2.17. *Let R be a CU ring and $e^2 = e \in R$. Then the corner ring eRe is CU.*

Proof. Let $eae \in eRe$. By assumption $eae = u + c$, for some $u \in U(R)$ and $c \in C(R)$. Then $eae - u$ is central. By commuting $eae - u$ with e, we have $eu = ue$. By Lemma 2.16, eue is invertible in eRe and ece is central in eRe. Hence $eae = ece + eue$ is the CU-decomposition of eae in eRe. Thus eRe is CU. ∎

As a direct consequence of Theorem 2.17, we have the following.

Proposition 2.18. *Let R be a ring. For any positive integer n, if $M_n(R)$ is a CU ring, then R is a CU ring.*

Proof. Let n be any positive integer and $e_{11} \in M_n(R)$ denote the $n \times n$ matrix unit with $(1,1)$ entry 1 elsewhere 0. By Theorem 2.17, $e_{11}M_n(R)e_{11}$ is a CU ring. Since $e_{11}M_n(R)e_{11}$ is isomorphic to R, R is a CU ring. ∎

The next result shows that being CU for rings is preserved under the direct products of rings.

Proposition 2.19. *Let $R = \prod_{i \in I} R_i$ be a direct product of rings. Then R is a CU ring if and only if R_i is a CU ring for each $i \in I$.*

Proof. We may assume that $I = \{1,2\}$ and $R = R_1 \times R_2$. Note that $C(R) = C(R_1) \times C(R_2)$ and $U(R) = U(R_1) \times U(R_2)$. For the necessity, let $r_1 \in R_1$. Then $(r_1, 0) = (c_1, c_2) + (u_1, u_2)$ where (u_1, u_2) is invertible in R and (c_1, c_2) is central in R. Hence $r_1 = c_1 + u_1$ is a CU-decomposition of $r_1 \in R_1$. So R_1 is a CU ring. A similar proof takes care for R_2 be CU. For the sufficiency, assume that R_1 and R_2 are CU. Let $(r_1, r_2) \in R$. By assumption r_1 and r_2 have CU-decompositions $r_1 = c_1 + u_1$ and $r_2 = c_2 + u_2$ where u_1 is invertible in R_1, c_1 is central in R_1 and u_2 is invertible in R_2, c_2 is central in R_2. Hence (r_1, r_2) has a CU-decomposition $(r_1, r_2) = (c_1, c_2) + (u_1, u_2)$. The same proof works for any index set I. ∎

Recall that a ring R is called *unit-central* [4] if all unit elements are central in R.

Lemma 2.20. *Every unit-central CU ring is commutative.*

Proof. Assume that R is a unit-central CU ring. Let $a \in R$ with $a = c + u$ where c is central and u is unit. By assumption u is central. So a is central. ∎

Recall that in [2], uniquely nil clean rings are defined. An element a in a ring R is called *uniquely nil clean* if there is a unique idempotent e such that $a - e$ is nilpotent. The ring R is *uniquely nil clean* if each of its elements is uniquely nil clean. It is proved that in a uniquely nil clean ring every idempotent is central [2, Lemma 5.5]. In fact, if e is an idempotent in a uniquely clean ring R, for any $r \in R$, then $e + (re - ere)$ can be written in two ways as a sum of an idempotent and a nilpotent as $e + (re - ere) = (e + (re - ere)) + 0 = e + (re - ere)$. Then $e = e + (re - ere)$ and $re - ere = 0$. Similarly, $er - ere = 0$. Hence e is central. Let R be a ring with involution $*$. In [7], a ring is called **-clean* if each of its elements is a sum of a unit and a projection, and R is *strongly *-clean* if each of its elements is a sum of a unit and a projection that commute with each other. It is proved that every strongly *-clean ring is abelian in [7, Lemma 2.1]. In [3], strongly nil *-clean rings are investigated. A ring is called *strongly nil *-clean* if every element of R is the sum of a projection and a nilpotent that commute with each other.

Proposition 2.21. (1) *Every uniquely nil clean ring is CU.*
(2) *Every strongly *-clean ring is CU.*
(3) *Every strongly nil *-clean ring is CU.*

Proof. (1) Let R be a uniquely nil clean ring and $a \in R$. Then there exists a unique idempotent e such that $(a+1) - e = b$ is nilpotent. Then $a = e + (b-1)$. By hypothesis e is central and $b-1$ is invertible. Hence R is a CU ring.
(2) Assume that R is a strongly *-clean ring. Let $a \in R$ with $a = u + p$ where u is unit and p is a projection with $up = pu$. Since R is abelian, p is central in R. Thus $a = u + p$ is a CU-decomposition of a.
(3) By [3, Proposition 2.5], every strongly nil *-clean ring is strongly *-clean ring. By (2), if R is a strongly nil *-clean ring, then it is a CU ring. ∎

In Example 3.6, we show that CU rings need not be uniquely nil clean.

3 Extensions of CU rings

In this section, we study some extensions of CU rings. In particular, we investigate under what conditions the Dorroh extension of R, the formal triangular matrix ring and some subrings of the ring of all $n \times n$ matrices $M_n(R)$ are CU.

Let R be a ring and $D(\mathbb{Z}, R)$ denote the *Dorroh extension* of R by the ring of integers $\mathbb{Z}$. Then $D(\mathbb{Z}, R)$ is the ring defined by the direct sum $\mathbb{Z} \oplus R$ with componentwise addition and multiplication $(n,r)(m,s) = (nm, ns + mr + rs)$ where $(n,r),\ (m,s) \in D(\mathbb{Z}, R)$. It is clear that $C(D(\mathbb{Z}, R)) = \mathbb{Z} \oplus C(R)$. The identity of $D(\mathbb{Z}, R)$ is $(1,0)$ and the set of invertible elements is

$$U(D(\mathbb{Z}, R)) = \{(1,u) \mid u+1 \in U(R)\} \cup \{(-1,u) \mid u-1 \in U(R)\}.$$

Theorem 3.1. *Let R be a ring. Then R is a CU ring if and only if $D(\mathbb{Z}, R)$ is CU.*

Proof. Assume that R is a CU ring. Let $(n,r) \in D(\mathbb{Z}, R)$. Since R is CU, $r = c + u$ where $u \in U(R)$ and $c \in C(R)$. Then $(n,r) = (n-1, c+1) + (1, u-1)$ is a CU-decomposition of (n,r). Conversely, let $r \in R$. Then $(0,r) = (-1,c) + (1,u)$ or $(0,r) = (1,c) + (-1,u)$. Hence $r = (c-1) + (u+1)$ or $r = (c+1) + (u-1)$. So R is CU. ∎

Let R be a ring and S a subring of R and

$$T[R,S] = \{(r_1, r_2, \cdots, r_n, s, s, \cdots) : r_i \in R, s \in S, n \geq 1, 1 \leq i \leq n\}.$$

Then $T[R,S]$ is a ring under the componentwise addition and multiplication. Note that $U(T[R,S]) = T[U(R), U(R) \cap U(S)]$ and $C(T,[R,S]) = T[C(R), C(R) \cap C(S)]$.

Proposition 3.2. *Let R be a ring and S a subring of R. Then the following are equivalent.*

1. *$T[R,S]$ is a CU ring.*
2. *R and S are CU.*

Proof. (1) $\Rightarrow$ (2) Assume that $T[R,S]$ is CU and let $a \in R$. Consider $X = (a, 0, 0, \cdots) \in T[R,S]$. From the assumption there exist an invertible element $U = (u_1, u_2, \cdots, u_m, t, t, \cdots)$ and a central element $C = (c_1, c_2, \cdots, c_n, s, s, \cdots)$ in $T[R,S]$ such that $X = C + U$. By this equality $a = c_1 + u_1$ is CU-decomposition of a. To see S is a CU ring, let $s \in S$. Then, $A = (0, 0, \cdots, 0, s, s, \cdots) \in T[R,S]$. Since $T[R,S]$ is CU, we have $A = C + U$ where $U = (u_1, u_2, \cdots, u_m, v, v, \cdots)$ is invertible and $C = (c_1, c_2, \cdots, c_m, w, w, \cdots)$ is central. Hence, $s = w + v$ is a CU-decomposition of s.
(2) $\Rightarrow$ (1) Let R and S be CU rings and $Y = (a_1, a_2, \cdots, a_n, s, s, s, \cdots)$ be an arbitrary element in $T[R,S]$. Then there exist $c_i \in C(R)$, $u_i \in U(R)$, $c \in C(R) \cap C(S)$ and $u \in U(R) \cap U(S)$ where $1 \leq i \leq n$, such that $a_i = c_i + u_i$ for all $1 \leq i \leq n$ and $s = c + u$. Then $Y = (u_1, u_2, \cdots, u_n, u, u, \cdots) + (c_1, c_2, \cdots, c_n, c, c, \cdots)$ is a CU decomposition of Y in $T[R,S]$. ∎

In the sequel, we investigate under what conditions subrings of $M_n(R)$ are CU rings.

The rings $H_{(s,t)}(R)$ **:** Let R be a ring, and let $s, t \in C(R)$. Let

$$H_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R, a - d = sc, d - f = te \right\}.$$

Then $H_{(s,t)}(R)$ is a subring of $M_3(R)$. Note that $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$ if and only if $A \in M_3(R)$ and $a - d = sc, d - f = te$ if and only if $A = \begin{bmatrix} sc + te + f & 0 & 0 \\ c & te + f & e \\ 0 & 0 & f \end{bmatrix}$.

Lemma 3.3. *Let R be a ring, and let $s,t \in C(R)$. Then*
$$C(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} sc+te+f & 0 & 0 \\ c & te+f & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid c,e,f \in C(R) \right\}.$$

Proof. Let $A = \begin{bmatrix} sc+te+f & 0 & 0 \\ c & te+f & e \\ 0 & 0 & f \end{bmatrix} \in C(H_{(s,t)}(R))$. Let $x \in R$ and $B = xe_{11} + xe_{22} + xe_{33} \in H_{(s,t)}(R)$ where e_{ij} denote the matrix units in $M_3(R)$. Then $AB = BA$ implies, among others, $cx = xc$, $ex = xe$ and $fx = xf$. Since s and t are central, all components of A are central.

Conversely, let $A = \begin{bmatrix} sc+te+f & 0 & 0 \\ c & te+f & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$ with c, e and f of A are central. Let $B = \begin{bmatrix} sy+tu+v & 0 & 0 \\ y & tu+v & u \\ 0 & 0 & v \end{bmatrix} \in H_{(s,t)}(R)$. We show that $AB = BA$. In fact $(3,3)$ component of AB is $fv = vf$ is the $(3,3)$ component of BA since f is central R. $(2,3)$ component of AB is $(te+f)u + ev$. Since $fu = uf$ and e is central, $(te+f)u = u(te+f)$. Hence $(te+f)u+ev$ is the $(2,3)$ component of BA. $(2,2)$ component of AB is $(te+f)(tu+v)$. Since $te+f$ is central and $(te+f)(tu+v) = (tu+v)(te+f)$, $(tu+v)(te+f)$ is the $(2,2)$ component of BA. $(2,1)$ component of AB is $c(sy+tu+v)+(te+f)y$, and then $c(sy+tu+v)+(te+f)y = y(sc+te+f)+(tu+v)c$ is $(2,1)$ component of BA. $(1,1)$ component of AB is $(sc+te+f)(sy+tu+v)$. Since $sc+te+f$ is central in R, $(sc+te+f)(sy+tu+v) = (sy+tu+v)(sc+te+f)$ is the $(1,1)$ component of BA. Hence $AB = BA$ for all $B \in H_{(s,t)}(R)$. Thus A is central in $H_{(s,t)}(R)$. ■

Lemma 3.4. *Let R be a ring, and let $s,t \in C(R)$. Then the set of all invertible elements of $H_{(s,t)}(R)$ is*

$$U(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid a,d,f \in U(R), c,e \in R \right\}.$$

Proof. Assume that a, d and f are invertible and let a^{-1}, d^{-1} and f^{-1} denote the inverses of a, d and f respectively. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$ and $B = \begin{bmatrix} a^{-1} & 0 & 0 \\ -d^{-1}ca^{-1} & d^{-1} & -d^{-1}ef^{-1} \\ 0 & 0 & f^{-1} \end{bmatrix} \in H_{(s,t)}(R)$. Then $AB = BA = I_n$. Since $a - d = sc$ if and only if $a^{-1} - d^{-1} = -sd^{-1}ca^{-1}$ and $d - f = te$ if and only if $d^{-1} - f^{-1} = -td^{-1}ef^{-1}$, $B = A^{-1} \in H_{(s,t)}(R)$.

Conversely, suppose that $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$ is invertible in $H_{(s,t)}(R)$

with inverse $B = \begin{bmatrix} x & 0 & 0 \\ y & z & u \\ 0 & 0 & v \end{bmatrix}$. Then $AB = BA = I_n$. Comparing entries we reach $ax = xa = 1$, $dz = zd = 1$ and $fv = vf = 1$. Hence a, d and f are invertible. ■

Theorem 3.5. *Let R be a ring, and let $s, t \in C(R) \cap J(R)$. Then R is a CU ring if and only if $H_{(s,t)}(R)$ is CU.*

Proof. Assume that R is CU. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$. Let $f = f_1 + f_2$, $e = e_1 + e_2$ and $c = c_1 + c_2$ denote the CU-decompositions of f, e and c where $f_1, e_1, c_1 \in U(R)$ and $f_2, e_2, c_2 \in C(R)$. Choose $d_2 = f_2 + te_2$ and $a_2 = d_2 + sc_2$. By Lemma 3.3, $C = \begin{bmatrix} a_2 & 0 & 0 \\ c_2 & d_2 & e_2 \\ 0 & 0 & f_2 \end{bmatrix} \in C(H_{(s,t)}(R))$. Moreover, $d - d_2 = d - f_2 - te_2 = d - f + f_1 - te_2 = f_1 - te - te_2$ is invertible since $f_1 \in U(R)$ and $te - te_2 \in J(R)$. Similarly $a - a_2 = d - d_2 + sc - sc_2 \in U(R)$. Let $U = \begin{bmatrix} a_1 & 0 & 0 \\ c_1 & d_1 & e_1 \\ 0 & 0 & f_1 \end{bmatrix}$ with $a_1 = a - a_2$ and $d_1 = d - d_2$. By Lemma 3.4, $U \in U(H_{(s,t)}(R))$. Hence $A = C + U$ is a CU-decomposition of A.
Conversely, suppose that $H_{(s,t)}(R)$ is CU and $a \in R$.
Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in H_{(s,t)}(R)$. There exist $C = \begin{bmatrix} x' & 0 & 0 \\ y' & z' & t' \\ 0 & 0 & u' \end{bmatrix} \in C(H_{(s,t)}(R))$ and $U = \begin{bmatrix} x & 0 & 0 \\ y & z & t \\ 0 & 0 & u \end{bmatrix} \in U(H_{(s,t)})(R)$ such that $A = U + C$. Then $x' \in C(R)$ and $x \in U(R)$. So $a = x' + x$ is a CU-decomposition of a. ■

We have proved that every uniquely nil clean ring is CU. There are CU rings that are not uniquely nil clean.

Example 3.6. *The ring $H_{(0,0)}(\mathbb{Z})$ is CU but not uniquely nil clean.*

Proof. By Theorem 3.5, $H_{(0,0)}(\mathbb{Z})$ is CU. Note that for $a \in \mathbb{Z}$ has a uniquely nil clean decomposition if and only if $a = 0$ or $a = 1$. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & a & e \\ 0 & 0 & a \end{bmatrix} \in H_{(0,0)}(R)$. Assume that A has a uniquely nil clean decomposition. There exist unique $E^2 = E = \begin{bmatrix} x & 0 & 0 \\ y & x & u \\ 0 & 0 & x \end{bmatrix} \in H_{(0,0)}(R)$ and $N = \begin{bmatrix} g & 0 & 0 \\ h & g & l \\ 0 & 0 & g \end{bmatrix} \in N(H_{(0,0)}(R)$ such that $A = E + N$. Then A has a uniquely nil clean decomposition. This is not the case for each $a \in \mathbb{Z}$. Hence $H_{(0,0)}(\mathbb{Z})$ is not uniquely nil clean. ■

Generalized matrix rings: Let R be ring and s a central element of R. The four tuple $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ becomes a ring denoted by $K_s(R)$ with addition defined componentwise and with multiplication defined in [5] by

$$\begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ y_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{bmatrix}.$$

Then $K_s(R)$ is called *generalized matrix ring over R.*

Lemma 3.7. *Let F be a field. Then*
(1) $U(K_0(F)) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_0(F) \mid a \neq 0, d \neq 0\}$.
(2) $C(K_0(F))$ *consists of all scalar matrices.*

Proof. (1) Clear from [8, Lemma 3.1].
(2) It follows from [6, Lemma 1.1]. ■

Theorem 3.8. *Let F be a field with* $\mid F \mid \geq 3$. *Then* $K_0(F)$ *is a CU ring.*

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any matrix. Since $\mid F \mid \geq 3$, we can find some $u \in F$ such that $a - u, d - u \in U(F)$. Let $C = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$. Then $A - C \in U(K_0(F))$. ■

Example 3.9. *Let F be a field with two elements. Then* $K_0(F)$ *is not CU.*

Proof. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in K_0(F)$. Suppose that A has a CU-decomposition such that $A = C + U$ where $U = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ is invertible of which its main diagonal entries must be nonzero and $C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ is central. $A = C + U$ implies $c + x = 1$ and $c + t = 0$. We complete the discussion by two cases :
Case I. $c = 0$. $t + c = 0$ implies $t = 0$. Invertibility of U requires $t = 1$. A contradiction.
Case II. $c = 1$. Then $t = 1$ and $x = 0$. Again a contradiction. Hence A does not have a CU-decomposition. ■

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