A short note on Isaacs–Navarro’s Theorem

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Abstract. In this short note, we give a character free proof to a result of Isaacs–Navarro.

Mathematics Subject Classification. 20D10, 20F28.

Keywords. Coprime action, Real elements, Quaternion-free groups.

1. Introduction. Let p be a prime and let K be a p' -group acting on a p -group P . If P is abelian, then it is known that $P = [P, K] \times C_P(K)$ by a result of Fitting [1, Theorem 4.34]. This result easily yields that if K fixes all elements of order p in P , then K acts trivially on P when P is abelian (see [1, Corollary 4.35]). Indeed, the assumption that P is abelian can be removed when p is odd:

Theorem A [1, Theorem 4.36]. *Let K be a p' -group acting on a p -group P where p is odd, and assume that K fixes all elements of order p in P . Then K acts trivially on P .*

It is well known that the same conclusion can be obtained for $p = 2$ by assuming further that K fixes every element of order 4 in P . The following result of Isaacs and Navarro (see [2, Theorem B]) shows that the extra assumption for $p = 2$ can be weakened to assume that K fixes every real element of order 4 in P .

Theorem B (Isaacs, Navarro). *Let K be a group of odd order that acts on a 2-group P , and assume that K fixes all elements of order 2 and all real elements of order 4 in P . Then K acts trivially on P .*

Recall that an element x of a group G is called a **real element** if there exists $y \in G$ such that $x^y = x^{-1}$. The original proof of this result depends on character theory and we give a character free proof of the result in this note. Before giving our alternative proof, we would like to give an application

of Theorem B. Note that a group is called **quaternion-free** if it has no section isomorphic to the quaternion group of order 8.

Corollary C [3, Lemma 2.3]. *Let K be a group of odd order that acts on a quaternion free 2-group P , and assume that K fixes all elements of order 2 in P . Then K acts trivially on P .*

Proof Suppose that K acts nontrivially on P . Then there exists a real element x of order 4 in P such that $[x, K] \neq 1$ by Theorem B. Pick $y \in P$ such that $x^y = x^{-1}$. If y is of order 2, then $\langle x, y \rangle$ is a dihedral group, and it is generated by elements of order 2. Then K acts trivially on $\langle x, y \rangle$ by hypothesis, which is not the case. It follows that y is of order at least 4, and so $\langle x, y \rangle / \langle y^4, x^2 y^{-2} \rangle \cong Q_8$ as it can be checked by the presentation of Q_8 . This contradiction completes the proof. $\square$

2. A new proof of Theorem B.

Lemma D *Let G be a group and $x, y \in G$ be of order 4 such that $x^2 = y^2$ and $[x, y]$ is an involution lying in $Z(G)$. Then xy is a real element of order 4.*

Proof Note that $(xy)^{-1} = y^{-1}x^{-1} = y^3x^3 = y(y^2x^2)x = yx = (xy)^x$, and so xy is a real element. Moreover, we get $(xy)^2 = xyxy = x^2y[y, x]y = x^2y^2[y, x] = [y, x] \neq 1$ by using the fact that $[y, x] = [x, y]^{-1} \in Z(G)$. It follows that xy is of order 4 as desired. $\square$

Proof of Theorem B Let P be a minimal counter example to the theorem. Then we see that P is non-abelian by [1, Corollary 4.35]. Let H be a proper K -invariant subgroup of P . Clearly every element of H of order 2 and every real element of H of order 4 is also fixed by K . Thus, we see that H satisfies the hypothesis, and so $[H, K] = 1$ by the minimality of P . We also see that $[P, K] = P$, since otherwise $[P, K]$ is a proper K -invariant subgroup of P , and so $[P, K, K] = 1$ and coprime action yields that $[P, K] = 1$ by [1, Lemma 4.29], which is not the case.

(1) $P' \leq Z(P)$.

Clearly, both P' and $[P, P']$ are proper K -invariant subgroups of P , and so we see that $[P', K, P] = 1$ and $[P, P', K] = 1$. Then three subgroup lemma yields that $[K, P, P'] = [P, P'] = 1$, that is, $P' \leq Z(P)$ as claimed.

(2) If H is a proper K -invariant subgroup of P , then $H \leq P'$.

Since K acts trivially on both P' and H , it also acts trivially on the group $P'H$. Thus, we see that $P'H < P$. Write $R = P'H$. Then $R/P' \leq C_{P/P'}(K)$. Note that $[P/P', K] = [P, K]P'/P' = P/P'$, and hence we obtain that $C_{P/P'}(K) = 1$ by Fitting's theorem (see [1, Theorem 4.34]). It follows that $R/P' = 1$, and hence $H \leq P'$ as desired.

(3) g^2 lies in P' for all $g \in P$.

Since $\Phi(P)$ is a proper K -invariant subgroup of P , we see that $\Phi(P) \leq P'$ by (2). Hence, we obtain that $g^2 \in \Phi(P) \leq P'$ for all $g \in P$.

(4) There is no real element of order 4 in P .

Let x be a real element of order 4 in P . Then by hypothesis, the group $\langle x \rangle$ is centralized by K , in particular, it is K -invariant. We observe that $x \in P'$

by (2), and so we get that $x \in Z(P)$ by (1). Then $x^g = x \neq x^{-1}$ for all $g \in P$. This contradiction shows that there is no real element of order 4.

(5) Every element of order 4 in P lies in P' .

Let x be an element of order 4 in P . We first claim that $[x, x^k] = 1$ for each $k \in K$. Set $y = x^k$ for some $k \in K$. Note that $(x^2)^k = x^2$ as x^2 has order 2 and K fixes every element of order 2 in P by our hypothesis. Then we see that $x^2 = y^2$. Suppose that $[x, y] \neq 1$. Since $x^2 \in P' \leq Z(P)$ by (3) and (1), we obtain that $1 = [x^2, y] = [x, y]^2$, and so $[x, y]$ is an involution lying in the center of P . It follows that xy is a real element of order 4 by Lemma D, which is not possible by (4). This contradiction shows that $[x, y] = 1$. Thus, we see that $\langle x^K \rangle$ is an abelian group. Note that $\langle x^K \rangle \neq P$ as P is non-abelian. It follows that $\langle x^K \rangle$ is a proper K -invariant subgroup of P , and hence $\langle x^K \rangle \leq P'$ by (2). Then $x \in P'$ as claimed.

Final contradiction Recall that $g^2 \in P' \leq Z(P)$ for all $g \in P$, and hence $1 = [g^2, y] = [g, y]^2$ for any $y \in P$. Then we see that each nontrivial commutator has order 2, and so we get that the exponent of P' is 2 since P' is abelian. Hence, we see that there is no element of order 4 in P by (5). It follows that the exponent of P is 2 and K acts trivially on P by our hypothesis. This contradiction completes the proof. $\square$

Acknowledgements. I would like to thank Prof. I. Martin Isaacs for taking my attention on this topic.

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Received: 8 March 2019