



D-bound and Bekenstein bound for McVittie solution surrounded by dark energy cosmological fields

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Received: 14 January 2020 / Accepted: 8 July 2020 / Published online: 18 July 2020

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Abstract The cosmological candidate fields for dark energy as quintessence, phantom and cosmological constant are studied in terms of an entropic hypothesis imposed on the McVittie solution surrounded by dark energy. We certify this hypothesis as “*D*-bound-Bekenstein bound identification” for dilute systems and use it as a criterion to determine which candidate of dark energy can satisfy this criterion for a dilute McVittie solution. It turns out that only the cosmological constant can pass this criterion successfully while the quintessence and phantom fields fail, as non-viable dark energy fields for this particular black hole solution. Moreover, assuming this black hole to possess the saturated entropy, the entropy-area law and the holographic principle can put two constraints on the radius R of the cosmological horizon. The first one shows that the Hubble radius is discrete such that for any arbitrary value of the black hole mass m_0 , the value of R is determined up to an integer number. The latter one shows that when a black hole is immersed in a cosmological background, the radius of the cosmological horizon is constrained as $R < \frac{1}{H}$.

1 Introduction

The cosmological constant is the simplest candidate for dark energy to describe the current accelerating expansion of the universe [1–4]. The corresponding Λ CDM model is in very good agreement with observations. However, Λ CDM model suffers from two well known “cosmological constant problem” [5] and “coincidence problem” [6–8]. Other alternative models of dark energy have then introduced such as quintessence model [9], with a canonical scalar field having a particular potential in the presence of matter and radiation which could interpret the late-time cosmic acceleration [10–13]. The other alternative for dark energy is phantom model with a phantom scalar field rolling up the potential because of the negative kinetic energy [14–19]. The equation of state parameter of the phantom field in the case of an exponential potential results in a big-rip singularity which is considered as a problem. To study more about weirdness of phantom and quintessence fields refer to [20–28]. In

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addition, the strange behaviour of quintessence and phantom fields in a particular interval of time, in this paper, is in agreement with the problematic constraint on the quintessence energy density in the early cosmological era and the late time cosmological behaviour of phantom field leading to a big rip singularity [13, 29]. Quintessence potentials are classified into “freezing models” and “thawing models” [30]. These potentials are imposed by particular features at early universe in order to have quintessence as dark energy source in late time universe [13, 31–36]. On the other hand, in order to have a consistent phantom scalar field theory which leads to big rip singularity in the late time universe, one needs to go beyond the framework of particle physics [17–19, 29, 37].

In this paper, using a thermodynamical approach, we intend to study the McVittie solution surrounded by cosmological constant, phantom and quintessence fields and show that the entropy bounds can exclude phantom and quintessence fields from being considered as dark energy candidates for this solution. The use of some relevant entropic limits can be considered as a powerful thermodynamical approach in the study of dark energy models. In fact, the equations of motion can perfectly predict the time-reversible behaviour of dynamical systems, nevertheless, for thermodynamical systems the time-reversibility is not guaranteed due to some entropic consideration. Dynamical black holes, surrounded by the cosmological fields, are considered as such thermodynamical systems, and one can use the relevant entropic considerations to study about the viability of these surrounding cosmological fields from thermodynamical point of view. In this regard, we study the entropic bounds in the McVittie solution surrounded by cosmological constant, phantom and quintessence fields and try to find a thermodynamical criterion by which one can single out the viable dark energy field that can satisfy this thermodynamical criterion.

The McVittie solution [38] is one of the exact solutions of Einstein’s general theory of relativity which describes a black hole evolving in time. This dynamical solution appears on a wide range of problems stemming from the perfect-fluid cosmology to the scalar field actions and modified theories of gravity [39]. This solution has the following properties: (i) it is a spherically symmetric, shear free and perfect fluid solution of the Einstein field equations, (ii) the near field is the Schwarzschild solution in the isotropic coordinates with the mass parameter m , and (iii) the far field limit is the Friedmann–Robertson–Walker (FRW) spacetime with the scale factor $a(t)$. Then, by setting $m = 0$ in the metric, one find FRW metric as the background, and by setting $a(t) = 1$ one arrives at the Schwarzschild solution, in the isotropic coordinates, as the local inhomogeneity. However, the global structure of McVittie spacetime has been a prominent issue for the controversial question of whether this solution describes a black hole embedded in an isotropic FRW universe or not [40–43]. The answer to this question has been found recently in [44, 45]. In [46, 47], it is discussed that how the particle motion and the structure of the event horizon in this solution ultimately depend on the cosmological history. For an expanding FRW background, the McVittie solution possesses a weak singularity at $\bar{r} = m/2$ which is a spacelike hypersurface and the past boundary of spacetime [40–42]. This locus manifests itself as a part of future boundary of spacetime when there exists a nonnegative cosmological constant. For the de Sitter background, the singularity $\bar{r} = m/2$ disappears and its 2-surface describes a true black hole horizon. To clarify more on these features, the McVittie solution has been generalized to the spatially non-flat backgrounds [48]. As the generalizations of the McVittie solution, there are also solutions representing the charged black holes in an expanding FRW universe [49–52], see also [53] for the general relativistic perfect fluid black hole solutions and [54, 55] for the classification and other possible generalizations. In derivation of the general solution in [54] (including the McVittie solution), the scale factor $a(t)$ arises as an integration constant when one integrate the field equations, see Sects. II and III of [54] where $q(t) = a(t)/\dot{a}(t)$ and the

energy-momentum is a perfect fluid. Then, to determine the arbitrary scale factor $a(t)$, one possibility is to consider an equation of state for the matter field supporting the Einstein field equations as it is usually done for the Friedmann equations, i.e. $p = \omega\rho$, or make an ansatz for its form for various cosmological eras according to the standard model of cosmology. In the case of McVittie solution describing an inhomogeneity in FLRW background [44, 45], the perfect fluid source represents the matter field filling the background FLRW universe obeying the Friedmann equations in the cosmic scale. Thus, this cosmic fluid can be radiation, dust, quintessence, cosmological constant or a phantom field regarding its equation of state. However, there has yet been no proof that a McVittie metric with a dark energy equation of state other than that of a cosmological constant (i.e. for $\omega \neq -1$) represents a true black hole that contains an event horizon. In the literature, what is mainly studied is the properties of apparent horizons. As an instance, the apparent horizons for the McVittie solution in a phantom background are discussed in [43]. For more discussion on McVittie solution in phantom and quintessence backgrounds, one may also see [56, 57].

In this paper, motivated by the fact that the McVittie solution may represent a dynamical black hole in a cosmic background and then it must obey the thermodynamical entropy constraints, we put this solution under the scrutiny of the D -bound [58] and Bekenstein bound [59–61]. We obtain these entropy bounds for the McVittie solution in a general form. Then, we discuss about the dilute system limit of these bounds for various cosmological background fields, those are the solutions to the Friedmann equations, and noting that both these bounds are direct results of the generalized second law of thermodynamics (GSL), we demand their identification in the dilute system limit, as a hypothesis of “ D -bound-Bekenstein bound identification” [58]. Here, we note that because the entropy bounds deal with the cosmological apparent horizon and the considered cosmological fields solves the standard Friedmann equations at this cosmological scale, then the McVittie solution and its entropic considerations are meaningful. We show that these entropy bounds put some restrictions on the parameters of the McVittie solution with different cosmological background fields. In the following, we review these entropy bounds which are essential for our purpose.

The Bekenstein bound claims that there is an upper bound on the entropy of any physical system. It states that for an isolated and stable thermodynamic system within an asymptotically flat space, there is a constraint on the universal entropy of the system as follows

$$S_m \leq 2\pi RE, \quad (1)$$

where E is the total energy and R is the radius of the system. Bekenstein bound has been considered in two forms, the empirical form [61–65] and the logical form [59, 60, 66, 67]. Its logical form is based on the generalized second law of thermodynamics (GSL) and the Geroch process. To study about quantum effects on this bound, one can refer to [68–70].

D -bound has been introduced by Bousso [58] and investigates a gravitational system by a gedanken experiment as follows. Suppose a black hole in the universe with a cosmological horizon. The total entropy of the system in a case when there is a matter system (a black hole) and a cosmological horizon reads as $S_m + S_h$ where S_m and $S_h = A_c/4$ are the entropy of the matter system and the cosmological horizon with the area A_c , respectively. Now, suppose that an observer moves away from the matter system until the matter system falls out of the apparent horizon of this observer. This observer is a witness of crossing the matter system from the apparent horizon in a thermodynamical process. Then, the total entropy of the final system is given by $S_0 = \frac{A_0}{4}$ where A_0 is the horizon area in the absence of the matter system. Now, according to the GSL, the observer can put an upper bound to the entropy of the matter

system by

$$S_m \leq \frac{1}{4}(A_0 - A_c). \quad (2)$$

This is the D -bound on the matter system which has been introduced firstly for the de Sitter space. The identification of the D -bound and the Bekenstein bound on the matter system has been shown for the de Sitter background. Also, it has been indicated that this identification can be generalized to the arbitrary dimensions [58]. Following the same method in [58], the D -bound entropy for the various possible black hole solutions on a four dimensional brane, and its identification with the Bekenstein bound have been investigated in [71]. It was shown that there are some differences in the D -bound entropy for the solutions on a brane within a higher dimensional bulk in comparison with the usual four dimensional black hole solutions. It is concluded that these differences are because of the extra loss of information due to the extra dimensions when black hole is crossing from the apparent horizon of the observer confined to the four dimensional brane. Bearing these in mind, it seems that the hypothesis of identification of D -bound with Bekenstein bound is a reasonable ingredient in a thermodynamic treatment of Black holes, with entropic considerations. Finally, we assert that since the D -bound is a direct result of GSL, it is also definable for a dynamical system like McVittie solution having rapidly evolving matter fields and causal horizon [72, 73].

The organization of this paper is as follows. In Sect. 2, we obtain the D -bound and Bekenstein bound for McVittie Solution in a general form. Then, we discuss about the dilute system limit for different cosmological backgrounds as the candidates for dark energy. In Sect. 3, using the entropy-area law and the holographic principle, we obtain two constraints on the cosmological horizon radius. The paper ends with a conclusion in Sect. 4.

2 D -bound and Bekenstein bound for McVittie Solution

In this section, we study the McVittie solution [38] and derive the corresponding D -bound and Bekenstein bound. The McVittie solution represents the embedding of the Schwarzschild solution within the FRW cosmological background which can be generally de Sitter or not. The line element of this solution in the isotropic coordinates reads as

$$ds^2 = -\frac{\left(1 - \frac{m(t)}{2\bar{r}}\right)^2}{\left(1 + \frac{m(t)}{2\bar{r}}\right)^2} dt^2 + a(t)^2 \left(1 + \frac{m(t)}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega_2^2). \quad (3)$$

Considering $G_{\mu\nu}$ as the Einstein tensor, the McVittie no-accretion condition is given by $G^{\bar{r}}_{\bar{t}} = 0 = T^{\bar{r}}_{\bar{t}}$ which forbids the accretion of the cosmic fluid onto the central black hole. This condition leads to the following differential equation

$$\frac{\dot{m}}{m} + \frac{\dot{a}}{a} = 0, \quad (4)$$

which can be solved as

$$m(t) = \frac{m_0}{a(t)}, \quad (5)$$

where m_0 is a positive constant and $a(0) = 1$.¹ The metric (3) can be written in the Schwarzschild coordinates for more convenience. To do this, one can define the areal radius

¹ In the modern language, Eq. (4) represents the constancy of the Hawking–Hayward mass m_H , i.e. $\dot{m}_H = 0$ [74]. Indeed, $m(t)$ here stands just as a metric coefficient in a particular coordinate system. In fact, one has to identify the Hawking–Hayward mass m_H [74] as the physically relevant mass, which eventually is

R as [76]

$$R = a(t)\bar{r} \left(1 + \frac{m(t)}{2\bar{r}} \right)^2, \quad (6)$$

and finds the identity

$$\frac{\left(1 - \frac{m(t)}{2\bar{r}} \right)^2}{\left(1 + \frac{m(t)}{2\bar{r}} \right)^2} = 1 - \frac{2m_0}{R}. \quad (7)$$

The Eq. (4) gives

$$H \left(1 + \frac{m}{2\bar{r}} \right) + \frac{\dot{m}}{\bar{r}} = H \left(1 - \frac{m}{2\bar{r}} \right) \quad (8)$$

where $H = \dot{a}/a$ is the Hubble parameter of the FRW background. Then, using (6), (7) and (8) the metric can be written as

$$ds^2 = - \left(1 - \frac{2m_0}{R} - H^2 R^2 \right) dt^2 + \frac{dR^2}{1 - \frac{2m_0}{R}} - \frac{2HR}{\sqrt{1 - \frac{2m_0}{R}}} dt dR + R^2 d\Omega_2^2. \quad (9)$$

To remove the cross term, we use the coordinate transformation

$$dT = \frac{1}{F} (dt + \beta dR), \quad (10)$$

where $F(t, R)$ is an integration factor for the above closed 1-form and $\beta(t, R)$ ensures that in the new coordinate system, the $dt dR$ component of the metric vanishes. The function $\beta(t, R)$ can be found as [76]

$$\beta(t, R) = \frac{HR}{\sqrt{1 - \frac{2m_0}{R}} \left(1 - \frac{2m_0}{R} - H^2 R^2 \right)}. \quad (11)$$

Then, our line element (3) becomes

$$ds^2 = - \left(1 - \frac{2m_0}{R} - H^2 R^2 \right) F^2 dT^2 + \frac{dR^2}{1 - \frac{2m_0}{R} - H^2 R^2} + R^2 d\Omega_2^2. \quad (12)$$

For $H = \text{constant}$, the integration factor $F(t, R)$ can be set to unity, and then this metric reduces to the Schwarzschild–de Sitter metric.

For this spacetime, the Misner–Sharp–Hernandez mass M_{MSH} [77, 78] representing the total mass within the radius R can be found as

$$M_{\text{MSH}} = \frac{4\pi G}{3} \rho R^3 + m_0, \quad (13)$$

where $\rho(t) = \frac{3}{8\pi} H^2(t)$ is the background cosmic fluid density. Thus, M_{MSH} includes both the energy of the background cosmic fluid inside the sphere of radius R and the mass of the local inhomogeneity m_0 .

To obtain the D -bound on the matter system in McVittie spacetime, we need to derive the physical horizons of the system, one as the inner horizon, which is an anti-trapping surface representing an inner apparent horizon [79, 80], and the other one as the cosmological apparent horizon. The locations of the horizons for the metric (12) are given by $g^{RR} = 0$ which leads to

related to the physical size of the central object or its corresponding horizon, in order to avoid making any coordinate-dependent statements on the mass and size [50, 74] or the temperature [75] of the central object.

$$H(t)^2 R^3 - R + 2m_0 = 0. \quad (14)$$

This equation has the following solutions

$$R_1 = \frac{2}{\sqrt{3}H} \sin(\psi), \quad (15)$$

$$R_2 = \frac{1}{H} \cos(\psi) - \frac{1}{\sqrt{3}H} \sin(\psi), \quad (16)$$

$$R_3 = -\frac{1}{H} \cos(\psi) - \frac{1}{\sqrt{3}H} \sin(\psi), \quad (17)$$

where $\sin(3\psi) = 3\sqrt{3}m_0H$. The solution R_3 is nonphysical because for an expanding universe with the positive H , R_3 becomes negative. There are strong evidences that, in the absence of event horizons, one can ascribe an entropy to the apparent horizons. The thermodynamic behaviour of these horizons has been considered extensively [81–96]. Therefore, using the Bekenstein–Hawking entropy-area law $S = A/4$, we ascribe the entropy to the apparent horizons R_1 and R_2 as πR_1^2 and πR_2^2 , respectively.

To derive D -bound (2) for the McVittie solution, we consider the radius r_0 of the system as $r_0 = R_2|_{m_0=0} = 1/H$, when the mass-energy of the system is only due to the cosmic fluid ($\rho(t)$) inside the system. In the presence of the matter inhomogeneity m_0 as well as the cosmic fluid, the radius of the cosmological horizon is $r_c = R_2$. Therefore, the initial entropy of the system is $S_m + S_\rho + \frac{A_c}{4}$ and the final entropy of the system becomes $S_\rho + \frac{A_0}{4}$. Thus, using the Bekenstein–Hawking entropy-area law, the D -bound (2) for the matter system of the McVittie solution reads as

$$S_m \leq \pi(r_0^2 - r_c^2), \quad (18)$$

where, substituting r_0 and r_c using (16), it takes the following form

$$S_m \leq \pi \left(\frac{2}{3H^2} \sin^2 \psi + \frac{2}{\sqrt{3}H^2} \cos \psi \sin \psi \right). \quad (19)$$

This bound is the D -bound for the McVittie solution for the generic backgrounds.

For the dilute system limit [58], the size of the local inhomogeneity R_1 must be negligible in comparison with the cosmic horizon R_2 . Indeed, this means that the local mass m_0 should be very small relative to H^{-1} indicating the cosmic Hubble horizon radius. Then, regarding (15) and (16), one realizes that the dilute system limit is equivalent to the approximation relations $\sin \psi \approx \psi$ and $\cos \psi \approx 1$ where $\psi = \sqrt{3}m_0H$. Using these relations, we obtain $R_1 \approx 2m_0$ and $R_2 \approx H^{-1} - m_0 \approx H^{-1}$ which indicate the approximate Schwarzschild and Hubble horizon radii, respectively. Therefore, the general D -bound (19) reduces to

$$S_m \leq \frac{2\pi m_0}{H}, \quad (20)$$

for the dilute systems.

Now, our aim is to obtain the Bekenstein bound for the McVittie solution. To do this, we follow Bousso [58] for the definition of the Bekenstein bound as

$$S_m \leq \pi r_g R, \quad (21)$$

where r_g is the gravitational radius of the matter system and R is the circumscribing radius of the system. For the McVittie solution, these radii correspond to R_1 and R_2 in (15) and

(16), respectively. Then, the Bekenstein bound reads as

$$S_m \leq \frac{2\pi}{\sqrt{3}H^2} \cos \psi \sin \psi - \frac{2\pi}{3H^2} \sin^2 \psi. \quad (22)$$

This form is the general form of the Bekenstein bound for the McVittie spacetime. Then, one realizes that in the general cosmological setup of McVittie solution, the D -bound and the Bekenstein bound do not coincide and indeed the latter is tighter. However, using the dilute system conditions, as discussed after Eq. (19), the above Bekenstein bound reduces to the form $S_m \leq \frac{2\pi m_0}{H}$.

We summarize our findings till now in the following remark.

Remark 1 The entropy D -bound and the Bekenstein bound do not coincide for a generic cosmological background in McVittie spacetime. Indeed, the latter bound is always tighter. However, for any general cosmological background satisfying the “dilute system limits”, these two bounds are identified in McVittie spacetime.

In the following subsections, we investigate these entropy bounds for the de Sitter, phantom and quintessence backgrounds in more details.

2.1 McVittie solution in de Sitter background and entropy bounds

Here, we consider the de Sitter background in which $H = \text{constant} \propto \sqrt{\Lambda}$ where Λ is the cosmological constant. Since the value of Λ is constant and very small, then one can always use the dilute system limit for the de Sitter background [58] which results in the coincidence of the D -bound and the Bekenstein bound. The interesting point about the cosmological constant is that the dilute system limit (i.e. $m_0 \ll 1/H$) is provided for both the early and late times. This turns out that the cosmological constant is a special fluid providing the coincidence of the D -bound and the Bekenstein bound for all cosmic times. In the following subsections, we show that this identification does not happen in general for the phantom and quintessence fields for all cosmological time intervals.

2.2 McVittie solution in phantom background and entropy bounds

The McVittie solution in the phantom background with the barotropic equation of state $\omega < -1$ has been studied in [97,98]. A phantom dominated universe evolves towards a finite time big rip singularity [99] as

$$a(t) = \frac{a_0}{(t_{\text{rip}} - t)^{\frac{2}{3|\omega+1|}}} \quad (23)$$

where a_0 is a constant. The derivation of Hubble parameter is straightforward and concisely is given by

$$H(t) = \frac{2}{3|\omega+1|} \frac{1}{t_{\text{rip}} - t}. \quad (24)$$

For the phantom background, the D -bound is the same as the general Eq. (19) where the Hubble parameter is replaced by (24). Regarding (15), (16) and (24), one can discuss about the dilute system limit and consequently the coincidence of the D -bound and the Bekenstein bound for a phantom background as follows.

- At the early times, both the local apparent horizon R_1 and the cosmological horizon R_2 exist, and they are located approximately at $2m_0$ and $1/H$, respectively. At the early times

when the phantom field is not the dominant field, the dilute system limit is provided. This is supported by the fact that for small t values, and since big rip time t_{rip} is large enough, then H is small and consequently the approximation relation $\sin(\psi) \approx \psi$ is valid.

- As time progresses, the phantom field dominates the cosmos such that H and consequently $\rho(t)$ grow up and diverge at the finite time big rip state. Then, the dilute system approximation fails for the late time cosmology when the phantom field dominates the universe.

The consequence of the above two points is that for the McVittie solution in the phantom background, the coincidence of the obtained general D -bound (19) and the Bekenstein bound (22) happens only for the early times when the phantom field is not the dominant field in the universe and consequently the dilute system limit is provided. As the cosmic time progresses and the phantom field dominates the universe, the dilute system limit fails and these two bounds deviate. This behaviour is one of the weird features of the cosmological phantom fluid, which also leads to the violation of the second law of thermodynamics in many ways [100–102]. Here, one notes that the asymptotic behaviour of McVittie horizons is far from trivial and differs significantly from an FLRW solution in extreme regimes, see [79, 80] where the effect of the cosmological expansion, for the different choices of scale factor, on the causal structure of the McVittie solution is studied.

2.3 McVittie solution in quintessence background and entropy bounds

For the quintessence background with $-1 < \omega < -1/3$, the scale factor of the FRW universe is given by [99]

$$a(t) = a_0 t^{\frac{2}{3(\omega+1)}}, \quad (25)$$

where a_0 is a constant. Thus, the Hubble parameter reads as

$$H(t) = \frac{2}{3(\omega+1)} \frac{1}{t}. \quad (26)$$

Here, regarding (15), (16) and (26), one notes to the following points.

- At the early times, the dilute system limit, $m_0 \ll 1/H$, fails for the solution (26) which represents divergent H and $\rho(t)$ for a quintessence field. Generally, quintessence potentials are classified into “freezing models” and “thawing models” [30]. These potentials are imposed by particular features at early universe in order to have quintessence as dark energy source in late time universe [13, 31–36].
- For the late times, H decreases and the dilute system limit $m_0 \ll 1/H$ is provided.

In contrast to the phantom field, for a quintessence field the dilute system limit is provided only for the late times, and consequently the coincidence of the D -bound (19) and the Bekenstein bound (22) happens only for the late times when the quintessence appears as the dark energy source of the expansion of the universe.

We conclude our findings in these three subsections in the following remark.

Remark 2 Derivation of the entropy D -bound and Bekenstein bound for the McVittie space-time turns out that the cosmological constant field with $\omega = -1$ is the unique cosmological field which provides the identification of the D -bound and the Bekenstein bound for all cosmic time intervals. Any deviation from the barotropic equation of state parameter $\omega = -1$ perturbs this identification. For the phantom field possessing $\omega < -1$, the identification of these two bounds exists only at the early times and is lost for the late times. For the

quintessence field with $-1 < \omega < -1/3$, there is no identification for the early universe but as the universe expands this identification emerges. Then, entropy criteria for McVittie spacetime imply the cosmological constant as the most viable dark energy candidate.

Before ending this section, it is worth mentioning that there are other matter fields like radiation and dark matter playing a role in the McVittie solution at the early times [44]. However, regarding our aim in this paper which is finding an entropic criterion on dark energy fields by comparing the D -bound with Bekenstein bound, the existence of a cosmological horizon is essential for constructing D -bound. Indeed, the existence, location and behaviour of horizons at early times are the essential requirements to construct the D -bound and discuss its identification with Bekenstein bound. One notes that the existences of the inner and outer apparent horizons of McVittie solution given in (15) and (16), respectively, are subjected to the condition $0 < \sin(\psi) < 1$ or $m_0 H(t) < \frac{1}{3\sqrt{3}}$ [43]. Therefore, there is a critical time for the existence of an apparent horizon for any background field determined by $m_0 H(t_c) = \frac{1}{3\sqrt{3}}$. As an instance, for a dust background with the Hubble parameter $H(t) = \frac{2}{3t}$, this critical time is $t_c = 2\sqrt{3}m_0$. Then, in the context of McVittie solution, there is no physical apparent horizon at the early times, i.e for $t < t_c = 2\sqrt{3}m_0$, in a dust-dominated background. This fact is also addressed in detail in [43], and one can observe from Fig. 1 in [43] that at the early times none of the inner and cosmological apparent horizons appear. Then, the construction of D -bound fails before the critical time $t_c = 2\sqrt{3}m_0$ when there is no cosmological horizon. The same argument applies for the radiation field possessing the Hubble parameter $H(t) = \frac{1}{2t}$. However, the situation changes when there is a phantom field in the background. For a phantom background, the Hubble parameter reads as $H(t) = \frac{2}{3|\omega+1|} \frac{1}{t_{\text{rip}}-t}$ and the time interval which the apparent horizons exist reads as $t < t_{\text{rip}} - \frac{2\sqrt{3}m_0}{|\omega+1|}$ where t_{rip} is the future cosmological big rip time. Hence, when the cosmological background possesses a phantom field, there are both the cosmological and local horizons in early times and one can construct D -bound and investigate its identification with Bekenstein bound. This point can be seen in Fig. 3 of [43] where both the inner and cosmological apparent horizons exist before t_{rip} . According to these considerations, we have addressed the phantom background at the early times. However, one notes that in the early times, the total dynamics of the McVittie solution are complicated, and it is governed by the total matter source that includes dark matter, radiation and dark energy sources in general.

3 Constraint on the radius of cosmological horizon in McVittie spacetime

In this section, we use discrete behaviour of entropy-area law to derive constraint on the radius of cosmological horizon in McVittie solution surrounded by cosmological fields. In the first subsection, we consider the discrete behaviour of entropy-area law and in the next subsection we attempt to put a constraint on the radius of cosmological horizon by Holographic principle and maximum entropy of the whole system.

3.1 Discrete behaviour of entropy-area law and radius of cosmological horizon

3.1.1 The case of de Sitter background ($\omega = -1$)

As we mentioned in Sect. 2.1, the D -bound or Bekenstein bound for $\omega = -1$ read as $S_m \leq \frac{2\pi m_0}{H}$ for generic matter systems. For a black hole which possesses the saturated

entropy, we have

$$S_m = \frac{2\pi m_0}{H}. \quad (27)$$

On the other hand, by putting the value of the entropy of black hole $S_m = \frac{A}{4l_p^2}$ in Eq. (27), we obtain²

$$\frac{2\pi m_0}{H} = \frac{A}{4l_p^2} = \frac{\alpha N}{4}, \quad (28)$$

where we have put $A = \alpha l_p^2 N$ as the area of black hole and N is an integer number. α is the proportionality constant and is an $\mathcal{O}(1)$ dimensionless coefficient.³ The proportionality constant α has been considered by various values. The requirement which demands the number of state e^S to be integer leads to $\alpha = 4 \ln q$ where q is an integer [104]. Various kinds of arguments impose different appropriate integers, such as $q = 2, 3$ [105–107]. Consistent highly damped quasinormal modes (QNMs) [108] and “holographic shell model” for BHs demand $\alpha = 8\pi$ [109–111] and $\alpha = 8 \ln 2$ [112], respectively. Finally, by considering all of these constraints, the interval $1 < \alpha < 30$ is a consistent and reasonable range [104–111]. Now, with respect to Eq. (28), the Hubble radius $R = 1/H$ is given by

$$R = \frac{\alpha}{8\pi m_0} N, \quad (29)$$

which shows that the Hubble radius is discrete such that for any arbitrary value of m_0 , the value of R is given up to an integer number.

3.1.2 The case of background with $\omega \neq -1$

In the phantom background, assuming that the matter system is a black hole, the Bekenstein entropy becomes

$$S_m = \frac{2\pi}{\sqrt{3}H^2} \cos \psi \sin \psi - \frac{2\pi}{3H^2} \sin^2 \psi. \quad (30)$$

By putting the value of the black hole’s entropy $S_m = \frac{A}{4l_p^2}$ in Eq. (30), we obtain

$$\frac{2\pi}{\sqrt{3}H^2} \cos \psi \sin \psi - \frac{2\pi}{3H^2} \sin^2 \psi = \frac{A}{4l_p^2} = \frac{\alpha N}{4}. \quad (31)$$

Here, unlike the previous case, we are not able to write H up to a discrete number for the late times. This is because of that $\sin(\psi)$ and $\cos(\psi)$ terms have nonlinear dependence on H and m_0 . Therefore, in this case, the radius of the cosmological horizon in the presence of black hole does not show any discrete behaviour in late times, but for early times the value of H in (24) is small and as we mentioned in Sect. 2.2 the system can be diluted we can consider this case as a dilute case (27), so the interpretations are like as $\omega = -1$ case.

In the quintessence background, we have the same interpretation as the cosmological constant phase $\omega = -1$ for the late times, but for the early times the system cannot be dilute. Then, the radius of cosmological horizon in the presence of black hole does not show any discrete behaviour for the early times.

² Note that in all of the other formulas we have put $l_p = 1$ but for the clarification here we have inserted the l_p in Eq. (28).

³ There is also an interesting method for calculation of the entropy of black hole by the method of graph theory [103].

3.2 Constraints via the holographic principle

One can think of maximum entropy of the system in McVittie spacetime. The maximum entropy of the system of black hole and FLRW background with a cosmological horizon is πR^2 and R is the system radius which is the radius of cosmological horizon R_2 . Now, according to the Bekenstein bound the entropy of the whole system is $2\pi M_{\text{MSH}} R_2$. Regarding the Bekenstein bound to reach maximum entropy, we have to consider the amount of M_{MSH} by $R = R_2$. In doing so, by using Eqs. (13) and (28) we have

$$M_{\text{MSH}} = \frac{H^2 R^3}{2} + m_0 = \frac{R}{2}. \quad (32)$$

Now, we derive m_0 as follows

$$2m_0 = R^2(1 - H^2 R^2). \quad (33)$$

Then, because of $m_0 \geq 0$ we have $R \leq \frac{1}{H}$. Note that we obtained Eq. (33) in the sense that the local inhomogeneity, namely m_0 part, is a black hole. If the m_0 part is not a black hole, we still have the same constraint $R \leq \frac{1}{H}$. Therefore, by imposing the holographic principle we obtain a constraint over cosmological horizon by Hubble radius.

We summarize our findings in this section in the following remark.

Remark 3 When the local matter system in the McVittie spacetime is a true black hole, the entropy-area law and holographic principle put two different constraints on the cosmological horizon radius R . These constraints are the results of the fact that a black hole admits the saturated entropy. The first one states that there is a correspondence between the inner apparent horizon area and the cosmological horizon area which results in that the Hubble radius is discrete such that for any arbitrary value of black hole mass m_0 , the value of R is given up to an integer number. The latter represents that when a black hole is placed in a cosmological background, the cosmological horizon radius is constrained as $R \leq \frac{1}{H}$. The equality case is provided only by disappearance of the black hole.

4 Conclusion

Based on the generalized second law of thermodynamics, we have imposed a hypothesis of “ D -bound-Bekenstein bound identification” for dilute systems on the McVittie spacetime surrounded by dark energy fields. We have found that

- This identification is realized for “all times” of cosmological constant domination $\omega = -1$, “early times” of phantom field domination $\omega < -1$, and “late times” of quintessence field domination $\omega > -1$.
- This identification is not realized for $\omega \neq -1$ in “late times” and “early times” for “phantom” and “quintessence” fields, respectively.

Therefore, since the cosmological constant preserves and respects this identification at “all times”, we can single out the cosmological constant as the viable dark energy, consistent with this hypothesis, and rule out the quintessence and phantom fields in the study of McVittie spacetime. This result may have important impact on the validity of the studies on the McVittie spacetime in the presence of “phantom” and “quintessence” fields.

We have used discrete behaviour of entropy-area law to derive constraint on the radius of cosmological horizon in McVittie solution surrounded by cosmological fields. These constraints reveal a discrete feature of the radius of cosmological horizon. The prominent point

here is that all of these considerations are valid when the system can be diluted. Regarding the McVittie solution surrounded by cosmological constant field, the system can be diluted, and the radius of Hubble horizon can be treated in a discrete way according to the discrete behaviour of inner apparent horizon. This result is true for the McVittie solution with phantom field at early times and quintessence field at late times. At early times for quintessence field and late times for phantom field, the system cannot be diluted, and the discrete behaviour of the inner apparent horizon does not manifest itself in the discrete radius of Hubble horizon.

Acknowledgements We appreciate the anonymous referee for the constructive comments that significantly improved the quality of the paper.

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