Continuum Mech. Thermodyn. (2019) 31:413–425
https://doi.org/10.1007/s00161-018-0695-x

ORIGINAL ARTICLE

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Application of energy measures in detection of local deviations in mechanical properties of structural elements

Received: 23 October 2017 / Accepted: 20 June 2018 / Published online: 28 June 2018

Abstract The identification of local damages in structural elements is considered. In the proposed formulation, the damages are represented as local changes in structural stiffness and they are defined by a set of parameters. The main effort of this work is directed to the use of global measures of energy to indicate the local changes of stiffness. An idea of use of additional design parameters in order to optimize the experiment and to enlarge the sensitivity of objective functional with respect to damage parameters is applied. In order to accomplish this goal, the distribution of energy within the structural domain is optimized. Two cases, namely the eigenproblem and the structure under the static external load, are discussed. Simple illustrative example of identification of damage is discussed, and numerical results are presented.

Keywords Damage identification · Damage detection · Small deviations · Optimization

1 Introduction

The problem of detection and analysis of deviations in structural elements have motivated the development of several different methods and techniques [1–10]. As a result, several techniques have been proposed to identify

Communicated by Francesco dell'Isola.

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the existence and to characterize the properties of deviations, such as wavelet-based analysis [11], changes in natural frequencies [12] or in the generalized flexibility matrix [13], and many other approaches can be found in the literature (for a review of the most important results, see also [14]). By the word *deviation*, we will indicate in this work any kind of difference with respect to the mechanical properties of a *perfect* structural element. The most immediate example is a damage (which is treated in the aforementioned literature), for instance modelled in terms of a local change of stiffness, but in general it is possible to consider also a local increase of mass or a local change of temperature. Since the vast literature in the modelling and identification theory of damages supplies us with several examples and models, in the following we will explicitly refer to damages in order to consider practical cases, but the proposed method is valid for any kind of deviation.

The interest in this research area has been fuelled also by the fast growth of the mechanical metamaterial field [15–17]. Metamaterials can be defined as materials characterized by (at least) two scale of description, whose macroscopic behaviour is obtained and controlled by tailoring the mechanical properties of the underlying microstructure [18–20]. Their macroscopic behaviour is usually described in terms of a continuum theory obtained by means of the so-called homogenization procedures [21–29]. Due to the high complexity of the microstructure, such continuum models (which may belong to the set of the so-called generalized continuum theories, see for instance [30–33]) are usually studied by means of numerical methods [34–38]. The resulting macroscopic behaviour is, in general, quite exotic [39–43] (an interesting example is given, for instance, by the so-called pantographic structure [44–52]). Metamaterials can exhibit, indeed, negative Poisson ratio, negative effective mass and other interesting properties that can be exploited in engineering applications (see [53] for a review). As a consequence, the analysis of damage in this kind of structures is extremely important and has pushed the interest of several researchers in this field [54–60].

Among many different techniques used in structure damage identification, one of the most established is based on the comparison of natural (and higher) eigenfrequencies of damaged and perfect structural elements [61–64]. This method has the practical advantage that can be easily performed by using a standard equipment for vibration analysis. However, it has the serious drawback that, in the hypothesis of small damages, the eigenvalues are not sensitive to local changes in structural stiffness. The main reason is that eigenfrequencies are global measures, and the analysis of a local property can be often difficult. It results that an experimental measure cannot appreciate the discrepancy due to the damage. Moreover, this kind of analysis cannot supply any information about the properties of the damage.

The aim of this work is to propose an extension of this method that allows overcoming these drawbacks. The essential point is that the effects of a possible damage strongly depends on the distribution of elastic energy of the structural element [65]. Therefore, a way to enlarge the damage effect is to modify the distribution of elastic energy in order to concentrate its large part in a small sub-domain where damage appears. To achieve modifications of the energy distribution, we introduce an additional system interacting with the element under examination. This subsystem, that can be for example an additional rigid or flexible support, additional mass, vibration absorber or other more complex structure, will modify the displacement field in the structural element in a fully experimental controlled way. The proposed method is articulate in two steps.

The first one is devoted to the detection of a deviation in the structure, and it is a modification of the standard approach. Moreover, it provides the optimal configuration of the additional system which maximize the effects of the damage on the energy distribution. In particular, the eigenvalue problem (or the analogous problem for other energy measure) has to be solved for the perfect structural element in interaction with different configurations of the additional system. This set of values will be then compared with the experimentally obtained eigenfrequencies (or other energy measures) of the damaged structure in interaction with the additional system in the *same* configurations of the perfect case. As we will show in the next sections, due to the action of the additional system the discrepancy between the two frequencies can be experimentally measured and, in particular, it will be maximum when the elastic energy distribution, modified by the action of the additional system, is peaked in a neighbourhood of the damage (for a similar application in biomechanical systems see [66]). In addition to eigenfrequencies, also other energy measurements can be used. In the following, we will show an explicit example of an elastic element, with a deviation in the stiffness, under a static load. In this case, the information about the energy distribution is obtained by considering the compliances of a damaged and a perfect structural element.

In the second step, by means of an optimization procedure, an analysis of the properties of the damage can be performed. In this identification problem, we assume that the damage can be described in terms of a set of parameters. As we will show in Sect. 4, the previous set of measurement of the frequencies of the damaged structure allows to define a set of iso-frequency surfaces in the damage parameter space. In an ideal case without measurement errors, these surfaces should intersect in one point which corresponds to the actual values of the parameters of the damage. In a more realistic case, instead, they will intersect in more than one point and we have to perform an optimization procedure to estimate the values of the damage parameters. It has to be remarked that the proposed method can be generalized in several directions. One of them is for sure the possibility to consider more than one damage, and it will be the aim of following investigations.

The work is organized in the following way. In Sects. 2 and 3, we will discuss the procedure devoted to the detection of a deviation in mechanical properties of a structural element by considering the eigenfrequencies analysis and the case of a static problem, respectively. In Sect. 4, we will formulate from the theoretical point of view the identification problem which allows to estimate the parameters of the damage. Finally, in Sect. 5 we will consider the explicit example of the identification of the damage in a statically loaded beam and we will show some numerical simulations.

2 Detection of deviations

In the standard analysis, fundamental or higher-order eigenfrequencies are measured and used to estimate the changes in structure and identify possible damage. Indeed, the change in eigenfrequency (or other energy measure) in damaged element depends to big extend on the distribution of elastic energy. In particular, the eigenfrequencies λ_i can be expressed in terms of energies by means of the Rayleigh quotient:

$$\lambda_i = \frac{U_i}{K_i}, \tag{1}$$

where U_i and K_i represent amplitudes of elastic and kinetic energies, respectively, and the index i represents the mode of oscillation. Being a deviation a local quantity, its effect on the total elastic energy could be in general too small to be detected by measuring eigenfrequencies or other energy measures. To amplify the effect of the deviation, we consider an additional system in interaction with the structural element. Due to the interaction with this subsystem, the energy distribution of the element will change in a more substantial way. In particular, from Eq. (1) we can see that the eigenfrequencies depend on the elastic energy which in turn, for a solid body, depends on the strain energy distribution. Being the strain, the symmetric part of the gradient of the displacement field, by means of the interaction with the external system (or other kind of external interactions which will modify the actual configuration), the resulting strain energy distribution will be modified. Since the effects of a damage on energy measures are enhanced if it is localized near the peak of the strain energy distribution, we can look for the particular configuration of the external system which maximize the effects of the damage on the eigenfrequency (or on other energy measure, as in the static case successively and explicitly discussed). In particular, when the eigenvalues analysis is considered, we will perform measurements of the eigenfrequencies of the damaged element by considering different configurations of the additional system, that we can fully control, and the discrepancy functional

$$\phi_\omega = \frac{\omega - \omega_\mathrm{d}}{\omega} \tag{2}$$

of the eigenfrequencies of the perfect and damaged structural element, in interaction with the same configurations of the subsystem, will be maximum when the large part of the elastic energy distribution is localized on the damage.

To motivate this procedure, let us consider the following example. A cantilever beam clamped at the left-hand end, $x = 0$ is loaded by static concentrated force F at the other end, $x = L$. The bending stiffness is $K = JE$, where J is the inertia moment and E is the Young modulus. Let us consider as deviation a reduced stiffness $K' = 0.7K$ in a sub-domain $d = L/100$ located at $x = 0.4L$. The contribution of energy located in the damaged region to the total energy integrated over the entire domain will be negligible (see Fig. 1). To enhance the effect of the damage, we consider two additional forces $P_1 = -20F$ and $P_2 = 14F$ located at $x = s$ and $x = s + 0.1L$, respectively. We examine the distribution of energy of the system for different positions $s = 0.2L$, $s = 0.4L$, and $s = 0.8L$ of the additional forces (see Fig. 2). In these cases, the contribution of energy located in the damaged region is equal to 0.5, 7, and 0.1%, respectively. Therefore, by comparing the eigenfrequencies of the damaged beam in interaction with the additional forces to the perfect case in the same conditions, it will be possible to spot the existence of the deviation.

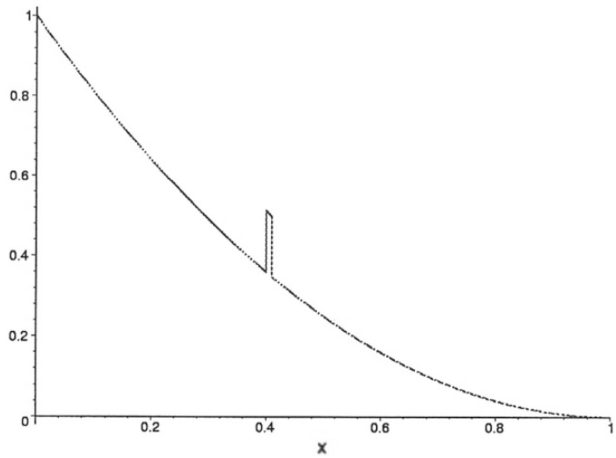

Fig. 1 Distribution of strain energy in damaged beam

3 The case of static problem

The eigenvalues analysis is, in several situation, the most suitable way to investigate the existence of a damage and its properties. However, there could be experimental conditions, like the study of elastic elements under static loads or the analysis of harmonic forced vibration in elastic and viscoelastic elements, for which other quantities related to the energy, like the work of external forces or generalized potential energy, can be measured and controlled in a simpler way. The proposed method allows to consider these different energy measures in a similar way.

Let us clarify this point by means of an example. The elastic body shown in Fig. 3 is under static load. The equilibrium equations with the associated boundary conditions are

$$\begin{aligned} \sigma_{ij,j} &= 0 \quad \text{in} \quad V \\ u_i &= 0 \quad \text{on} \quad S^u \\ \sigma_{ij} n_j &= t_i^0 \quad \text{on} \quad S^T \end{aligned} \tag{3}$$

where σ_{ij} is the stress tensor, u_i is the displacement field and n_j is the normal vector to the surface. The role of the additional subsystem is now played by a concentrated force F whose position $l = s$ can vary along the line l in Fig. 3 . The potential energy can be introduced in the form

$$\Pi = \int_V U(e_{ij})\,\mathrm{d}V - g, \tag{4}$$

where $U(e_{ij})$ is the density of strain energy, $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ being the strain, and g is the potential of the external forces

$$g = \int_{S^T} t_i^0 u_i\,\mathrm{d}S + Fw, \tag{5}$$

where Fw denotes the work done by the additional force F, w being the component of the displacement u in the direction of the force F. Its analogous expression for the damaged body will be

$$g_\mathrm{d} = \int_{S^T} t_i^0 u_{i_\mathrm{d}}\,\mathrm{d}S + Fw_\mathrm{d} \tag{6}$$

where the subscript d highlights that these quantities refer to the damaged body. The objective functional that has to be optimized, i.e. the analogous of functional (2) in the eigenproblem case, represents the compliance difference of a perfect and damaged element:

$$\phi_g = \frac{g - g_\mathrm{d}}{g}. \tag{7}$$

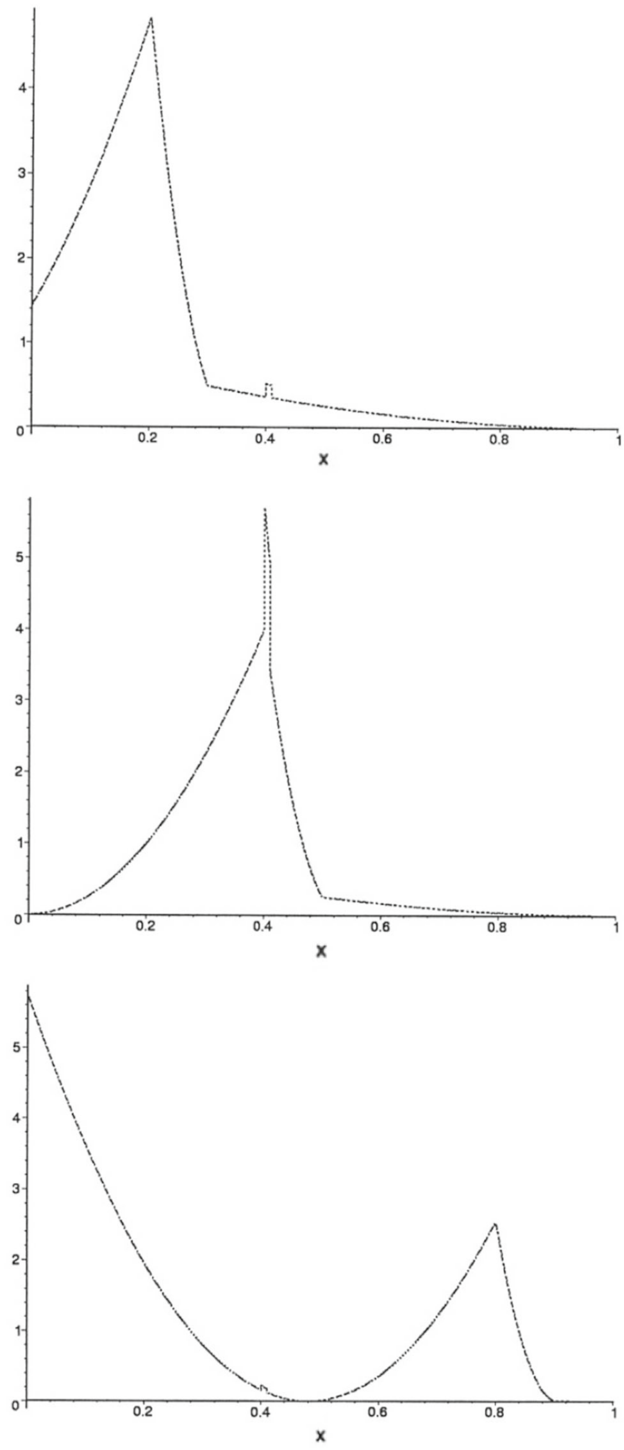

Fig. 2 Distribution of strain energy in damaged beam with two additional forces applied at different positions: $s = 0.2L$, $s = 0.4L$, and $0.8L$, respectively

We will now derive the necessary conditions for the extremum of this objective functional. Since we are looking for the optimal configuration (in the sense explained in Sect. 2) of the external system, which is fully characterized by the intensity of the force F and its application position s, we will consider the variation of functional (7)

$$\delta\phi_g = \frac{g_{\mathrm{d}}\delta g - g\delta g_{\mathrm{d}}}{g^2} \tag{8}$$

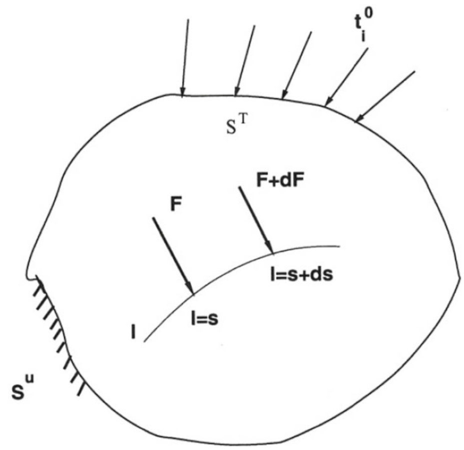

Fig. 3 Body under static loading and additional force of variable position and value

with respect to the parameters of the external system F and s, together with the variation $\delta_u \Pi = 0$. The explicit expressions of the variations are:

$$\begin{aligned}\delta g &= \delta_F g + \delta_s g \\ &= \left.\frac{\mathrm{d}w}{\mathrm{d}l}\right|_{l=s} F\,\delta s + w\big|_{l=s}\,\delta F = F w'\,\delta s + w\,\delta F,\end{aligned} \tag{9}$$

$$\begin{aligned}\delta g_{\mathrm{d}} &= \delta_F g_{\mathrm{d}} + \delta_s g_{\mathrm{d}} \\ &= \left.\frac{\mathrm{d}w_{\mathrm{d}}}{\mathrm{d}l}\right|_{l=s} F\,\delta s + w_{\mathrm{d}}\big|_{l=s}\,\delta F = F w'_{\mathrm{d}}\,\delta s + w_{\mathrm{d}}\,\delta F.\end{aligned} \tag{10}$$

Therefore, the variation (8) of functional (7) becomes:

$$\begin{aligned}\delta\phi_g = -\frac{1}{2g^2}\Bigg\{&F\left[w'\int_{S^T} t_i^0 u_{i_{\mathrm{d}}}\,\mathrm{d}S - w'_{\mathrm{d}}\int_{S^T} t_i^0 u_i\,\mathrm{d}S + F(w_{\mathrm{d}}w' - ww'_{\mathrm{d}})\right]\delta s \\ &+\left[w\int_{S^T} t_i^0 u_{i_{\mathrm{d}}}\,\mathrm{d}S - w_{\mathrm{d}}\int_{S^T} t_i^0 u_i\,\mathrm{d}S\right]\delta F\Bigg\}.\end{aligned} \tag{11}$$

The resulting necessary conditions for the extreme of objective functional (8) are

$$w\int_{S^T} t_i^0 u_{i_{\mathrm{d}}}\,\mathrm{d}S - w_{\mathrm{d}}\int_{S^T} t_i^0 u_i\,\mathrm{d}S = 0 \tag{12}$$

with respect to the variation of F, and

$$w'\int_{S^T} t_i^0 u_{i_{\mathrm{d}}}\,\mathrm{d}S - w'_{\mathrm{d}}\int_{S^T} t_i^0 u_i\,\mathrm{d}S + F(w_{\mathrm{d}}w' - ww'_{\mathrm{d}}) = 0 \tag{13}$$

with respect to the variation of s. The solutions of these equations represent the set of values of the additional subsystem that enhance the difference between the damaged and perfect case. In Sect. 5, we give an explicit example to further clarify the meaning of these relations.

4 Formulation of the identification problem

The procedure shown in the previous sections allows also to acquire information about the properties of the deviation. We do not aim to model the deviation but, once a model is chosen, the existence of a set of M parameters $\{d_m\}_{m=1}^M$, describing the deviation is assumed. By means of the following identification problem, it is possible to estimate the values of these parameters by means of an optimization procedure. Also for this second step, it is possible to consider, in a similar way, different observables associated with the energy, depending on the particular experimental conditions. In the following, we will explicitly formulate the identification problem for the eigenvalues analysis and for the case of static problems.

To formulate the identification problem[1] in the case of eigenfrequencies analysis, let us start from the eigenvalue problem

$$(K - \lambda_i M)X_i = 0, \tag{14}$$

where K and M denote the stiffness and mass matrices of the structural element, respectively, and $\lambda_i = \omega_i^2$ is the ith eigenvalue of the ith eigenvector X_i. Let us assume that the damage results in a modification of the stiffness. In particular, once a model of the damage is assumed, the stiffness of the damaged structure $\tilde{K}$ can be written as

$$\tilde{K}(d_1, \ldots, d_\mathrm{M}) = K - K_\mathrm{d}(d_1, \ldots, d_\mathrm{M}), \tag{15}$$

where K_d is the damage model depending on the damage parameters. Let us consider now that a set of L experimental measures of the eigenfrequencies, say $\{\bar{\lambda}_l\}_{l=1}^L$ of the damaged structural element for different configuration of the additional system are available. Then from Eq. (14), it is possible to define a set of L functions on the space of damage parameters as:

$$F_l^\lambda\left(d_1^{(l)}, \ldots, d_\mathrm{M}^{(l)}\right) = \frac{(X_i^l)^T \tilde{K} X_i^l}{(X_i^l)^T M X_i^l} - \bar{\lambda}_l = 0, \quad l = 1, \ldots, L. \tag{16}$$

The condition

$$F_l^\lambda\left(d_1^{(l)}, \ldots, d_\mathrm{M}^{(l)}\right) = 0, \quad \forall l \in [1; L] \tag{17}$$

will identify a set of $M - 1$-dimensional surfaces $\{S_l\}_{l=1}^L$ in the space of parameters, and they represent the values of the parameters that give rise to the same value of the eigenfrequency.

An analogous definition can be given in the case of static problems (see Sect. 3), once a set of measurement $\{\bar{g}_j\}_{l=1}^L$ of the work of external forces g for different configurations of the additional subsystem is given. Indeed, it is possible to define the following functions, depending on the damage parameters:

$$F_l^g\left(d_1^{(l)}, \ldots, d_M^{(l)}\right) = \bar{g}_l + 2\Pi_\mathrm{d}\left(d_1^{(l)}, \ldots, d_M^{(l)}\right), \quad l = 1, \ldots, L. \tag{18}$$

Again, the condition

$$F_l^g\left(d_1^{(l)}, \ldots, d_M^{(l)}\right) = 0, \quad \forall\, l \in [1; L] \tag{19}$$

will define a set of $M - 1$-dimensional surfaces in the space of parameters, representing now the values of the damage parameters giving rise to the same value of the energy.

In both cases, in principle the curves should intersect in one point identifying the actual values of the damage parameters. However, due to the finite precision of experimental measurements, the surfaces will not intersected in one point (see Fig. 4) and an optimization procedure has to be performed to estimate the values of the parameters.

In particular, we want to find the point in the space of parameters that minimizes the sum of the distances from the surfaces, i.e.

$$\psi(d^{(0)}) = \sum_{l=1}\left(\inf_{p_l \in S_l} \mathfrak{D}(d^{(0)}, p_l) + \beta_l F_l(d^{(l)})\right), \tag{20}$$

[1] From now on, we will explicitly refer to a damage of the stiffness, but the same procedure can be carried on for any deviation from the properties of a perfect beam.

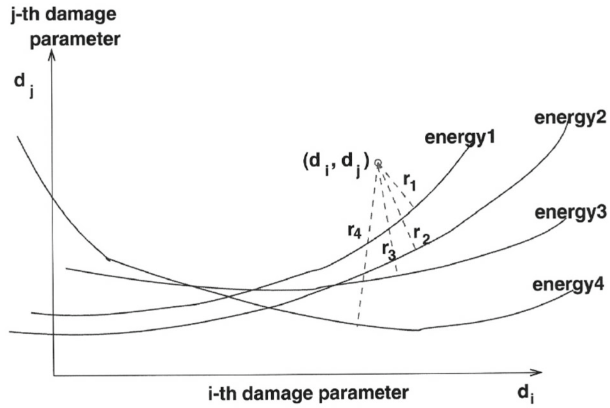

Fig. 4 The space of damage parameters and lines of constant energy measures associated with experimental measurements

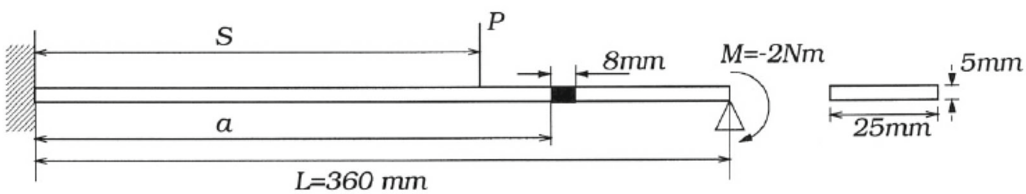

Fig. 5 Examined damaged beam with an additional force

where $d^{(0)}$ indicates the unknown set of actual values of the parameters, $\mathfrak{D}(\cdot, \cdot)$ is a distance measure and $\{\beta_l\}_{l=1}^{L}$ is a set of Lagrange multipliers. To clarify the identification procedure in the next section, we will discuss an explicit examples.

5 Identification of the damage in a statically loaded beam

To illustrate the identification procedure, an explicit example in the case of a statically loaded beam is examined numerically. Let us consider (see Fig. 5) a beam clamped at one end $x = 0$ and simply supported at the other one at $x = L = 360$ mm.

An external bending moment M is applied at the end $x = L$. In order to control the displacement field, an additional force P was applied. The value P and the position s of the force can be used to optimize the experiment conditions in order to enhance the differences between the perfect and the actual beam. The optimality conditions in the considered case can be obtained by solving Eqs. (12) and (13), that in our case read,

$$\phi_{\mathrm{d}} w - \phi w_{\mathrm{d}} = 0\,, \tag{21}$$

$$P(w'_{\mathrm{d}} w - w_{\mathrm{d}} w') + M(w'_{\mathrm{d}} \phi - w' \phi_{\mathrm{d}}) = 0\,, \tag{22}$$

where ϕ is the angle associated to the moment M. In Fig. 6, the values of the additional force P for different application points are shown.

The damage is localized in a region of 8 mm at a distance $a = 122.4$ mm from the clamped end, and it results in a $b = 20$ GPa reduction of the stiffness. In this case, the space of parameters has dimension two and it is spanned by the values of (a, b), i.e. the position and the value of the stiffness reduction, respectively. We consider $L = 4$ different configurations of the external force. In particular, we will consider the following positions of the force $s = 90$ mm, $s = 120$ mm, $s = 180$ mm, and $s = 240$ mm. The modules of the forces P associated to these positions are such that they satisfy the optimality conditions obtained by solving Eqs. (21) and (22). Once a set of measurements of the work of external forces $\{\bar{g}_l\}_{l=1}^{4}$ were simulated for these different configurations of the external forces, by following the identification procedure we are now able to define the following one dimensional surfaces in the space of damage parameters:

$$\tilde{F}_l^g(a^{(l)}, b^{(l)}) = \bar{g}_l + 2\Pi_{\mathrm{d}}(a^{(l)}, b^{(l)}) = 0, \quad l = 1, 2, 3, 4. \tag{23}$$

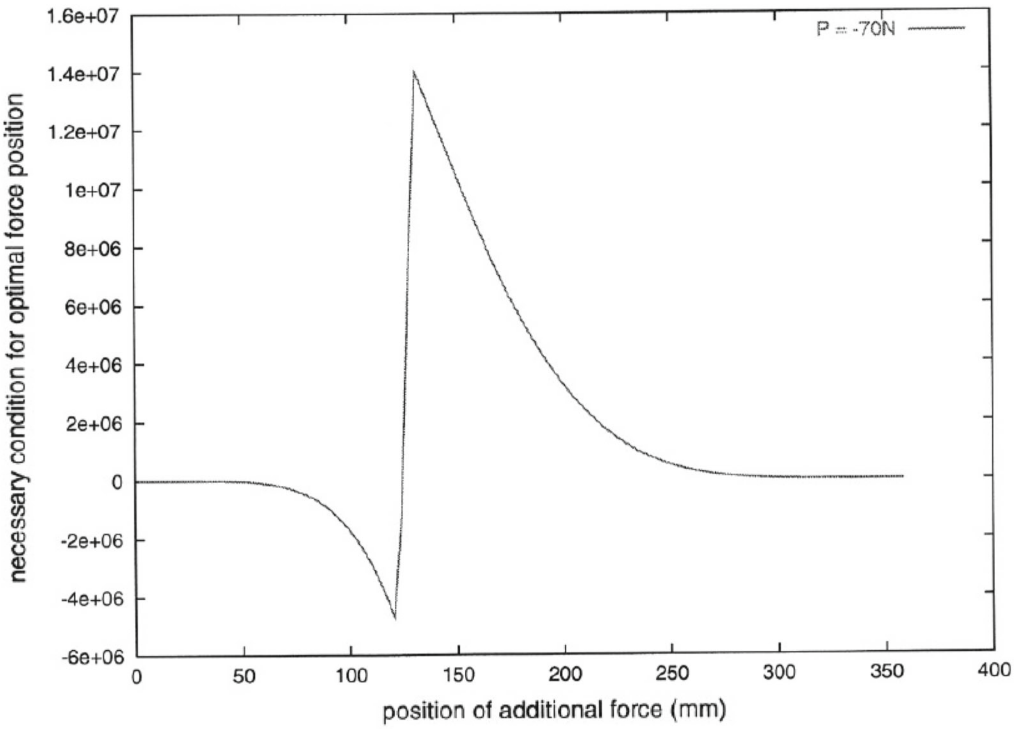

Fig. 6 Values of the necessary condition for variable force position

As it is shown in Fig. 7, the one-dimensional surfaces intersected in several points. Therefore, in order to estimate the actual values of the damage parameters (a^0, b^0) we have to optimize functional (20) which in this case reads:

$$\psi(a^0, b^0) = \sum_{l=1}^{3} \left(\inf_{p_l \in S_l} \mathfrak{D}\big((a^0, b^0), p_l\big) + \beta_l F_l(a^{(l)}, b^{(l)}) \right). \tag{24}$$

The results of the identification procedure are the following. Assumed position of the damage: 122.4 mm, determined position of damage: $a^0 = 121.62$ mm. Assumed stiffness of damage: 20 GPa, identified stiffness of the damage: $b^0 = 19.97$ GPa.

6 Conclusions and perspectives

In order to avoid the difficulty associated with small sensitivity of eigenvalues with respect to local variations of stiffness, an extension of the usual approach has been proposed. The general idea is based on the observation that by application of additional forces to the system the displacement field can be controlled and by this means the distribution of elastic energy can be modified to concentrate significant part of energy in sub-domain where the damage is located. The proposed method allows also to consider in a similar way other different observables associated to the energy, like the work of external forces in the static case. It follows from the investigations that effect of damage can be significantly increased, even for moderate and small damages. The explicit example of a cantilever Euler beam has been discussed, and it was shown that the for the optimal configuration of the additional subsystem the contribution of energy located in the damaged region reaches the 7%. Once a damage model is selected, i.e. we can describe it in terms of a set of parameters, it is possible to estimate the properties of the damage by means of an optimization procedure. This was called identification problem and, again, allows to consider in a similar way different observables related to the energy. The identification of the damage in a statically loaded Euler beam was explicitly discussed by means of numerical simulations. In particular, the optimal experimental conditions to set in order to enhance the differences between the perfect and damaged beam were computed. Then, by means of a set of measures of the work of external forces on the static beam corresponding to different configurations of the additional subsystem, a set of surfaces in the

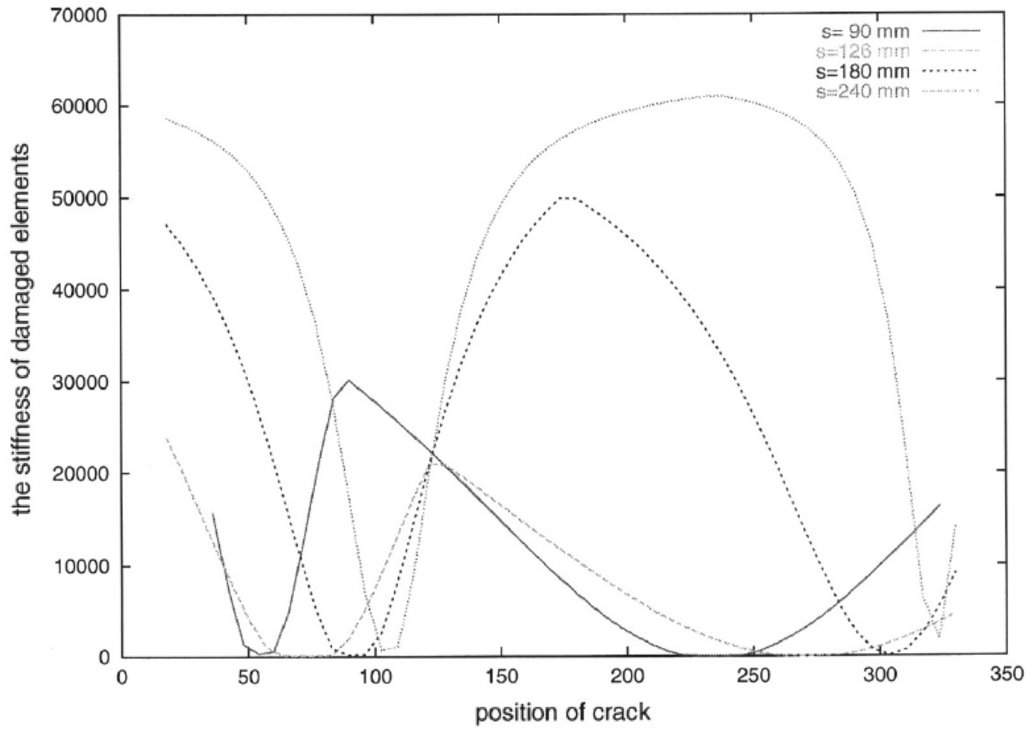

Fig. 7 The curves in the space of damage parameters associated with different values of measured work of external forces

space of damage parameters were defined. The actual parameters of the damage were estimated in this explicit example by finding the point in the space of damage parameters which minimizes the sum of the distances from these surfaces.

Starting from these results, further investigations can be considered. It could be of interest to study the robustness of the proposed method by means of examples with a more complex damage models, i.e. with a higher-dimensional parameter space as well as generalize it to conceive also more damages. Indeed, although it should be possible to treat the case of more damages by enlarging the parameter space, it has to be investigated how to implement the optimization problem for the external system in this case. Moreover, an interesting analysis could be performed by considering the effects of more general models on the space of parameters. In particular, it is conceivable to consider viscoelastic materials (see for example [67]), Timoshenko beam and generalized continua [68–71] beam models like higher gradient continua (see [72] for a comprehensive review). These models show behaviours that cannot be explained in terms of classical Cauchy theories [73–76] and they could require a larger number of parameters to describe a deviation in the mechanical properties of the considered structural element. An explicit interesting example could be discussed in the framework of multi-scale materials, like pantographic structures [77], where several applications [78–81] of the detection and analysis of damage have been studied.

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