On the real, rational, bounded, unit interpolation problem in $\mathcal{H}_\infty$ and its applications to strong stabilization

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Abstract

One of the most challenging problems in feedback control is strong stabilization, i.e. stabilization by a stable controller. This problem has been shown to be equivalent to finding a finite dimensional, real, rational and bounded unit in H_∞ satisfying certain interpolation conditions. The problem is transformed into a classical Nevanlinna–Pick interpolation problem by using a predetermined structure for the unit interpolating function and analysed through the associated Pick matrix. Sufficient conditions for the existence of the bounded unit interpolating function are derived. Based on these conditions, an algorithm is proposed to compute the unit interpolating function through an optimal solution to the Nevanlinna–Pick problem. The conservatism caused by the sufficient conditions is illustrated through strong stabilization examples taken from the literature.

Keywords

Unit interpolation, rational interpolation, strong stabilization, stable controller

Introduction

This paper studies the Real, Rational, Bounded, Unit Interpolation Problem (RRBUIP), which is closely related to strong stabilization and simultaneous stabilization problems in feedback control theory (Abdallah et al., 1995; Bredemann, 1995; Wakaiki et al., 2013; Xin and Liu, 2013). In Hara and Vidyasagar (1990), the bounded unit interpolation problem is defined and its connections with sensitivity shaping and robust stabilization by a stable controller are discussed. The definition of RRBUIP is as follows.

Problem definition: Given interpolation data $\mathcal{A} = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$ and $\mathcal{B} = \{\beta_0, \beta_1, \dots, \beta_k\}$ where $\alpha_i \in \mathbb{C}_+$ and $\beta_i \in \mathbb{C}$ for $i \in \{0, 1, \dots, k\}$, find a finite dimensional, real, rational function $F(s)$ such that

- I1. $F \in \mathcal{H}_\infty$
- I2. $F^{-1} \in \mathcal{H}_\infty$
- I3. $F(\alpha_i) = \beta_i$ for all $i \in \{0, 1, \dots, k\}$
- I4. $\|F\|_\infty < 1$

where $\mathcal{H}_\infty$ is the set of all bounded analytic functions on $\mathbb{C}_+$. Naturally, the set $\mathcal{A}$ consists of distinct elements and in order to obtain solutions with real coefficients, we assume that set $\mathcal{A}$ and $\mathcal{B}$ are conjugate symmetric, i.e. if $\alpha_j \in \mathcal{A}$ and $\beta_j \in \mathcal{B}$ then $\bar{\alpha}_j \in \mathcal{A}$, $\bar{\beta}_j \in \mathcal{B}$, and $F(\bar{\alpha}_j) = \bar{\beta}_j$ for all $j \in \{0, 1, \dots, k\}$ where $\bar{\alpha}_j$ and $\bar{\beta}_j$ are the complex conjugate of α_j and β_j , respectively.

Note that if we only consider I1, I3 and I4, the problem reduces to a classical Nevanlinna–Pick interpolation problem which is known to be solvable if and only if the associated Pick matrix is positive definite. The reader can be directed to Ball et al. (1990) for the details and all suboptimal solutions to this problem. In the literature, there are some other

suboptimal solution methods for the Nevanlinna–Pick interpolation problem which involve some mappings (Möbius transforms or conformal maps) (Khargonekar and Tannenbaum, 1985; Doyle et al., 1992; Foias et al., 1996; Zeren and Özbay, 1998). Recently, an optimal solution to the Nevanlinna–Pick interpolation problem has been described in Yücesoy and Özbay (2016), without any mappings, through an eigenvalue–eigenvector decomposition. The method of this paper is going to make use of this optimal solution method to the Nevanlinna–Pick interpolation problem.

In feedback control theory, the stabilization of a plant by a stable controller is called strong stabilization. The motivation for strong stability comes from a robustness to sensor failures (Doyle et al., 1992; Ünal and Iftar, 2012b) and possibility of controller verification in open loop (van de Wal et al., 2002).

There is a range of literature surrounding the strong stabilization of finite dimensional systems (Campos–Delgado and Zhou, 2003; Cheng et al., 2007, 2011; Gümüşsoy and Özbay, 2009; Gündeş and Özbay, 2011; Petersen, 2009). For infinite dimensional systems, sensitivity shaping by a stable controller has been studied in Gümüşsoy and Özbay (2008). Most of these methods rely on finding an interpolating function F which satisfies I1, I2 and I3. This problem can be called a unit

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interpolation problem and there are some alternative approaches to solve this problem or a relaxed version of it (positive real interpolation) (Ball et al., 1990; Doyle et al., 1992; Vidyasagar, 1985). Another paper which studies stable and $\mathcal{H}_\infty$ controller design for multi-input multi-output systems with multiple input/output time delays, making use of the small gain theorem, is Ünal and İftar (2012c). Readers are also directed to Luy et al. (2014) and Luy (2017) for recent advances in $\mathcal{H}_\infty$ optimal control methods in state-space representations with applications.

It is essential to note that a parity interlacing property (PIP) is necessary for a plant to be strongly stabilized. In other words, plants without an even number of poles between any pair of right half plane zeros on the extended positive real axis cannot be stabilized by a stable controller. In addition, according to Ünal and İftar (2012a), a PIP is also sufficient for time delay systems with added restrictions.

To the best of our knowledge, robust stabilization by a stable controller for infinite dimensional systems is an open research problem and one of the most recent contributions has been made in Wakaiki et al. (2013), where it is shown that robust stabilization of an infinite dimensional system by an infinite dimensional stable controller can be reduced to a bounded unit interpolation problem. In general, it is easy to show that for the finite dimensional case, robustly stabilizing stable controller design can be reduced to bounded unit interpolation.

Assume that $P = N/D$ is the finite dimensional plant and $N, D \in \mathcal{H}_\infty$ is a co-prime factorization of the plant where $|D(j\omega)| = 1$ for all ω . Then, it is known that $C = (X + DQ)/(Y - NQ)$ is the parametrization of all stabilizing controllers provided that $X, Y, Q \in \mathcal{H}_\infty$ and $XN + DY = 1$. Let us define $R = Y - NQ$; in this case $R \in \mathcal{H}_\infty$ has to satisfy the interpolation conditions $R(z_i) = 1/D(z_i)$ for internal stability of the feedback loop where z_i stands for the zeros of the plant P in $\mathbb{C}_+$. It is obvious that $R^{-1} \in \mathcal{H}_\infty$ is necessary for strong stabilization. For robust stabilization

$$\|WT\|_\infty = \|W(1 - DR)\|_\infty < 1$$

is required, where W is the multiplicative plant uncertainty (Doyle et al., 1992). Following the ideas of Wakaiki et al. (2013) it is easy to see that for $\|WT\|_\infty < 1$ it is sufficient to have

$$|R(j\omega)| < (1 - |W(j\omega)|)/|W(j\omega)|$$

for all ω provided that $\|W\|_\infty < 1$. In addition to these, assume that there exists a W_a which satisfies $W_a, W_a^{-1} \in \mathcal{H}_\infty$ and

$$|W_a(j\omega)| < (1 - |W(j\omega)|)/|W(j\omega)|$$

for all ω . Hence, we can find a robustly stabilizing stable controller if it is possible to find a bounded unit F such that $F, F^{-1} \in \mathcal{H}_\infty$, $\|F\|_\infty < 1$ and it satisfies the interpolation conditions $F(z_i) = 1/(W_a(z_i)D(z_i))$. If such a function F exists, then $R = FW_a$ gives the desired controller.

In light of the above discussion, this paper aims to find a solution to the finite dimensional, real, rational, bounded unit interpolation problem in $\mathcal{H}_\infty$, since robust stabilization using a stable controller can be reduced to this type of problem.

Finite dimensionality of the interpolating function is crucial for practical purposes. The necessary and sufficient conditions for an infinite dimensional bounded unit interpolating function is given in Ball and Helton (1979) and Tannenbaum (1982) through a modified Pick matrix. In Gümüsoy and Özbay (2009) and Özbay (2010), a solution method for the infinite dimensional case is discussed. This paper aims to find a sufficient condition for the finite dimensional case and to derive an algorithm for the desired interpolating function. The conservatism of the proposed method is also compared to the infinite dimensional case as well.

The rest of the paper is organized as follows: Section ‘Known solutions of bounded unit interpolation’ briefly explains the known finite dimensional solutions of the bounded unit interpolation problem together with the relevant literature about the positive real interpolation problem and generalized entropy criteria. Section ‘Solution through optimal Nevanlinna–Pick interpolant’ is the novel contribution of this paper as it explains the method used to generate a bounded unit interpolating function having a predetermined form if the necessary conditions are satisfied. At the end of the section, an algorithm is provided as a summary of the proposed method for practical purposes. Two simple interpolation problems from previous literature are also revisited in this section in order to illustrate the performance comparison of the proposed method. Four different illustrative examples from literature on strong stabilization are revisited and solved by the proposed method in section ‘Examples’. The final section concludes the paper with a discussion and possible future studies.

Known solutions of bounded unit interpolation

In Abdallah et al. (1995), sufficient conditions to find a solution for RRBUIP are derived. The conservatism of these conditions is represented by a two-point interpolation problem in Abdallah et al. (1995) and by a three-point interpolation problem in Bredemann (1995). Both examples will be revisited in the next section and solved by the proposed method of this paper to give a comparison. The method of Abdallah et al. (1995) solves the bounded rational real unit interpolation problems with interpolants of a higher degree than the method proposed in this paper.

In Yücesoy and Özbay (2015), a method to solve bounded rational real unit interpolation problems with only real interpolation data is introduced. This method modifies the algorithm defined in Doyle et al. (1992) and Vidyasagar (1985). The original algorithm is designed to find an interpolating unit without a bound on the infinity norm. The modification decreases the norm of the interpolating function at the cost of increasing the order. The effectiveness of the method on two-point interpolation problems with real data was shown in Yücesoy and Özbay (2015). However, this method lacks the ability to solve problems with complex interpolation data.

In Ball et al. (1990), the parametrization of all solutions to the positive real interpolation problem is defined in terms of four transfer functions defined by the interpolation data. Note that any positive real rational function F is also a unit

in $\mathcal{H}_\infty$; (i.e. $F, F^{-1} \in \mathcal{H}_\infty$). This method is capable of handling both real and complex interpolation data; however, there exists no insight on how to bound the infinity norm of the resulting interpolating function.

There have been some efforts in the literature which try to solve the bounded unit interpolation problem through positive real functions: Byrnes et al. (2001) and Georgiou (1999) formulated the problem of positive real interpolation as a maximization problem with a generalized entropy criterion. The dual of this problem is a convex optimization problem in a finite dimensional space. The bound on the infinity norm of the interpolating function is modelled as a constraint to the minimization problem; Fanizza et al. (2007) utilizes these ideas to find a passive finite dimensional approximate for originally passive systems by analytic interpolation. The method from Fanizza et al. (2007) produces positive real interpolating functions with a finite dimension which closely approximates the frequency response of the original system. Furthermore, Karlsson et al. (2010) also uses the same approach towards analytic interpolation and solves the finite dimensional bounded interpolation problem with a possible non-minimum phase but stable interpolating function. Although all of these studies are related to an analytic interpolation problem, none of them directly addresses RRBUIP.

Solution through optimal Nevanlinna–Pick interpolant

In this paper we consider a form of $F(s)$ given as

$$F(s) = [\hat{F}(s)]^n \quad (1)$$

where $\hat{F}(s) = \frac{G(s)+1}{G(s)+\rho}$ and n is a positive integer with $\rho \in \mathbb{R}$ and $G \in \mathcal{H}_\infty$.

Proposition 1. $\hat{F}$ defined by (1) is a unit function in $\mathcal{H}_\infty$, i.e. $\hat{F} \in \mathcal{H}_\infty$ and $\hat{F}^{-1} \in \mathcal{H}_\infty$, if $\rho > 1$, $G \in \mathcal{H}_\infty$ with $\|G\|_\infty < 1$. Moreover, under these conditions, we have $\|\hat{F}\|_\infty < 1$.

Proof. The fact that $\hat{F}$ is a unit in $\mathcal{H}_\infty$ comes from the small gain theorem. In order to prove that $\|\hat{F}\|_\infty < 1$, let us follow the definition of the norm

$$\|\hat{F}(s)\|_\infty = \sup_{\omega} |\hat{F}(j\omega)| = \sup_{\omega} \left| \frac{G(j\omega)+1}{G(j\omega)+\rho} \right| \quad (2)$$

Using (2) we can rewrite the statement $\|\hat{F}\|_\infty < 1$ as

$$\sup_{\omega} \left| \frac{x(\omega) + jy(\omega) + 1}{x(\omega) + jy(\omega) + \rho} \right| < 1 \Leftrightarrow \frac{(x(\omega) + 1)^2 + y^2(\omega)}{(x(\omega) + \rho)^2 + y^2(\omega)} < 1, \forall \omega \quad (3)$$

where $G(j\omega) = x(\omega) + jy(\omega)$, and $x(\omega), y(\omega) \in \mathbb{R}$ for all ω . By simple algebra and assuming $\rho = 1 + \varepsilon$ for some $\varepsilon > 0$, we need to prove

$$2\varepsilon(x(\omega) + 1) + \varepsilon^2 > 0, \forall \omega \quad (4)$$

Note that the condition

$$\|G(s)\|_\infty = \sup_{\omega} |G(j\omega)| = \sup_{\omega} \sqrt{x^2(\omega) + y^2(\omega)} < 1 \quad (5)$$

implies that $|x(\omega)| < 1$ for all ω . Putting together $\varepsilon > 0$ and $(x(\omega) + 1) > 0$ for all ω , (4) is proven.

Conditions of Proposition 1 correspond to I1, I2 and I4 of RRBUIP.

Proposition 2. If $\hat{F}$ satisfies conditions I1, I2 and I4, then these conditions also hold for any positive integer power of $\hat{F}$, i.e. for $F = \hat{F}^n$ for some positive integer n .

Proof. The case for conditions I1 and I2 is straightforward since F has the same zeros and poles as $\hat{F}$ with multiplicity n . For I4, we can rewrite $\hat{F}(j\omega) = r(\omega)e^{j\theta(\omega)}$ and $F(j\omega) = r^n(\omega)e^{jn\theta(\omega)}$ where $r(\omega) \in \mathbb{R}$ and $\theta(\omega) \in \mathbb{R}$ for all ω . With this interpretation

$$\|\hat{F}\|_\infty = \sup_{\omega} |\hat{F}(j\omega)| = \sup_{\omega} |r(\omega)| < 1 \quad (6)$$

implies that $|r(\omega)| < 1$ for all ω . Hence, we can conclude that

$$\|F\|_\infty = \sup_{\omega} |F(j\omega)| = \sup_{\omega} |r^n(\omega)| = \sup_{\omega} |r(\omega)|^n < 1 \quad (7)$$

since $|r(\omega)| < 1$ for all ω and $n > 0$.

Let us consider the arguments in Ohta et al. (2001) for I1, I2 and I3. Given the interpolation data $\mathcal{A}$ and $\mathcal{B}$ as in RRBUIP, a unit interpolating function of degree kn_0 exists for a positive integer n_0 if the following Nevanlinna–Pick matrix

$$P_{ij} = \left[\frac{\beta_i^{1/n} + \bar{\beta}_j^{1/n}}{\alpha_i + \bar{\alpha}_j} \right]_{i,j \in \{0,1,\dots,k\}} \quad (8)$$

is a positive definite matrix for $n = n_0$. As explained in Ohta et al. (2001), the n -th root is calculated in such a way that if $\beta_i \in \mathcal{B}$ and $\beta_j \in \mathcal{B}$ are conjugate pairs, so are $\beta_i^{1/n}$ and $\beta_j^{1/n}$. All possible combinations of the n -th roots of complex interpolation pairs should be checked to decide if P is a positive definite matrix. It is proven in Ohta et al. (2001) that every unit interpolation problem has a solution in the integer interval $n_0 \leq n < \infty$ if the problem is feasible for some integer $n_0 > 0$. Note that this condition is only for the existence of an interpolating unit; however, it says nothing about the infinity norm (I4) of the interpolating function. The following proposition defines a sufficient condition for the solution of RRBUIP.

Proposition 3. In order to solve the problem defined by RRBUIP, let R be a Pick matrix defined as

$$R_{ij} = \left[\frac{1 - \gamma_i \bar{\gamma}_j}{\alpha_i + \bar{\alpha}_j} \right]_{i,j \in \{0,1,\dots,k\}} \quad (9)$$

where

$$\gamma_i = \frac{\rho \beta_i^{1/n} - 1}{1 - \beta_i^{1/n}} \quad (10)$$

for $i \in \{0,1,\dots,k\}$. If R is positive definite for some $\rho > 1$ and $n = n_0$, where n_0 is a positive integer, then a real rational

bounded unit interpolating function F with degree kn_0 exists and it satisfies all conditions I1, I2, I3 and I4.

Proof. To prove this proposition, let us first note that if R is positive definite for some integer $n = n_0 > 0$, then it is possible to find a rational function $G \in \mathcal{H}_\infty$ of order k which satisfies the interpolation conditions $G(\alpha_i) = \gamma_i$ for all $i \in \{0, 1, \dots, k\}$ and $\|G\|_\infty < 1$. For the calculation of optimal $G(s)$, we refer to Yücesoy and Özbay (2016). By using this G , we can write $\hat{F} = (G + 1)/(G + \rho)$ as in (1) and this $\hat{F}$ satisfies $\hat{F}(\alpha_i) = \beta_i^{1/n_0}$ for all $i \in \{0, 1, \dots, k\}$. Note that $\hat{F}$ has degree k and it satisfies I1, I2 and I4 by Proposition 1. For the final step, if we write $F = \hat{F}^{n_0}$, then it satisfies I3; i.e. $F(\alpha_i) = \beta_i$ for all $i \in \{0, 1, \dots, k\}$. F also satisfies I1, I2 and I4 by Proposition 2, hence F is a solution of RRBUIP with degree kn_0 .

It is important to note that having R be positive definite is a sufficient condition to have a solution for the real rational bounded unit interpolation problem provided that the necessary conditions (parity interlacing property and $|\beta_i| < 1$ for all $i \in \{0, 1, \dots, k\}$) are satisfied.

Proposition 3 has two parameters, ρ and n , in order to satisfy R being positive definite. In general, we need to conduct a search on ρ vs. n planes to find the region in which R is positive definite. However, in this study we want to find the lowest possible degree interpolating function; i.e. minimum possible n . In order to achieve this, throughout this paper we will first find the smallest possible n for which R can be made positive definite. But first, let us figure out the effect of $\rho = 1 + \varepsilon$ on $\|\hat{F}\|_\infty$.

Proposition 4. If $\varepsilon = \varepsilon_0$ solves RRBUIP for some positive integer $n = n_0$ then there exists $\varepsilon_1 > \varepsilon_0$ for which the problem is feasible.

Proof. Let us assume that $R^{(0)}$, which is defined by (9) for $\rho = 1 + \varepsilon_0$, is positive definite (i.e. $R^{(0)} > \phi I$ where I is the identity matrix of proper size and $\phi > 0$).

Write $R^{(1)}$ using (9) for $\rho = 1 + \varepsilon_1$ as

$$\begin{aligned} R_{ij}^{(1)} &= \left[\frac{1 - (\varepsilon_0 w_i - 1 + \delta w_i)(\varepsilon_0 \bar{w}_j - 1 + \delta \bar{w}_j)}{\alpha_i + \bar{\alpha}_j} \right] \\ &= R_{ij}^{(0)} + \delta \left[\frac{w_i + \bar{w}_j - 2\varepsilon_0 w_i \bar{w}_j}{\alpha_i + \bar{\alpha}_j} \right] - \delta^2 \left[\frac{w_i \bar{w}_j}{\alpha_i + \bar{\alpha}_j} \right] \quad (11) \\ &= R_{ij}^{(0)} + \delta \Delta_{ij}^{(1)} + \delta^2 \Delta_{ij}^{(2)} \\ R^{(1)} &> I(\phi - \delta \Delta - \delta^2 \Delta) \end{aligned}$$

where

$$\varepsilon_1 = \varepsilon_0 + \delta, \delta > 0, w_i = \frac{\beta_i^{1/n_0}}{1 - \beta_i^{1/n_0}}$$

$$\Delta = \max(\|\Delta^{(1)}\|_\infty, \|\Delta^{(2)}\|_\infty)$$

For $\delta = 0$, we know that $\phi > 0$, hence $R^{(1)} = R^{(0)}$ and $R^{(0)}$ is positive definite by assumption. As δ increases, the right hand side of (11) decreases. However, $R^{(1)}$ is positive definite until it reaches zero. Assume that $\delta_f > 0$ is the point which makes the right hand side of (11) zero. Hence, it is proven that the

problem is feasible when $\delta \in [0, \delta_f) \rightarrow \varepsilon \in [\varepsilon_0, \varepsilon_1)$ where $\varepsilon_1 = \varepsilon_0 + \delta_f > \varepsilon_0$.

Proposition 5. If the RRBUIP is feasible for some $n = n_0$ and $\varepsilon = \varepsilon_0$, then it is possible to decrease the norm of the interpolating function by some $\varepsilon_1 > \varepsilon_0$ if ε_1 also solves the interpolation problem.

Proof. The result is obtained directly from the proofs of Propositions 1 and 4.

Putting all these together, we can divide the problem into two parts.

1. Fix ε as some sufficiently small number and search linearly over n and find the smallest possible n_0 which makes R in (9) positive definite.
2. Using the idea in Proposition 5, fix $n = n_0$ this time and conduct a search on ε to find largest possible ε for which R in (9) stays positive definite.

This interpretation leads us to the smallest degree solution of the RRBUIP within the framework of the proposed method (however, there may be other solutions that give lower-degree solutions). The proposed method is summarized in Algorithm 1 in detail.

In order to understand the conservatism introduced by this sufficient condition, we can compare it to some other sufficient conditions from the literature. In Abdallah et al. (1995), a method to generate bounded unit interpolating functions is introduced. The interpolation problem of

$$(\alpha, \beta) = \{(1, 0.29984), (2, 0.130588)\}$$

is solved by a 5th order unit interpolating function having an infinity norm of 0.8473. By the method proposed in this paper, it is possible to solve the same problem with a 3rd

Algorithm 1. Bounded unit interpolation

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1: Interpolation Data:  $(\alpha_i, \beta_i), i \in \{0, 1, \dots, k\}$ 
2: Maximum Degree Desired:  $n_{max}$ 
3: Continue if PIP is satisfied, jump to Step 19 if not.
4: Continue if all  $|\beta_i| < 1$  for all  $i \in \{0, 1, \dots, k\}$ , jump to Step 19 if not.
5:  $\rho_1 = 1 + \varepsilon$  where  $\varepsilon = 10^{-3}$ .
6:  $M = \text{floor}(n_{max}/k)$ 
7:  $n = 0$ 
8: while  $n < M$  do
9:    $n = n + 1$ 
10:  Calculate  $\gamma_i$  for all  $i \in \{0, 1, \dots, k\}$  using  $\rho_1$  and  $n$  as in (10)
11:  if  $R$  in (9) is positive definite then
12:    Set  $n_0 = n$ 
13:    Set  $\rho_2$  as a big number. (in most practical cases  $\rho_2 = 100$  is sufficiently large)
14:    Binary search on  $\rho \in [\rho_1, \rho_2]$  by using  $n_0$  to find the range over which  $R$  is positive definite  $\rightarrow (\rho_{low}, \rho_{high})$ 
15:    Use Yücesoy and Özbay (2016) to calculate the optimal interpolating function for given  $\rho_{high}$  and  $n_0$ .
16:    return
17:  end if
18: end while
19: No feasible interpolating function exists, exit

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order bounded unit function having an infinity norm of 0.9745. It is important to note that since the infinity norm of both solutions remains below 1, having a smaller degree is an advantage to the proposed method. A 28th order unit is designed by the same method in Bredemann (1995) to solve the interpolation problem with the data

$$(\alpha, \beta) = \{(1, 0.1), (3, 0.2), (5, 0.15)\}.$$

It is indeed possible to solve this problem with an 18th order unit by using the method in this paper.

Examples

The test cases for the proposed algorithm and the conservatism caused by the proposed sufficient condition will be explained by four different examples.

Example 1

Let us revisit the example in Wakaiki et al. (2013) with a slight modification. The plant definition and co-prime factorization of the plant is given as

$$\begin{aligned} P(s) &= \frac{(s - \alpha)(s + 1)(s - 4e^{-s} + 1)}{(s - 10)(s - 15)(e^{-s} + 0.2s + 0.1)} \\ M_n(s) &= \frac{(s - \alpha)(s - p)}{(s + \alpha)(s + p)} \\ M_d(s) &= \frac{(s - 10)(s - 15)(s^2 - 1.4446s + 4.9233)}{(s + 10)(s + 15)(s^2 + 1.4446s + 4.9233)} \\ N_0(s) &= P(s)M_d(s)/M_n(s) \end{aligned} \quad (12)$$

where $p = 0.7990$ is the only zero of the term $(s - 4e^{-s} + 1)$ in $\mathbb{C}_+$. Note that N_0 is outer (i.e. $N_0, 1/N_0 \in \mathcal{H}_\infty$). Let us assume further that we are given a robustness weight of

$$W(s) = K \frac{s + 1}{s + 10}$$

(i.e. $\|WT\|_\infty < 1$ for $T = PC/(1 + PC)$ is required for robust stability) which satisfies $\|W\|_\infty < 1$ when $K < 1$ and there exists a finite dimensional outer approximation W_s such that $|W_s(j\omega)| < 1 - |W(j\omega)|$ for all ω . It has been proven that for such a plant P , a robustly stabilizing stable controller can be designed if it is possible to find a bounded unit interpolating function U such that

$$U(z_i) = \frac{W(z_i)}{M_d(z_i)W_s(z_i)}$$

where $z_1 = \alpha$ and $z_2 = p$ are the only simple zeros of the plant in $\mathbb{C}_+$ (see Wakaiki et al. (2013) for details). The maximum allowable uncertainty bound (i.e. K_{max}) calculated with the method defined in Wakaiki et al. (2013) for each value of α is given in Figure 1. Note that this bound shows the maximum

Figure 1. Maximum allowable multiplicative uncertainty with respect to the location of the unstable zero, see Wakaiki et al. (2013) for details.

value of K where the problem is solvable by an infinite dimensional bounded interpolating function U using their method.

We should also note that Yücesoy and Özbay (2015) has attempted to find finite dimensional bounded interpolating functions for this problem. The disadvantage of the method in Yücesoy and Özbay (2015) is that it only applies to real interpolation data. Otherwise, it gives a good approximation of the maximum allowable uncertainty bound with a 5th order U for each α . Figure 1 also shows the maximum allowable uncertainty bound calculated by a 3rd and 5th order U which is designed by the proposed method of this paper. It is clear that the results in this method are similar to the results in Yücesoy and Özbay (2015) and in addition, the newly proposed method is also capable of handling complex interpolation data. This is a superior feature of the proposed method. It is also important to note that the proposed method approximates the infinite dimensional behaviour more accurately as the order of the interpolating function increases. This is a natural and expected feature of an interpolation method.

Example 2

Let us consider a different example as shown below

$$\begin{aligned} P(s) &= \frac{(s - 100)(s - 1 - j\omega)(s - 1 + j\omega)}{(s - 10)(s + 1)(s + 10)} \\ M_n(s) &= \frac{(s - 100)(s - 1 - j\omega)(s - 1 + j\omega)}{(s + 100)(s + 1 - j\omega)(s + 1 + j\omega)} \\ M_d(s) &= \frac{(s - 10)}{(s + 10)} \end{aligned} \quad (13)$$

$$N_0(s) = P(s)M_d(s)/M_n(s)$$

Note that P has two complex zeros and one real zero in $\mathbb{C}_+$. Because of the complex zeros, the method of Yücesoy and Özbay (2015) is not applicable. Figure 2 shows the maximum allowable uncertainty bound for each value of ω using an infinite dimensional interpolator, a 4th order interpolator and an 8th order interpolator. The controller design method and

Figure 2. Maximum allowable multiplicative uncertainty with respect to real part of the unstable zeros.

robustness weight W are the same as in Example 1. As expected, the maximum allowable uncertainty bound approaches the infinite dimensional interpolator case as the degree of the interpolator increases.

Example 3

This example is taken from Zeren and Özbay (2000), where a method is introduced to design finite dimensional stable controllers which have the same degree as the plant. The example is given to illustrate the MIMO case of the proposed method. In Zeren and Özbay (2000), a MIMO plant P is defined as

$$P = \begin{bmatrix} \frac{(s+1)(s-2-j\theta)(s-2+j\theta)}{(s+2+j)(s+2-j)(s-1)(s-5)}, \\ \frac{(s+5)(s-2-j\theta)(s-2+j\theta)}{(s+2+j)(s+2-j)(s-1)(s-5)} \end{bmatrix} \quad (14)$$

It was shown that, as θ decreases, it becomes more difficult to find a stable controller using their method and indeed, for $\theta < 12$, their solution becomes numerically fragile.

Let us assume that we rewrite the plant P as

$$P = [P_1, P_2]$$

and define

$$P_0 = \frac{(s+3)(s-2-j\theta)(s-2+j\theta)}{(s+2+j)(s+2-j)(s-1)(s-5)} \quad (15)$$

where $P_1 = P_0(1 + W_1)$ and $P_2 = P_0(1 + W_2)$. Then we can also define $W = 2(10^{-5}s + 1)/(s + 3)$ where

$$|W(j\omega)| > |W_1(j\omega)| = |W_2(j\omega)|$$

is satisfied as shown in Figure 3.

Note that if it is possible to find a stable controller C which internally stabilizes P_0 and satisfies $\|WT\|_\infty < 1$ for $T = P_0C/(1 + P_0C)$, then this C will strongly stabilize P . Since $\|W\|_\infty < 1$ as in Figure 3, then we can apply the ideas in

Figure 3. Bode magnitude plots of W_1 , W_2 , W .

Figure 4. Degree of the interpolator with respect to imaginary part of the unstable zeros, see Zeren and Özbay (2000) for details.

Wakaiki et al. (2013) to find finite dimensional stable C . One important observation is that since P_0 is strictly proper, the controller will be improper. However, it is always possible to adjust an improper controller to make it bi-proper without losing stability (see Xin and Liu (2013) for details).

Figure 4 shows the order of the bounded unit interpolating function which was designed by the proposed method of this paper with respect to θ (i.e. imaginary part of the zeros of P_0 in $\mathbb{C}_+$). This point was first discussed in Smith and Söndergeld (1986), that is, the degree of the unit interpolating function increases as the PIP comes closer to violation (i.e. as θ decreases).

As seen from Figure 4, the proposed method in this paper is capable of finding a stable controller for some relatively small values of θ (i.e. $\theta < 12$), whereas the original study was not able to give a numerically stable solution strategy. However, the degree of the controller becomes impractically high as $\theta \rightarrow 0$. The biggest disadvantage of the proposed method is that it can find a 4th order interpolating function at its best, which yields a 6th order controller, whereas the method from Zeren and Özbay (2000) can only find a 4th order controller. Further studies can be undertaken to find conditions which will focus on the degree of the resulting controller.

Figure 5. Degree of the interpolating function with respect to norm of the weighted sensitivity, see Gümüşsoy and Özbay (2007) for details.

Example 4

In Gümüşsoy and Özbay (2007), a method to design stable controllers for sensitivity minimization is proposed by bounded unit interpolation. Let us revisit an example from that paper. We need to find a real, rational transfer function F such that $F(s_i) = w_i/\gamma$ for $i = 1, 2$ where F is also a bounded unit function, $s_{1,2} = 0.3125 \pm 0.8548j$ and $w_{1,2} = 0.79 \mp 0.42j$. Gümüşsoy and Özbay (2007) have proposed a search algorithm to find F and they showed that for $\gamma > 1.08$ it is always possible to find a third order F satisfying all conditions.

Using the proposed method in this paper, as shown in Figure 5, it is possible to find some high degree F for $\gamma > 1.088$. Despite this disadvantage, for $\gamma > 1.124$ the proposed method from this paper is capable of finding F of degree three or less. This might be an advantage in designing low order controllers despite some performance degradation - i.e. for larger γ .

Conclusion

An alternative approach to solve the finite dimensional, real, rational, bounded unit interpolation problem has been proposed. The proposed approach starts with a predetermined form for the interpolating function given by (1) and converts the bounded unit interpolation problem into the classical Nevanlinna–Pick interpolation problem by utilizing the given form. Sufficient conditions are derived by using the associated Pick matrix of the transformed problem on top of the well-known necessary conditions for bounded unit interpolation in $\mathcal{H}_\infty$ (e.g. PIP).

The performance of the proposed approach is compared to other methods from the literature over two different examples: a two-point and a three-point bounded unit interpolation problem. This method from literature addresses the same problem and it was observed that the proposed method is able to find lower-degree interpolating functions in comparison to this approach.

The conservatism caused by the proposed method is discussed in four different strong stabilization problems.

Example 1 is a simple modification of the problem studied in Wakaiki et al. (2013). The same example was also studied in Yücesoy and Özbay (2015), which suggests an interpolation method to interpolate only real interpolation data. The method in this paper performs as well as the method in Yücesoy and Özbay (2015) and additionally has the ability to handle complex interpolation data. Example 2 was created in order to discuss the performance of the proposed method when the complex interpolation data is involved. It is clear from this example that the proposed method can handle complex interpolation data as well. We can also see that, as expected, the proposed method approximates the performance of the infinite dimensional interpolating function better as the allowable degree of the final interpolating function increases. Examples 3 and 4 were considered in order to compare the performance of the proposed method through known examples from strong stabilization literature. The degree of the interpolating function increases rapidly as the problem data comes closer to violating the necessary condition (i.e. parity interlacing property), as expected. This behaviour conforms to the discussions in relevant papers. The proposed method is also able to find a controller in the infeasible region of Zeren and Özbay (2000) at the expense of an increase in the controller degree. The proposed method is also more capable of finding lower-degree controllers than the one given in Gümüşsoy and Özbay (2007) with the expense of a small degradation in the $\mathcal{H}_\infty$ performance.

One main disadvantage of the proposed interpolation algorithm is that it can only find interpolating functions having order $\hat{k}n_0$ where n_0 is a positive integer and $\hat{k}$ is the total number of interpolation conditions, i.e. $\hat{k} = k + 1$. For some smaller sized problems (having 2-3 interpolation points) such as the examples in this paper, this is not a significant problem (indeed this is not a problem at all when there are exactly two interpolation points). However, this can be a major disadvantage when the size of the problem increases. In order to circumvent this, some future work can be conducted in order to find conditions under which the norm of F will remain less than 1, when ρ in (1) is a unit function in $\mathcal{H}_\infty$ instead of being scalar.

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