# Dynamic $L_2$ Output Feedback Stabilization of LPV Systems With Piecewise Constant Parameters Subject to Spontaneous Poissonian Jumps

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***Abstract*—This letter addresses the $L_2$ output feedback stabilization of linear parameter varying systems, where the parameters are assumed to be stochastic piecewise constants under spontaneous Poissonian jumps. We provide sufficient conditions in terms of linear matrix inequalities (LMIs) for the existence of a full-order output feedback controller. Such LMIs, however, can be computationally intractable due to the presence of integral terms. Nevertheless, we show that these LMIs can be equivalently represented by an integral-free LMI, which is computationally tractable. Finally, we provide analytical formulas to construct the controller and illustrate the applicability of the results through examples.**

***Index Terms*—LPV systems, output-feedback, stochastic hybrid systems.**

## I. Introduction

THE FRAMEWORK of linear parameter varying (LPV) systems, [1], [2], finds a wide range of real-world applications such as automotive systems [3], [4], aperiodic sampled-data systems [5], and aerospace systems [6]. Morever, LPV systems offer the plausibility of designing gain scheduled controllers [1], [7], [8]. In the recent past, a lot of literature has been dedicated to the study of LPV systems as evidenced by [1], [2], [9]–[14], and the references therein.

For the robust analysis and control of LPV systems, worst-case analysis is usually considered. In this regard, the parameters are either assumed to vary arbitrarily fast, i.e., with unbounded derivatives, or they are assumed to be continuous, i.e., their derivative is bounded. The efficiency of the existing methods for output feedback stabilization are limited due to the generality of the assumptions on the varying parameters. Therefore, by restricting the class of varying parameters, we aim to improve the efficacy of the existing methods.

The notion of quadratic stability is often defined for the case of arbitrarily fast parameter variation, where quadratic Lyapunov functions are considered that are parameter-independent. However, this technique is very conservative for the case of bounded variation rates, where parameter-dependent Lyapunov functions are preferred. Such parameter-dependent Lyapunov functions yield stability conditions that are less conservative [10].

In this letter, we consider a class of parameters that are piecewise constant. Such a class can be considered as a generalization of switched linear systems, where the system modes take values in an interval rather than a finite set. The stability analysis of LPV systems with piecewise constant parameters has been presented in [10]. However, the jumps in the parameter trajectories are assumed to be deterministic and they follow a minimium dwell-time condition, a notion introduced in [15], [16]. Recently, the focus of the research has been to consider LPV systems with stochastically evolving parameters, [17], [18]. Such a consideration matches real scenarios where adrupt variations in the system structure can occur, for example, component failures, haphazard disturbances, varying interconnections between subsystems, and variations in the operating point of a non-linear plant. These systems are well-modeled by Markov jump linear systems (MJLS), which is a class of stochastic dynamical systems. Several papers have been devoted to the stability and control of MJLS, for instance, [19], and [20].

The class of LPV systems considered in this letter generalizes the framework of MJLS with finite or infinite countable set to the case where the mode takes values in uncountable bounded set. The stability analysis of LPV systems with piecewise constant parameters under Poissonian jumps has recently been proposed in [21] where a state feedback controller has been designed by employing a bounded real lemma to ensure satisfactory $L_2$ performance.

By complementing the work of [21], we aim to design a parameter-dependent dynamic output feedback controller for stochastic hybrid LPV systems. The structure of such a controller is full in a sense that no internal structure is considered. This letter is novel because, unlike [21], the controller only requires the knowledge of the system output and not the system state. We assume that the jumps in the parameter trajectories are spontaneous and follow a Poissonian distribution, i.e., the time between two successive jumps is exponentially distributed. The resulting system is a piecewise deterministic Markov process, also known as a stochastic

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hybrid system, [17], [22]. In this framework, the deterministic part consists of the state dynamics and the parameters of the LPV system; whereas the stochastic part considers Markovian update rule for the parameters. The Markovian update rule consists of the jump time and the parameter values.

*Main contribution:* We provide sufficient conditions for the output feedback controller synthesis that takes the minimization of a performance norm into consideration. The synthesis conditions are formulated as infinite dimensional parameter-dependent LMIs. This involves several steps. First, we show how the sufficient conditions can be presented as a feasibility problem of two simultaneous LMIs. Such LMIs are computationally intractable due to the presence of an integral term. Second, we follow the result presented in [21], and [23], and provide a convex infinite-dimensional semi-definite program in order to tackle the intractability issue. We assume that parameter dependent decision variables in LMIs are polynomials, which is quite a reasonable assumption. It is because the parameters take values in a compact set and the polynomials can be used to approximate any continuous function in a compact set, [21].

The general framework of stochastic hybrid LPV systems and related preliminaries are presented in Section II. The main results on output feedback controller synthesis are provided in Section III, whereas Section IV provides analytical formulas for computing the controller matrices. The effectiveness of our approach is illustrated via examples in Section V. Finally, Section VI provides concluding remarks.

*Notation:* The notation will be simplified whenever no confusion can arise from the context. $I_n$ denotes the identity matrix of dimension $n$. Let $\mathbb{R}$ represent the set of real numbers, and let $\mathbb{R}^{n\times m}$ denote real matrices of dimension $n\times m$. The cone of symmetric (positive definite) matrices of dimension $n$ is denoted by $\mathbb{S}^n(\mathbb{S}^n_{\succ 0})$. For $A, B \in \mathbb{S}^n$, the expression $A \prec (\preceq) B$ means that $A-B$ is negative (semi)definite. For some square matrix $A$, we define $Sym[A] = A + A^T$.The Lebesgue measure of a compact set $\mathscr{B}$ is denoted by $\mu(\mathscr{B})$. For a given matrix-valued function $R(\rho) \in \mathbb{R}^n$ and $S(\rho) \in \mathbb{R}^{n\times m}$, we write $R \succ 0$ over $\mathcal{N}(S)$, whenever there exist $\epsilon \succ 0$ such that $R \succeq \epsilon I_n$ over $\mathcal{N}(S), \forall \rho \in \mathscr{B}$, where $\mathcal{N}(\cdot)$ denotes the null-space associated to the given matrix. We define the standard congruence transformation as $\mathfrak{C}(H,L) := L^T H L$, where '$T$' is the transpose. $\mathbb{E}[\,\cdot\,]$ denotes the standard expectation of a random process and $\mathbb{P}[\,\cdot\,]$ is the probability. $||\cdot||$ defines the standard Euclidean norm in $\mathbb{R}^n$.

## II. System Description and Preliminaries

We consider LPV systems with stochastic piecewise constant parameters subject to Poissonian jumps described by the following stochastic hybrid dynamics

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + E(\rho)w(t)\\ z(t) &= C(\rho)x(t) + D(\rho)u(t) + F(\rho)w(t)\\ y(t) &= C_y(\rho)x(t) + F_y(\rho)w(t)\\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_w}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and $z \in \mathbb{R}^{n_z}$ are the state, the exogenous input, the control input, the measured output, and the regulated output, respectively. The parameter vector $\rho(t)$ is piecewise constant (i.e., $\dot{\rho} = 0$ between the jumps) and randomly change its values with a finite jump intensity. We define $\rho(t)$ as

$$\mathbb{P}[\rho(t+h) \in \mathcal{B} | \rho(t) = \rho] = \kappa(\rho, \mathcal{B})h + o(h) \tag{2}$$

where $\rho \in \mathscr{B}$, $\mathcal{B} \subset \mathscr{B} - \rho$ is measurable, and $\kappa : \mathscr{B} \times \mathscr{B} \to \mathbb{R}_{\geq 0}$ is the *instantaneous jump rate* such that $\rho \mapsto \kappa(\rho, A)$ is measurable and $A \mapsto \kappa(\rho, A)$ is a positive measure. Particularly, $\kappa(\rho, d\theta)$ are transition rates and $\bar{\lambda}(\rho) = \int_{\mathscr{B}} \kappa(\rho, d\theta)$ are intensities. It is clear from its definition that the process $(x(t), \rho(t))_{t\geq 0}$ is a Markov process. For the sake of simplicity, we assume that $\kappa(\rho, d\theta) = \lambda(\rho,\theta)d\theta$ where $\lambda$ is a polynomial function. Similarly, we assume that the matrices in (1) are polynomial functions of $\rho$, making the overall system a polynomial stochastic hybrid system.

We introduce the following definitions and results from [21] which will be substantial in proving our main results.

*Definition 1:* The system (1)-(2) is mean-square stable (MSS) if for an arbitrary initial condition $(x_0, \rho_0)$, we have $\mathbb{E}[||x(t)||_2^2] \to 0$ as $t \to \infty$.

*Definition 2:* The $L_2$-norm of a signal $w : [0,\ \infty) \mapsto \mathbb{R}^n$ is

$$||w||_{L_2} = \left(\int_0^\infty \mathbb{E}[||w(s)||_2^2 ds]\right)^{\frac{1}{2}}.$$

If $||w||_{L_2} < \infty$, then the signal is of finite energy and $w \in L_2$.

*Definition 3 [21]:* The (stochastic) $L_2$ gain of the map $L_2 \ni w \mapsto z \in L_2$ with $u \equiv 0$ and $x(0) = 0$ induced by the system (1)-(2) is

$$||w \mapsto z||_{L_2 - L_2} = \sup_{||w||_{L_2}=1} ||z||_{L_2}.$$

*Theorem 1 ($L_2$ Performance-Bounded Real Lemma [21]):* Assume that there exist a matrix-valued function $\mathscr{P} : \mathscr{B} \mapsto \mathbb{S}^n_{\succ 0}$, and a scalar $\gamma \succ 0$ such that the LMI

$$\begin{bmatrix} Sym\big[\mathscr{P}(\rho)A(\rho)\big] + \mathscr{I} & \mathscr{P}(\rho)E(\rho) & C(\rho)^T \\ * & -\gamma^2 I_p & F(\rho)^T \\ * & * & -I_q \end{bmatrix} \prec 0, \tag{3}$$

holds for all $\rho \in \mathscr{B}$, where $\mathscr{I} = \int_{\mathscr{B}} \lambda(\rho,\theta)(\mathscr{P}(\theta) - \mathscr{P}(\rho))d\theta$. Then, the system (1)-(2) with $u, w \equiv 0$ is MSS with $L_2$-gain of $w \mapsto z$ less than $\gamma$.

## III. Main Results

This section presents the dynamic output feedback controller synthesis for stochastic hybrid system (1)-(2). Sufficient conditions for the existence of the $L_2$ controller are provided. We aim to design an output feedback controller of the form

$$\mathcal{K} : \begin{bmatrix} \dot{x}_k(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \mathcal{K}^{11}(\rho) & \mathcal{K}^{12}(\rho) \\ \mathcal{K}^{21}(\rho) & \mathcal{K}^{22}(\rho) \end{bmatrix} \begin{bmatrix} x_k(t) \\ y(t) \end{bmatrix}, \tag{4}$$

where $\mathcal{K}(\rho) : \mathscr{B} \mapsto \mathbb{R}^{(k+n_u)\times(k+n_y)}$.

The augmentation of the controller (4) with plant (1) forms the following closed-loop system

$$\begin{aligned} x_{cl}(t) &= A_{cl}(\rho)x_{cl}(t) + B_{cl}(\rho)w(t)\\ z(t) &= C_{cl}(\rho)x_{cl}(t) + D_{cl}(\rho)w(t) \end{aligned} \tag{5}$$

where $x_{cl}(t)^T = [x(t)^T x_k(t)^T]^T$ is the augmented closed-loop state with $x_{cl} \in \mathbb{R}^{\hat{n}}$ for all $\rho \in \mathscr{B}$ where $\hat{n} = n + k$. The matrices in (5) are parameterized as

$$A_{cl}(\rho) = \begin{bmatrix} A(\rho) + B(\rho)\mathcal{K}^{22}(\rho)C_y(\rho) & B(\rho)\mathcal{K}^{21}(\rho) \\ \mathcal{K}^{12}(\rho)C_y(\rho) & \mathcal{K}^{11}(\rho) \end{bmatrix},$$

$$\begin{bmatrix} Sym[XA+JC_y]+\mu(\mathscr{B})\lambda(\rho,\theta)[X(\theta)-X(\rho)]+Z(\rho,\theta) & XE+JF_y & (C+DUC_y)^T \\ * & -\gamma^2 I_{n_w} & F+DUF_y \\ * & * & -I_{n_z} \end{bmatrix} \prec 0 \tag{8a}$$

$$\begin{bmatrix} Sym[AY+BF]-\bar{\lambda}Y+R(\rho,\theta) & E+BUF_y & (CY+DF)^T & \mu(\mathscr{B})\lambda(\theta,\rho)^{1/2}Y \\ * & -\gamma^2 I_{n_w} & (F+DUF_y)^T & 0_{n_w\times n} \\ * & * & -I_{n_z} & 0_{n_z\times n} \\ * & * & * & -\mu(\mathscr{B})Y(\theta) \end{bmatrix} \prec 0 \tag{8b}$$

$$\begin{bmatrix} Y & 0_n \\ 0_n & X \end{bmatrix} \succ 0 \tag{8c}$$

$$B_{cl}(\rho) = \begin{bmatrix} E(\rho)+B(\rho)\mathcal{K}^{22}(\rho)F_y(\rho) \\ \mathcal{K}^{12}F_y(\rho) \end{bmatrix},$$
$$C_{cl}(\rho) = \begin{bmatrix} C(\rho)+D(\rho)\mathcal{K}^{22}(\rho)C_y(\rho) & D(\rho)\mathcal{K}^{21}(\rho) \end{bmatrix},$$
$$D_{cl}(\rho) = \begin{bmatrix} F(\rho)+D(\rho)\mathcal{K}^{22}(\rho)F_y(\rho) \end{bmatrix},$$

or, equivalently,

$$A_{cl}(\rho) = \hat{A}(\rho)+\bar{B}(\rho)\mathcal{K}(\rho)\bar{C}_y(\rho),$$
$$B_{cl}(\rho) = \hat{B}(\rho)+\bar{B}(\rho)\mathcal{K}(\rho)\bar{F}_y(\rho),$$
$$C_{cl}(\rho) = \hat{C}(\rho)+\bar{D}(\rho)\mathcal{K}(\rho)\bar{C}_y(\rho),$$
$$D_{cl}(\rho) = \hat{D}(\rho)+\bar{D}(\rho)\mathcal{K}(\rho)\bar{F}_y(\rho),$$

with

$$\begin{bmatrix} \hat{A}(\rho) & \hat{B}(\rho) & \bar{B}(\rho) \\ \hat{C}(\rho) & \hat{D}(\rho) & \bar{D}(\rho) \\ \bar{C}_y(\rho) & \bar{F}_y(\rho) & \star \end{bmatrix} = \left[\begin{array}{cc|c|cc} A(\rho) & 0 & E(\rho) & 0 & B(\rho) \\ 0 & 0_k & 0_{k\times n_w} & I_k & 0 \\ \hline C(\rho) & 0_{n_z\times k} & F(\rho) & 0 & D(\rho) \\ \hline 0 & I_k & 0 & \star & \star \\ C_y(\rho) & 0 & F_y(\rho) & \star & \star \end{array}\right],$$

where $\star$ represents an entry which has no importance. In the sequel, we drop the dependence of $\rho$ for the sake of clarity of presentation.

*Proposition 1:* Consider the controller $\mathcal{K}$ given in (4) and a performance bound $\gamma \succ 0$. Then, the closed-loop system (5) is MSS satisfying $||w \mapsto z||_{L_2-L_2} \prec \gamma$ if there exists a matrix-valued function $P : \mathscr{B} \mapsto \mathbb{S}^{\hat{n}}_{>0}$ such that

$$\mathcal{M} + (\mathcal{H}\mathcal{P})^T\mathcal{K}\mathcal{J} + \mathcal{J}^T\mathcal{K}^T(\mathcal{H}\mathcal{P}) \prec 0,$$

where $\mathcal{M} : \mathscr{B} \times \mathscr{B} \mapsto \mathbb{R}^{\hat{n}+n_w+n_z}$ is given as

$$\mathcal{M} = \begin{bmatrix} Sym[P\hat{A}]+\mathscr{H} & P\hat{B} & \hat{C}^T \\ * & -\gamma^2 I_{n_w} & \hat{D}^T \\ * & * & -I_{n_z} \end{bmatrix},$$

and $\mathscr{H} = \int_{\mathscr{B}} \lambda(\rho,\theta)(P(\theta)-P(\rho))d\theta$, holds for $\rho \in \mathscr{B}$. Moreover, $\mathcal{P} : \mathscr{B} \mapsto \mathbb{S}^{\hat{n}+n_w+n_z}_{>0}$, $\mathcal{H} : \mathscr{B} \mapsto \mathbb{R}^{(k+n_u)\times(\hat{n}+n_z+n_w)}$, and $\mathcal{J} : \mathscr{B} \mapsto \mathbb{R}^{(k+n_y)\times(\hat{n}+n_z+n_w)}$ are given by

$$\mathcal{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & I_{n_w} & 0 \\ 0 & 0 & I_{n_z} \end{bmatrix},$$
$$\mathcal{H} = \begin{bmatrix} \bar{B}^T & 0_{(k+n_u)\times n_w} & \bar{D}^T \end{bmatrix},$$
$$\mathcal{J} = \begin{bmatrix} \bar{C}_y & \bar{F}_y & 0_{(k+n_y)\times n_z} \end{bmatrix}.$$

*Proof:* The proof follows directly from Theorem 1. ■

*Theorem 2:* Assume that there exist matrix-valued functions $X : \mathscr{B} \mapsto \mathbb{S}^n_{>0}$, $Y : \mathscr{B} \mapsto \mathbb{S}^n_{>0}$, $J : \mathscr{B} \mapsto \mathbb{R}^{n\times n_y}$, $F : \mathscr{B} \mapsto \mathbb{R}^{n_u\times n}$, $U : \mathscr{B} \mapsto \mathbb{R}^{n_u\times n_y}$, $Z : \mathscr{B} \times \mathscr{B} \mapsto \mathbb{S}^n$, and $R : \mathscr{B} \times \mathscr{B} \mapsto \mathbb{S}^n$ such that the following integral equality constraints hold

$$\int_{\mathscr{B}} Z(\rho,\theta)d\theta = 0, \text{ and} \int_{\mathscr{B}} R(\rho,\theta)d\theta = 0, \tag{*}$$

for $\rho \in \mathscr{B}$, and LMIs (8a), (8b), and (8c), as shown at the top of this page, are satisfied for $\rho, \theta \in \mathscr{B}$, where $\mu(\mathscr{B})$ is the Lebesgue measure of the compact set $\mathscr{B}$. Then, there exists an output-feedback controller $\mathcal{K}$ of the form (4) such that the closed-loop system (5) is MSS in the absence of disturbance $w$ and such that the $L_2$-gain of the map $w \mapsto z$ is less than $\gamma$.

*Proof:* Let us define a matrix-valued function $P$ and its inverse $S = P^{-1}$ as

$$P := \begin{bmatrix} X & Y^{-1}-X \\ Y^{-1}-X & X-Y^{-1} \end{bmatrix}, \; S := P^{-1} = \begin{bmatrix} Y & Y \\ Y & \star \end{bmatrix}. \tag{6}$$

We drop the dependence of $\rho$ when there is no confusion possible. The LMI (8c) implies $X - Y^{-1} \succ 0$, and

$$X - (Y^{-1}-X)(X-Y^{-1})^{-1}(Y^{-1}-X) = Y^{-1} \succ 0.$$

This guarantees that $P \succ 0$ with the particular choice in (6). Now let us introduce $\mathcal{I} = [0_n \; I_n]$, and let us make explicit affine dependence of $A_{cl}$ on $\mathcal{K}^{11}$:

$$A_{cl} = \begin{bmatrix} A+B\mathcal{K}^{22}C_y & B\mathcal{K}^{21} \\ \mathcal{K}^{12}C_y & 0_n \end{bmatrix} + \begin{bmatrix} 0_n & 0 \\ 0 & \mathcal{K}^{11} \end{bmatrix} =: \underline{A}_{cl} + \mathcal{I}^T\mathcal{K}^{11}\mathcal{I}.$$

Now, define $\mathcal{F} = [I_n \; 0_n]^T$, and since $\mathcal{N}(diag(\mathcal{F}, I_{n_w}, I_{n_z})) = \{0\}$, the uniform definiteness is preserved under the congruence transformation $\mathfrak{C}(\cdot, diag(\mathcal{F}, I_{n_w}, I_{n_z}))$ on (3) with $A \to \underline{A}_{cl}$, $E \to B_{cl}$, $C \to C_{cl}$, and $F \to D_{cl}$. With the change of variables $U := \mathcal{K}^{22}$, $J := XBU + (Y^{-1}-X)\mathcal{K}^{12}$, we have

$$\begin{bmatrix} Sym[XA+JC_y]+\mathscr{X} & XE+JF_y & (C+DUC_y)^T \\ * & -\gamma^2 I_{n_w} & (F+DUF_y)^T \\ * & * & -I_{n_z} \end{bmatrix} \prec 0, \tag{7}$$

where $\mathscr{X} = \int_{\mathscr{B}} \lambda(\rho,\theta)(X(\theta)-X(\rho))d\theta$.

We follow the same methodology to prove the second inequality. Applying the congruence transformation $\Sigma := \mathfrak{C}(\mathcal{M}, diag(S, I_{n_w}, I_{n_z})) \equiv \mathcal{C}(\mathcal{M}, \mathcal{P}^{-1})$ followed by $\mathfrak{C}(\Sigma, diag(\mathcal{F}, I_{n_w}, I_{n_z}))$, we have

$$\begin{bmatrix} Sym[AY+BF]+\mathscr{Y} & E+BUF_y & (CY+DF)^T \\ * & -\gamma^2 I_{n_w} & (F+DUF_y)^T \\ * & * & -I_{n_z} \end{bmatrix} \prec 0, \tag{9}$$

where $\mathscr{Y} = -\bar{\lambda}(\rho)Y(\rho) + \int_{\mathscr{B}} \lambda(\rho,\theta)Y(\rho)Y^{-1}(\theta)Y(\rho)d\theta$, and $\bar{\lambda}(\rho) := \int_{\mathscr{B}} \lambda(\rho,\theta)d\theta$.

The integral terms in LMIs (7), and (9) renders the LMIs intractable. To tackle this problem, we employ a result from [23] that stipulates that the conditions in LMIs (8a), and (8b) are equivalent to (7), and (9) for all $\rho, \theta \in \mathscr{B}$, if and only if there exist two matrix-valued functions $R : \mathscr{B} \times \mathscr{B} \mapsto \mathbb{S}^n$, and $Z : \mathscr{B} \times \mathscr{B} \mapsto \mathbb{S}^n$ such that $\int_{\mathscr{B}} R(\theta,\rho)d\theta = 0$, and $\int_{\mathscr{B}} Z(\theta,\rho)d\theta = 0$, hold for $\rho \in \mathscr{B}$. Hence the integrals $\mathscr{X}$, and $\mathscr{Y}$ in (7), and (9) can be replaced by $\mu(\mathscr{B})\lambda(\rho,\theta)(X(\theta) - X(\rho)) + Z(\rho,\theta)$, and $-\bar{\lambda}(\rho)Y + \mu(\mathscr{B})\lambda(\rho,\theta)Y(\rho)Y^{-1}(\theta)Y(\rho) + R(\rho,\theta)$, respectively. A Schur complement followed by the change of variables $F := UC_yY + \mathcal{K}^{21}Y$ yields the matrix inequality (8b). ∎

The conditions formulated in the Theorem 2 are infinite-dimensional semi-definite programs and can not be solved directly. To make them tractable, one can employ gridding methods that will result in an approximate finite-dimensional semi-definite program.

*Remark 1:* By considering a generic choice of Lyapunov function $P = \begin{bmatrix} X & P_2 \\ P_2^T & P_3 \end{bmatrix}$ and its inverse $S := P^{-1} = \begin{bmatrix} Y & S_2 \\ S_2^T & S_3 \end{bmatrix}$, the equality $S = P^{-1}$ leads to a coupling condition $X = (Y - S_2S_3^{-1}S_2)^{-1}$ which is non-linear and non-convex in nature and renders intractability.

## IV. Controller Construction

This section deals with the construction of controller matrices. The approach is inspired by the work of [19] which provides formulas for the computation of controller matrices.

*Theorem 3:* Suppose there exist matrix-valued functions $X : \mathscr{B} \mapsto \mathbb{S}^n_{>0}$, $Y : \mathscr{B} \mapsto \mathbb{S}^n_{>0}$, $J : \mathscr{B} \mapsto \mathbb{R}^{n\times n_y}$, $F : \mathscr{B} \mapsto \mathbb{R}^{n_u\times n}$, and $U : \mathscr{B} \mapsto \mathbb{R}^{n_u\times n_y}$ satisfying simultaneous LMIs (8a), (8b), and (8c) and integral equality constraints in (*). Then, the following full-order LPV controller guarantees the MSS of the closed-loop (5) with the performance bound $\gamma \succ 0$:

$$\begin{aligned}
\mathcal{K}^{12} &= (Y^{-1} - X)^{-1}(J - XBU),\\
\mathcal{K}^{21} &= (F - UC_yY)Y^{-1},\\
\mathcal{K}^{22} &= U,\\
\mathcal{K}^{11} &= -(Y^{-1} - X)^{-1}\Big\{X(AY + BF) + (J - XBU)C_yY\\
&\quad + \tilde{A}^T - \int_{\mathscr{B}} \lambda(\theta,\rho)X(\theta)d\theta + \bar{\lambda}I + \tilde{C}^T(CY + DF)\\
&\quad + \big[XE + JF_y + \tilde{C}^T\tilde{D}\big] \times (\gamma^2 I + \tilde{D}^T\tilde{D})^{-1}\\
&\quad \times \big[\tilde{B} + (CY + DF)^T\tilde{D}\big]^T\Big\}Y^{-1},
\end{aligned}$$

where

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} A & E \\ C & F \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} U \begin{bmatrix} C_y & F_y \end{bmatrix},$$

for $\rho \in \mathscr{B}$.

*Proof:* If LMIs (8a)-(8c) are feasible subject to integral equality constraints in (*), then there exist $X$ and $Y$ such that Theorem 2 is satisfied. Moreover, there exists $P$ of the form (6) ensuring the existence of $L_2$ controller satisfying performance bound $\gamma \succ 0$. From Theorem 2, we have the following parametrization of matrices

$$\begin{aligned}
&J = XBU + (Y^{-1} - X)\mathcal{K}^{12}, \ F = UC_yY + \mathcal{K}^{21}Y,\\
&U = \mathcal{K}^{22},
\end{aligned}$$

which together with (6) yield $\mathcal{K}^{12}, \mathcal{K}^{21}$, and $\mathcal{K}^{22}$. For $\mathcal{K}^{11}$, notice that feasibility of (3) is equivalent to $N^\gamma(P) \prec 0$, where

$$\begin{aligned}
N^\gamma(P) &:= A_{cl}^TP + PA_{cl} + \mathscr{H} + C_{cl}^TC_{cl}\\
&\quad + (PB_{cl} - C_{cl}^T\tilde{D})(\gamma^2 I + \tilde{D}^T\tilde{D})^{-1}(B_{cl}^TP - \tilde{D}C_{cl}),
\end{aligned}$$

for $\rho \in \mathscr{B}$. Theorem 2 guarantees that $\gamma^2 I + \tilde{D}^T\tilde{D} \prec 0$, where $\tilde{D} = F + DUF_y$. Define

$$\mathfrak{R} = \begin{bmatrix} \mathfrak{R}^{11} & \mathfrak{R}^{21^T} \\ \mathfrak{R}^{21} & \mathfrak{R}^{22} \end{bmatrix} = \mathcal{Y}^T N^\gamma(P)\mathcal{Y}, \ \text{where } \mathcal{Y} := \begin{bmatrix} Y & I \\ Y & 0 \end{bmatrix}.$$

Applying the congruence transformation $\mathfrak{C}(N^\gamma(P), \mathcal{Y})$, it is easy to see that $\mathfrak{R}^{11}, \mathfrak{R}^{22} \prec 0$, and finally, setting $\mathfrak{R}^{21} = 0$ yields $\mathcal{K}^{11}$. This concludes the proof. ∎

## V. Simulations

In this section, we provide two illustrations to demonstrate the effectiveness of our approach. First, we present an academic example where we consider nonzero $D(\rho)$ and $F(\rho)$. Second, we consider a real application of VTOL helicopter.

### *A. Example 1-Academic Example*

Consider the system (1)-(2) with the following matrices

$$A(\rho) = \begin{bmatrix} 3-\rho & 1 \\ 2-\rho & 2+\rho \end{bmatrix}, \ E(\rho) = \begin{bmatrix} 0 \\ 1+\rho \end{bmatrix}, \ B(\rho) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C(\rho) = C_y(\rho) = [0\ 1], \ F(\rho) = \rho, \ D(\rho) = 0.2, \ F_y(\rho) = 0.$$

Choosing the parameter space $\mathscr{B} = [0\ 1]$, $\lambda(\rho,\theta) = 100$, $\gamma = 6$, we solve the LMIs (8a)-(8c) by determnistic gridding method with fifty points via YALMIP [24], [25], and SeDuMi solver [26], see [21] for computational aspects. Second order polynomial matrices are used for computing both $\rho$ and $(\rho,\theta)$ dependent unknown matrices and we compute the controller matrices of the form (4). The controller matrices are given in (10), and (11a), as shown at the bottom of the next page, and their variation as a function of parameter $\rho(t)$ is provided in Fig. 1.

$$\begin{aligned}
\mathcal{K}^{12}(\rho) &= \frac{1}{b(\rho)}\begin{bmatrix} 0.0028737\rho^4 - 0.18174\rho^3 + 74.02\rho^2 + 7258.5\rho + 78390 \\ -0.00533\rho^4 - 0.2735\rho^3 + 56.522\rho^2 + 1871.7\rho + 15182 \end{bmatrix},\\
\mathcal{K}^{21}(\rho) &= \frac{1}{c(\rho)}\begin{bmatrix} -0.0005661\rho^2 - 0.73281\rho - 33.408 \\ 0.00048475\rho^2 - 0.20983\rho - 14.465 \end{bmatrix}^T,\\
\mathcal{K}^{22}(\rho) &= 8.7333 - 0.083551\rho,
\end{aligned} \tag{10}$$

where $b(\rho) = 0.025941\rho^2 + 5.481\rho + 53.697$ and $c(\rho) = -2.8204e - 8\rho^2 + 0.0065797\rho + 0.62452$.

For the simulation purpose subject to disturbance, we set $w(t) = H(t) - H(t-1)$, where $H(\cdot)$ is the Heavyside step function. Fig. 2 (top) demonstrates the evolution of the states with an initial condition $[1,\ -1]^T$ subject to disturbance $w(t)$ and a typical stochastic trajectory depicted at the bottom of the same figure. For designing the stochastic parameter trajectory, exponentially distributed jump intervals have been considered, and on each jump the value of the parameter is drawn from a standard uniform distribution $\mathcal{U}(0,1)$.

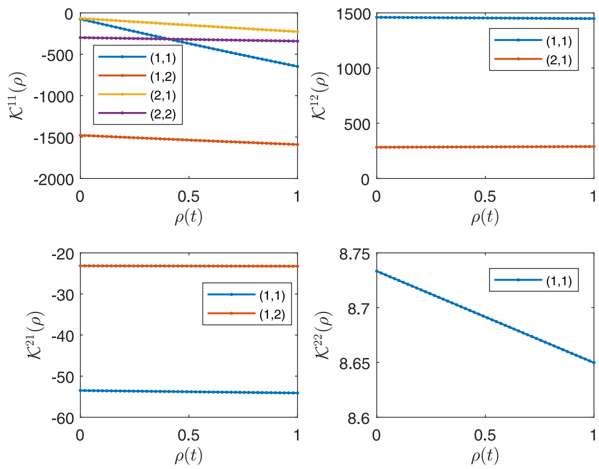


Fig. 1. Controller matrices as a function of $\rho(t)$.

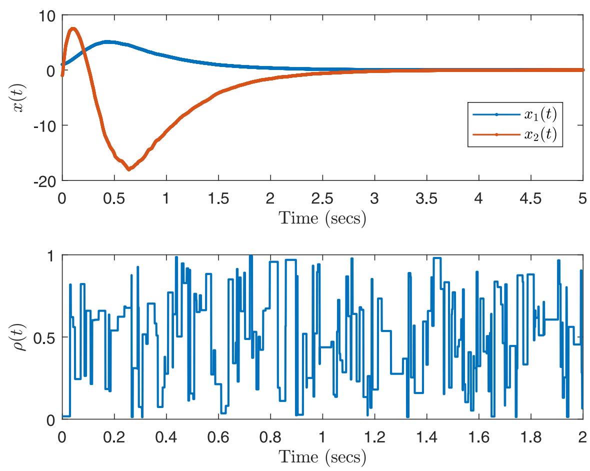


Fig. 2. Evolution of the states of the closed-loop with disturbance (top) subject to a typical stochastic trajectory.

## B. Example 2-VTOL Helicopter Model

We applied the controller synthesis technique to a VTOL helicopter model presented in [20]. In [20], only three modes (finite set) have been considered, however, we modeled the system as an LPV system subject to spontaneous Poissonian jumps taking values in uncountable bounded set. The dynamics of the system can be given as

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + Ew(t)\\ z(t) &= Cx(t) + Du(t) + Fw(t)\\ y(t) &= C_y x(t) + F_y w(t)\end{aligned} \tag{12}$$

where $\rho$ is a time-varying parameter and the state variables $x(t)^T = [x_1\ x_2\ x_3\ x_4]^T$ are horizontal velocity, the vertical

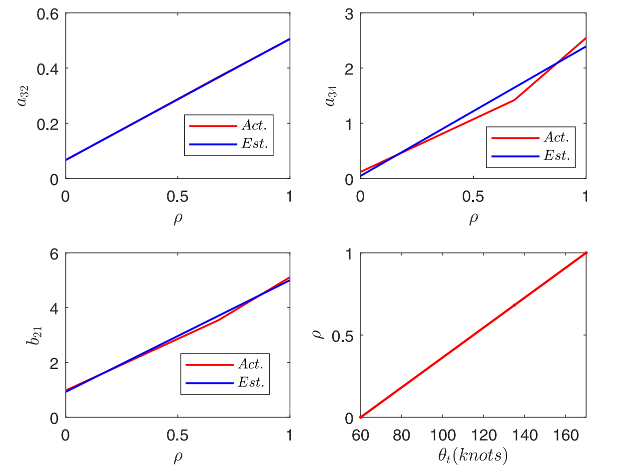


Fig. 3. Comparison of the actual and estimated entries of state and input matrices.

velocity, the pitch rate, and the pitch angle, respectively. The matrices in (12) are given as

$$A(\rho) = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555\\ 0.0482 & -1.01 & 0.0024 & -4.0208\\ 0.1002 & a_{32}(\rho) & -0.7071 & a_{34}(\rho)\\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B(\rho) = \begin{bmatrix} 0.4422 & 0.1761\\ b_{21}(\rho) & -7.5922\\ -5.52 & 4.49\\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} I_{4\times 4} & 0_{4\times 1}\end{bmatrix},$$

$$C = \begin{bmatrix} I_{4\times 4}\\ 0_{2\times 4}\end{bmatrix},\quad D = \begin{bmatrix} 0_{4\times 2}\\ I_{2\times 2}\end{bmatrix},\quad F = 0_{6\times 5},$$

$$C_y = \begin{bmatrix} 0 & 1 & 0 & 0\end{bmatrix},\quad F_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$a_{32}(\rho) = 0.0670 + 0.4390\rho,\ a_{34}(\rho) = 0.0479 + 2.3440\rho,$$

$$b_{21}(\rho) = 0.9263 + 4.0760\rho,$$

and the parameter $\rho$ is defined as $\rho = 0.0091(\theta_t - 60)$, where $\theta_t \in [60\ 170]$ is airspeed in knots. It is easy to see that the parameter space $\mathscr{B} = [0\ 1]$ is a bounded closed set. Fig. 3 shows the variation of $a_{32}, a_{34}$, and $b_{21}$ with respect to $\rho$ and moreover, also compares it with actual values given in [20]. We choose $\lambda(\rho,\theta) = 5$, $\gamma = 12$, and fifty gridding points to solve the inequalities in Theorem 2. Polynomials of order 2 have been employed to compute both $\rho$ and $(\rho,\theta)$ dependent unknown matrices in Theorem 2. We compute the controller matrices of the form (4) by using Theorem 3, where matrices are given as in (11b), as shown at the bottom of the previous page, and (13). The stabilizing state trajectories in Fig. 4 (top) for an arbitrary initial condition subject to a stochastic parameter trajectory shown in Fig. 4 (bottom)

$$\mathcal{K}^{11}(\rho) = \frac{1}{a(\rho)}\begin{bmatrix} 774.06\rho^4 + 20719.0\rho^3 + 32645.0\rho^2 + 751433.0\rho + 89260.0 & 271.59\rho^4 + 9536.1\rho^3 + 59789.0\rho^2 + 343122.0\rho + 1.7854e6\\ 373.35\rho^4 + 6055.2\rho^3 + 16050.0\rho^2 + 212566.0\rho + 79482.0 & 138.14\rho^4 + 2655.9\rho^3 + 15074.0\rho^2 + 93707.0\rho + 360844.0 \end{bmatrix}$$

$$a(\rho) = 0.052182\rho^4 + 3.7755\rho^3 + 35.416\rho^2 + 135.95\rho + 1207.2 \tag{11a}$$

$$\mathcal{K}^{11}(\rho) = \frac{1}{a(\rho)}\begin{bmatrix} 2.92e12\rho^2{+}1.77e14\rho{+}5.23e13 & -7.37e11\rho^2{-}2.46e14\rho{-}1.13e16 & -3.16e12\rho^2{-}2.6e14\rho{+}2.2e14 & -6.72e12\rho^2{-}5.88e14\rho{+}3.29e14\\ -3.2946e11\rho^2{-}2.8267e13\rho{+}1.26e13 & 7.71e11\rho^2{+}1.39e14\rho{+}7.13e15 & 7.8e11\rho^2{+}9.22e13\rho{-}1.53e14 & 1.99e12\rho^2{+}2.42e14\rho{-}2.63e14\\ 1.37e12\rho^2{+}7.38e13\rho{-}9.32e13 & -4.28e11\rho^2{-}1.18e14\rho{-}5.13e15 & -1.5e12\rho^2{-}1.13e14\rho{+}1.93e14 & -3.35e12\rho^2{-}2.81e14\rho{+}3.41e14\\ 1.24e12\rho^2{+}7.88e13\rho{+}1e13 & -2.5361e11\rho^2{-}1.12e14\rho{-}5.43e15 & -1.33e12\rho^2{-}1.19e14\rho{+}1e14 & -2.84e12\rho^2{-}2.72e14\rho{+}1.73e14 \end{bmatrix}$$

$$a(\rho) = -7.8377e9\rho^2 + 6.8309e10\rho + 1.0473e13 \tag{11b}$$

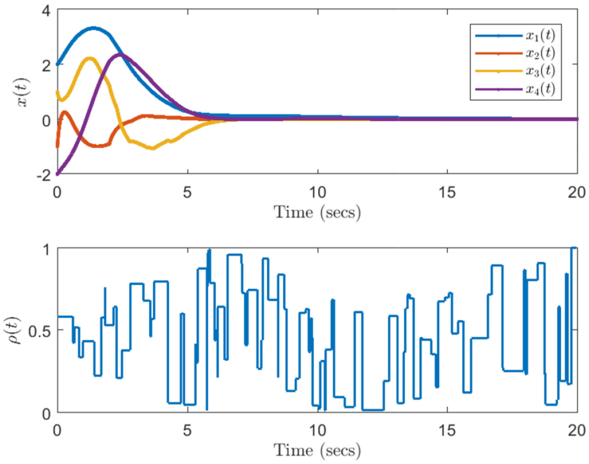


Fig. 4. Evolution of the states of the closed-loop with disturbance (top) subject to a typical parameter trajectory.

under the disturbance $w(t) = [v(t)\ v(t)\ v(t)\ v(t)\ v(t)]^T$, where $v(t) = 0.5(H(t) - H(t-5))$, support the applicability of the approach.

$$\mathcal{K}^{12}(\rho) = \frac{1}{b(\rho)}\begin{bmatrix} -2244.5\rho - 265900.0 \\ 1451.4\rho + 165233.0 \\ -1073.0\rho - 119811.0 \\ -964.48\rho - 127266.0 \end{bmatrix}, \mathcal{K}^{22}(\rho) = 0_{2\times 1},$$

$$\mathcal{K}^{21}(\rho) = \frac{1}{c(\rho)}\begin{bmatrix} -0.017448\rho - 1.1789 & 0.0034003\rho + 0.083679 \\ 0.26009 - 0.0059397\rho & 0.0043718\rho + 0.64351 \\ 0.013744\rho + 1.001 & 0.065493 - 0.001407\rho \\ 0.026131\rho + 1.7793 & -0.0053901\rho - 0.4661 \end{bmatrix}^T, \tag{13}$$

where $b(\rho) = 1.1365\rho + 244.76$, and $c(\rho) = 0.0012988\rho + 0.69104$.

*Remark 2:* The integral equality constraints on $Z$ and $R$ in Theorem 2 can be easily implemented using YALMIP as they are simply equality constraints on the coefficients on the matrix polynomials $Z$ and $R$.

## VI. Concluding Remarks

We provided sufficient conditions for the existence of a full-order output-feedback LPV controller for LPV systems with piecewise constant parameters subject to Poissonian jumps. These systems generalizes the framework of MJLS with finite/infinite countable sets to bounded uncountable sets. We formulated the sufficient conditions as convex infinite-dimensional LMIs and employed standard approximations for the relaxation of semi-definite program. Incorporating the parameter variation during the analysis not only reduces the conservatism of the existing methods, but also makes them more efficient.

The future prospect includes the design of a static $L_2$ output-feedback compensator and its control theoretic applications.

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