Operator Theory:
Advances and Applications, Vol. 113

On the Zeros of Tails of Power Series

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Dedicated to the memory of S. A. Vinogradov

1. Introduction

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1.1}$$

be a power series with a positive radius of convergence. Let

$$s_n(z) = \sum_{k=0}^{n} a_k z^k, \qquad t_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$$

be its nth section and nth tail, respectively.

By now, the distribution of the zeros of $s_n(z)$ has been studied in detail; see, e.g., [2, 4, 6, 7], where further references can be found. The distribution of the zeros of the tails $t_n(z)$ has been paid less attention. The behavior of the zeros of the tails of some *concrete* power series was considered in [1, 3, 8, 12–14]. Many important results related to the tails of *general* power series were obtained in [2] and, especially, in [11]; however, only the *moduli* of the zeros of tails were treated there.

Some facts related to the *arguments* of the zeros of tails of power series with *infinite* radius of covergence were obtained in [10]. These facts show that some restrictions on the arguments of the zeros imply a bound for the growth of the entire function (1.1). Our aim in this paper is to obtain a similar result for power series with a *finite* radius of convergence. The main result is as follows.

Theorem. *Let $f(z)$ be a power series* (1.1) *convergent in the unit disc* $\mathbf{D}$. *Suppose there exist two different tails, $t_m(z)$ and $t_n(z)$, such that all zeros of $t_m(z)t_n(z)$ lie on a finite system of radii of* $\mathbf{D}$. *Then*

$$\log M(r, f) = O\left(\frac{1}{(1-r)^2}\right), \quad r \to 1, \tag{1.2}$$

where $M(r, f) = \max\{|f(z)| : |z| = r\}$.

Supported in part by INTAS 96-0858 grant.

The following example shows that the bound (1.2) is the best possible in the sense of order:

$$f(z) = P(z) + z^{n+1} \cos \frac{z}{(1-z)^2}, \quad z \in \mathbf{D}, \quad n \in \mathbf{N},$$

where $P(z)$ is a polynomial of degree $< n+1$. In this example, the tails

$$t_n(z) = z^{n+1} \cos \frac{z}{(1-z)^2}, \quad t_{n+1}(z) = z^{n+1} \left(\cos \frac{z}{(1-z)^2} - 1 \right)$$

have only real zeros in $\mathbf{D}$.

2. Proof of the theorem

Consider the function

$$q(z) = \frac{t_n(z)}{s_m(z) - s_n(z)}, \quad z \in \mathbf{D}, \tag{2.1}$$

and observe that

$$q(z) - 1 = \frac{t_m(z)}{s_m(z) - s_n(z)}.$$

Since $s_m(z) - s_n(z)$ has finitely many zeros, the condition of the theorem implies that all roots of the three equations

$$q(z) = 0, \quad q(z) = \infty, \quad q(z) = 1 \tag{2.2}$$

lie on a finite system of radii of $\mathbf{D}$. Denoting these radii by

$$\{z : \arg z = \alpha_j, \ 0 \le |z| < 1\}, \quad 0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_p < 2\pi,$$

we consider the sectors

$$S_j = \{z : \alpha_j < \arg z < \alpha_{j+1}, \ 0 < |z| < 1\}, \quad j = 1, \ldots, p, \quad \alpha_{p+1} = \alpha_1 + 2\pi.$$

In order to investigate the behavior of $q(z)$ and $f(z)$ in S_j, we need the function

$$w(\zeta; \gamma) = \frac{(2\zeta^{\pi/\gamma} - i)(2 - i\zeta^{\pi/\gamma})}{(2\zeta^{\pi/\gamma} + i)(2 + i\zeta^{\pi/\gamma})}, \quad 0 < \gamma \le 2\pi, \ 0 < \arg \zeta < \gamma.$$

This function conformally maps the sector $\{\zeta : 0 < \arg \zeta < \gamma, \ 0 < |\zeta| < 1\}$ onto $\mathbf{D}$. A direct calculation shows that

$$\begin{aligned} 1 - |w(\rho e^{i\theta}; \gamma)| &> (1 - |w(\rho e^{i\theta}; \gamma)|^2)/2 \\ &= \frac{12\rho^{\pi/\gamma}(1 - \rho^{2\pi/\gamma}) \sin(\pi\theta/\gamma)}{(4 + \rho^{2\pi/\gamma} - 4\rho^{\pi/\gamma} \sin(\pi\theta/\gamma))(1 + 4\rho^{2\pi/\gamma} + 4\rho^{\pi/\gamma} \sin(\pi\theta/\gamma))} \\ &> (4/15)\rho^{\pi/\gamma}(1 - \rho^{2\pi/\gamma}) \sin(\pi\theta/\gamma) \\ &> C(1 - \rho) \sin(\pi\theta/\gamma), \quad 0 < \theta < \gamma, \ 1/2 \le \rho < 1, \end{aligned} \tag{2.3}$$

where $C > 0$ is a constant independent of ρ and θ.

The function

$$w(ze^{-i\alpha_j};\alpha_{j+1}-\alpha_j), \quad j \in \{1,2,\dots,p\},$$

conformally maps S_j onto $\mathbf{D}$. Let $z_j(w)$ be the function performing the inverse conformal mapping. Then the function

$$q(z_j(w)), \quad w \in \mathbf{D},$$

is analytic in $\mathbf{D}$ and does not assume the values 0 and 1 there. By the Schottky theorem (see, e.g., [5, p. 60]), we have the following estimate:

$$\log^+ |q(z_j(w))| = O\left(\frac{1}{1-|w|}\right), \quad |w| \to 1. \tag{2.4}$$

Since (2.1) implies $f(z) = s_n(z) + (s_m(z) - s_n(z))q(z)$, we see that

$$f(z_j(w)) = s_n(z_j(w)) + [s_m(z_j(w)) - s_n(z_j(w))]q(z_j(w)).$$

The functions $s_m(z_j(w))$ and $s_n(z_j(w))$ are bounded in $\mathbf{D}$; therefore, (2.4) implies that

$$\log^+ |f(z_j(w))| = O\left(\frac{1}{1-|w|}\right), \quad |w| \to 1.$$

Substituting

$$w = w(ze^{-i\alpha_j};\alpha_{j+1}-\alpha_j), \quad z \in S_j,$$

we get

$$\log^+ |f(z)| \le \frac{C_1}{1-|w(ze^{-i\alpha_j};\alpha_{j+1}-\alpha_j)|}, \quad z \in S_j,$$

and, taking (2.3) into account,

$$\log^+ |f(re^{i\varphi})| \le \frac{C_2}{(1-r)\sin[\pi(\varphi-\alpha_j)/(\alpha_{j+1}-\alpha_j)]}, \quad \alpha_j < \varphi < \alpha_{j+1}, \ \frac{1}{2} \le r < 1,$$

where the constants $C_1, C_2 > 0$ are independent of $z = re^{i\varphi}$. Evidently, the validity of the latter inequality for $j = 1,2,\dots,p$, implies that

$$\log^+ |f(re^{i\varphi})| \le \frac{C}{(1-r)\prod_{j=1}^{p} |\varphi-\alpha_j|}, \quad 0 \le \varphi < 2\pi, \ 0 < r < 1, \tag{2.5}$$

where $C > 0$ is independent of $re^{i\varphi}$.

To deduce (1.2) from (2.5), it suffices to apply the following lemma with

$$h(\zeta) = f(\zeta e^{i\alpha_j}), \ j = 1,2,\dots,p; \quad \delta = (1/2)\min\{(\alpha_{j+1}-\alpha_j) : 1 \le j \le p\}.$$

Lemma. *Let $h(\zeta)$ be a function analytic in the sector*

$$S = \{\zeta : |\arg\zeta| \le \delta \le \pi, \ 0 \le |\zeta| < 1\}$$

and satisfying the condition

$$\log^+ |h(\zeta)| \le \frac{B_1}{(1-|\zeta|)|\arg\zeta|}, \quad \zeta \in S, \tag{2.6}$$

where $B_1 > 0$ *is a constant independent of* ζ. *Then*

$$\log^+ |h(\zeta)| = O\left(\frac{1}{(1-|\zeta|)^2}\right), \quad \zeta \in S,\ |\zeta| \to 1. \tag{2.7}$$

This lemma is contained implicitly in the well-known Sjöberg–Levinson–Domar log log-theorem (see, e.g., [9, p. 376]). We present its proof here for the reader's convenience.

Without loss of generality, we can assume that δ is small, namely, $\sin\delta \le \sqrt{2}/4$, because it suffices to prove (2.7) in any sector with smaller δ.

We fix R, $1/2 \le R \le 1$, and consider the function

$$h_R(\zeta) = h(\zeta)\exp\left\{-\frac{B_2}{(R-\zeta)^2}\right\}, \quad \zeta \in S\setminus\{R\}, \quad h_R(R) := 0,$$

where $B_2 > 0$ is a constant to be chosen later. This function is analytic in $S\setminus\{R\}$. Moreover, since

$$|h_R(\zeta)| = |h(\zeta)|\exp\left\{-\frac{B_2\cos(2\arg(R-\zeta))}{|R-\zeta|^2}\right\}, \tag{2.8}$$

it is continuous on

$$S_R := S\bigcap\{\zeta : |\arg(R-\zeta)| \le \frac{\pi}{8}\},$$

for $R < 1$. We show that the constant B_2 can be chosen independent of R and so large that

$$\max\{|h_R(\zeta)| :\ \zeta \in S_R\} \le B_3, \quad \text{for} \quad R < 1, \tag{2.9}$$

where the constant $B_3 > 0$ is also independent of R.

Putting

$$L_R^{(1)} = \partial S_R\bigcap\{\zeta : |\arg\zeta| = \delta\}, \qquad L_R^{(2)} = \partial S_R\bigcap\{\zeta : |\arg(R-\zeta)| = \pi/8\},$$
$$M_R^{(k)} = \max\{|h_R(\zeta)| : \zeta \in L_R^{(k)}\}, \quad k = 1, 2,$$

we have

$$\max\{|h_R(\zeta)| : \zeta \in \partial S_R\} = \max\{M_R^{(1)}, M_R^{(2)}\}.$$

An elementary geometric calculation shows that for $\zeta \in L_R^{(1)}$ and $R < 1$ we have

$$|\zeta| \le \left(R\sin\frac{\pi}{8}\right)/\sin\left(\delta+\frac{\pi}{8}\right) < \left(\sin\frac{\pi}{8}\right)/\sin\left(\delta+\frac{\pi}{8}\right) < 1.$$

Therefore, $M_R^{(1)}$ is bounded by a constant independent of R.

Next, (2.6) and (2.8) imply the inequality

$$M_R^{(2)} \le \max\left\{\exp\left(\frac{B_1}{(1-|\zeta|)|\arg\zeta|} - \frac{B_2\cos(\pi/4)}{|R-\zeta|^2}\right) : \zeta \in L_R^{(2)}\right\}.$$

We set

$$R - \zeta = t\exp\left(\pm\frac{i\pi}{8}\right),\ t > 0, \quad \text{for} \quad \zeta \in L_R^{(2)}, \quad \frac{1}{2} \le R < 1.$$

Elementary geometric calculations show that, for these ζ,

$$0 \le t \le (R\sin\delta)/\sin\frac{\pi}{8} < R\cos\frac{\pi}{8},$$

$$1-|\zeta| > (1-|\zeta|^2)/2 > (R^2-|\zeta|^2)/2 = t(2R\cos\frac{\pi}{8}-t)/2 > (tR\cos\frac{\pi}{8})/2 > \frac{1}{8}t,$$

and, moreover, there is a positive constant B_4 independent of R and B_4 such that

$$|\arg\zeta| \ge B_4 t.$$

Therefore,

$$M_R^{(2)} \le \max\left\{\exp\left(\frac{8B_1}{B_4t^2} - \frac{B_2\cos(\pi/4)}{t^2}\right) \,:\, 0<t<\infty\right\}.$$

We conclude that if $B_2 \ge 8B_1/(B_4\cos(\pi/4))$, then $M_R^{(2)} \le 1$. Thus, (2.9) is true.

Fix any $\zeta \in \operatorname{int} S_1$. Since

$$\operatorname{int} S_1 \subset \bigcup_{1/2<R<1} S_R,$$

letting $R \uparrow 1$ in (2.9), we get

$$|h_1(\zeta)| \le B_3, \quad \zeta \in \operatorname{int} S_1.$$

Hence, for $\zeta \in \operatorname{int} S_1$,

$$|h(\zeta)| = \left|h_1(\zeta)\exp\frac{B_2}{(1-\zeta)^2}\right| \le B_3\exp\frac{B_2}{(1-|\zeta|)^2},$$

i.e., (2.7) is true. For $\zeta \in S\setminus\operatorname{int} S_1$ we have $|\arg\zeta| \ge B_5(1-|\zeta|)$, so that the validity of (2.7) is an immediate consequence of (2.6).

3. Some remarks

1°. One of the results of [10] mentioned in the Introduction is the following.

Let $f(z)$ be a power series (1.1) *with an infinite radius of convergence. Suppose that there exists* (i) *a finite system of rays*

$$P = \bigcup_{j=1}^{p}\{z : \arg z = \alpha_j,\ 0 \le |z| < \infty\}, \quad 0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_p < 2\pi, \qquad (3.1)$$

and (ii) *three pairwise different tails $t_m(z), t_n(z), t_l(z)$, such that all but finitely many zeros of $t_m(z)t_n(z)t_l(z)$ lie on P. Then*

$$\log M(r,f) = O(r^{\pi/\gamma}\log r), \quad r\uparrow\infty, \qquad (3.2)$$

where $\gamma = \min\{(\alpha_{j+1}-\alpha_j) : 1\le j\le p\}$ $(\alpha_{p+1}=\alpha_1+2\pi)$.

This result can be refined by replacing three tails in (ii) by two tails, $t_m(z)$ and $t_n(z)$, say. For the proof, it suffices to define $q(z)$ by the equation

$$q(z) = \frac{t_n(z)}{s_m(z)-s_n(z)}, \quad z \in \mathbf{C}$$

(but not by the equation on the last line on p. 1259 of [10]). The function $q(z)$ defined in this way is meromorphic in $\mathbf{C}$, and all but finitely many roots of the equations (2.2) lie on the rays (3.1). The remaining part of the proof in [10] can be repeated with only trivial changes.

2°. As mentioned in [10], estimate (3.2) can be refined in the following way by using the Nevanlinna theory for an angle:

$$\log M(r, f) = O(r^{\pi/\gamma}), \quad r \uparrow \infty. \tag{3.3}$$

This remark remains valid if one replaces three tails by two tails.

3°. Estimate (1.2) does not depend on the system of radii mentioned in the theorem, meanwhile (3.2) and (3.3) depend on the system of rays (3.1). The example

$$f(z) = P(z) + z^{n+1} \cos z^{p/2}, \quad n, p \in \mathbf{N},$$

where $P(z)$ is a polynomial of degree $< n+1$, and all zeros of $t_n(z)$ and $t_{n+1}(z)$ lie on the rays (3.1) with $\alpha_j = 2(j-1)\pi/p$, $j = 1, 2, \ldots, p$, $\gamma = 2\pi/p$, shows that this dependence is essential and, moreover, (3.3) is sharp in the sense of order.

4°. The question as to whether (1.2) can be sharpened if there is a finite system of radii such that more than two different tails (e.g., all tails) have all but finitely many zeros situated on that system remains open. A result related to the case of an infinite radius of convergence was obtained in [10].

5°. If we assume that there exist two different tails, $t_m(z)$ and $t_n(z)$, that have finitely many zeros in $\mathbf{D}$, then (1.2) can be improved; namely,

$$\log M(r, f) = O\left(\frac{1}{1-r}\right), \quad r \to 1. \tag{3.4}$$

The following example (due to A. M. Vishnyakova) shows that the bound (3.4) is also sharp in the sense of order:

$$f(z) = P(z) + z^{n+1} \exp \frac{2iz}{1+z^2}, \quad z \in \mathbf{D}, \quad n \in \mathbf{N},$$

where $P(z)$ is a polynomial of degree $< n+1$. In this example, both $t_m(z)$ and $t_n(z)$ have only one multiple zero at $z = 0$ in $\mathbf{D}$.

For the proof of (3.4), we observe that the function $q(z)$ defined by (2.1) admits values 0 and 1 only finitely many times in $\mathbf{D}$ and apply the Schottky theorem in the annulus $\{z : \rho < |z| < 1\}$ where $q(z)$ does not admit 0 and 1 at all.

I thank A. I. Il′inskii and A. M. Vishnyakova for their careful reading a preliminary version of this paper and for valuable remarks.

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