Chapter 5

DIFFUSION APPROXIMATION FOR PROCESSES WITH SEMI-MARKOV SWITCHES AND APPLICATIONS IN QUEUEING MODELS

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Abstract Stochastic processes with semi-Markov switches (or in semi-Markov environment) and general Switching processes are considered. In case of asymptotically ergodic environment functional Averaging Principle and Diffusion Approximation types theorems for trajectory of the process are proved. In case of asymptotically consolidated environment a convergence to a solution of a differential or stochastic differential equation with Markov switches is studied. Applications to the analysis of random movements with fast semi-Markov switches and semi-Markov queueing systems in case of heavy traffic conditions are considered.

Keywords: Semi-Markov process, switching process, averaging principle, diffusion approximation, consolidation of states, queueing models, random walks.

1. INTRODUCTION

In various models appearing at study of complex stochastic systems such as state-dependent queueing systems and networks, information and computer systems, production and manufacturing systems, etc., we come to a necessity to consider systems working in different scales of time (slow and fast) and such that their local transition characteristics can be dependent on a current value of some another stochastic process (external random environment, discrete interference of chance, stochas-

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tic failures and switches or in general some functional on a trajectory of a system).

An operation of a wide range of these systems can be described in terms of so called Switching Stochastic Processes and in particular in terms of processes with semi-Markov switches.

The main property of a Switching Process (*SP*) is that the character of its operation varies spontaneously (switches) at certain epochs of time which can be random functionals of a previous trajectory.

SP's appear at study of queueing systems and networks, branching and migration processes in a random environment, at the analysis of stochastic dynamical systems with random perturbations, random movements and other various applications.

Taking into account a high dimension and a complex structure, exact analytic solutions for these processes can be obtained only for special rare cases, and methods of a direct stochastic simulation work usually slow and do not give a possibility of parametric investigation of a system. Therefore asymptotic methods play the basic role at the investigation and approximate analytic modelling.

Different asymptotic approaches for various classes of complex stochastic systems are considered in books of Buslenko et al. (1973), Kovalenko (1980), Anisimov et al. (1987), Basharin et al. (1989), and papers of Harrison (1995), Harrison and Williams (1996), Mandelbaum and Pats (1998).

In the paper we give a general description of *SP*'s, consider some important subclasses of *SP*'s paying the main attention to processes with semi-Markov switches (*PSMS*), investigate results of Averaging Principle (*AP*) and Diffusion approximation (*DA*) types and consider models of asymptotic decreasing dimension and consolidation of the state space for *PSMS*.

Applications to the analysis of random movement and state-dependent queueing models in semi-Markov environment in cases when the states of the environment can be asymptotically averaged or the environment allows an asymptotic consolidation of its state space are considered.

2. SWITCHING STOCHASTIC PROCESSES

2.1 PRELIMINARY REMARKS

SP's are described as two-component processes $(x(t), \zeta(t))$, $t \geq 0$, with the property existing a sequence of epochs $t_1 < t_2 < \dots$ such that on each interval $[t_k, t_{k+1})$, $x(t) = x(t_k)$ and the behavior of the process $\zeta(t)$ depends on the value $(x(t_k), \zeta(t_k))$ only. The epochs t_k are switching times and $x(t)$ is the discrete switching component (see Anisimov, 1977, 1978, 1988a).

SP's can be described in terms of constructive characteristics and they are very suitable in analyzing and asymptotic investigation of complex stochastic systems with "rare" and "fast" switches (Anisimov, 1978, 1988a, 1994-1996).

We mention that switching times may be determined by external factors (for instance, in the case when a system is operating in some random environment) and also by inner and interconnected factors. In general switching times may be some random functionals of the previous trajectory of the system.

According to A.N. Kolmogorov, *SP*'s are the special class of random processes with discrete interference of chance or processes with discrete component. A wide range of processes with discrete component have been studied by different authors: Markov processes homogeneous on the 2nd component (Ežov and Skorokhod, 1969), processes with independent increments and semi-Markov switches (Anisimov, 1973, 1978), piecewise Markov aggregates (Buslenko et. al., 1973), Markov processes with semi-Markov interference of chance (Gikhman and Skorokhod, 1973), and Markov and semi-Markov evolutions (Griego and Hersh, 1969; Hersh, 1974; Kertz, 1978ab; Kurtz, 1972, 1973; Papanicolaou and Hersh, 1972; Pinsky, 1975; Korolyuk and Swishchuk, 1986, 1994).

Law of Large Numbers and CLT for special classes of random evolutions were proved by many authors (Griego and Hersh, 1969; Hersh and Papanicolaou, 1972; Kurtz, 1973, Kertz, 1978ab; Pinsky, 1975; Anisimov, 1973; Korolyuk and Turbin, 1978; Watkins, 1984; Korolyuk and Swishchuk, 1986, 1994). These results are mostly devoted to the analysis of processes with independent increments in a Markov or semi-Markov environment.

Limit theorems for general scheme of *SP*'s were studied in the author's papers in the following directions.

Theorems about convergence of a one *SP* to another are proved in the class of *SP*'s when the number of switches does not tend to infinity ("rare" switches) (see Anisimov, 1978, 1988ab). As in usual a limiting *SP* has a more simple structure and depends on less number of parameters, these results give us the possibility to decrease dimension asymptotically and consolidate in some sense the state space of *SP*.

On the base of these results the theory of asymptotic consolidation (merging) of state space and decreasing dimension for Markov and semi-Markov processes (homogeneous as well as non-homogeneous in time) was constructed (Anisimov, 1973, 1988a).

Asymptotic consolidation of states in particular means the following. Suppose that some Markov process (*MP*) or semi-Markov process (*SMP*) has transition probabilities of different orders and its state space can be divided to regions such that transition probabilities between them are small in some sense and the states in each region asymptotically communicate. Then under rather general conditions additive functionals on the process can be weakly approximated by processes with independent increments in Markov or semi-Markov environment with number of states equals to the number of regions.

Several results devoted to the asymptotic analysis of integral functionals and flows of rare events on trajectories of *SMP*'s operating in different scales of time are obtained in (Anisimov, 1973, 1977, 1978a). Applications to the asymptotic analysis of queueing models and multiprocessor computer systems in conditions of fast service can be found in Anisimov (1988a, 1996a), Anisimov et al. (1987), Anisimov and Sztrik (1989), Sztrik and Kouvatsos (1991).

The next direction of the investigations devoted to the case of fast switches (number of switches tends to infinity). In that case, if the increments on each switching interval are small, it is reasonable to expect taking into account the recurrent character of an operation of the process, that under some general conditions a process trajectory converges to a solution of some ordinary differential equation – *Averaging Principle* (*AP*), and the normed deviation weakly converges to some diffusion process - *Diffusion Approximation* (*DA*). Results of this type for different subclasses of *SP*'s were proved by Anisimov (1992, 1994, 1995), Anisimov and Aliev (1990).

Applications of these results to study an asymptotic behaviour of characteristics for Markov queueing systems and networks under transient conditions and with large number of calls were investigated by Anisimov (1992, 1995, 1996a), Anisimov and Lebedev (1992).

Now we consider a description of some important classes of *SP*'s such as Recurrent Processes of a Semi-Markov type, Processes with Semi-Markov Switches and give a general description of *SP*'s.

2.2 RECURRENT PROCESSES OF A SEMI-MARKOV TYPE

Let $\mathcal{F}_k = \{(\xi_k(\alpha), \tau_k(\alpha)), \alpha \in \mathcal{R}^r\}, k \geq 0$, be jointly independent families of random variables with values in $\mathcal{R}^r \times [0, \infty)$ and S_0 be independent of $\mathcal{F}_k, k \geq 0$ random variable in $\mathcal{R}^r$. We assume the measurability in α of variables introduced concerning σ-algebra $\mathcal{B}_{\mathcal{R}^r}$. Denote

$$t_0 = 0, \quad t_{k+1} = t_k + \tau_k(S_k), \quad S_{k+1} = S_k + \xi_k(S_k), \quad k \geq 0 \qquad (5.1)$$

and put

$$S(t) = S_k \quad \text{as} \quad t_k \le t < t_{k+1}, \quad t \ge 0. \tag{5.2}$$

Then a process $S(t)$ forms a Recurrent Process of a Semi-Markov type (*RPSM*) (see Anisimov, 1992, Anisimov and Aliev, 1990).
In the homogeneous case (distributions of families $\mathcal{F}_k$ do not depend on the parameter k) the process $S(t)$ is a homogeneous *SMP*. If also distributions of families $\mathcal{F}_k$ do not depend on the parameter α, epochs $t_k, k \ge 0$ form a recurrent flow and $S(t)$ is a generalized renewal process. If variables $\tau_k(\alpha)$ have exponential distributions, the process $S(t)$ is a *MP*.

2.3 RECURRENT PROCESS OF A SEMI-MARKOV TYPE WITH ADDITIONAL MARKOV SWITCHES

Let $\mathcal{F}_k = \{(\xi_k(x,\alpha), \tau_k(x,\alpha)), x \in X, \alpha \in \mathcal{R}^r\}, k \ge 0$ be jointly independent families of random variables with values in $\mathcal{R}^r \times [0,\infty)$, and let $x_l, l \ge 0$ be a *MP*, independent of $\mathcal{F}_k, k \ge 0$, with values in X, (x_0, S_0) be an initial value. We assume here and further the measurability in the pair (x,a) of variables introduced concerning σ-algebra $\mathcal{B}_X \times \mathcal{B}_{R^r}$ and put

$$t_0 = 0,\ t_{k+1} = t_k + \tau_k(x_k, S_k), S_{k+1} = S_k + \xi_k(x_k, S_k),\ k \ge 0, \tag{5.3}$$

$$S(t) = S_k, \quad x(t) = x_k \quad \text{as} \quad t_k \le t < t_{k+1}, \quad t \ge 0. \tag{5.4}$$

Then the process $(x(t), S(t))$ forms a *RPSM* with additional Markov switches. We assume that *RPSM* is regular, i.e. the component $x(t)$ has a finite number of jumps on each finite interval with probability one. If the distributions of variables $\tau_k(x,\alpha)$ do not depend on parameters (α, k), then the process $x(t)$ is a *SMP*.

2.4 GENERAL CASE OF RPSM

Let $\mathcal{F}_k = \{(\xi_k(x,\alpha), \tau_k(x,\alpha), \beta_k(x,\alpha)), x \in X, \alpha \in \mathcal{R}^r\}, k \ge 0$ be jointly independent families of random variables with values in $\mathcal{R}^r \times [0,\infty) \times X$, X be some measurable space, (x_0, S_0) be an initial value. We put

$$t_0 = 0, \quad t_{k+1} = t_k + \tau_k(x_k, S_k),$$

$$S_{k+1} = S_k + \xi_k(x_k, S_k), \quad x_{k+1} = \beta_k(x_k, S_k), \quad k \ge 0, \tag{5.5}$$

$$S(t) = S_k,\ x(t) = x_k \quad \text{as} \quad t_k \le t < t_{k+1},\quad t \ge 0. \tag{5.6}$$

Then the pair $(x(t), S(t))$, $t \ge 0$ forms a general *RPSM*. We mention that in this case we have feedback between both components $x(t)$ and $S(t)$. In particular, when distributions of the variables $\beta_k(x,\alpha)$ do not depend on the parameter α, the sequence x_k forms a *MP* and we obtain the previous case of *RPSM* with Markov switches.

2.5 PROCESSES WITH SEMI-MARKOV SWITCHES

Now we consider an operation of some random process in a semi-Markov environment. Let $\mathcal{F}_k = \{\zeta_k(t,x,\alpha), t \ge 0, x \in X, \alpha \in \mathcal{R}^r\}$, $k \ge 0$ be jointly independent parametric families of random processes, where $\zeta_k(t,x,\alpha)$ at each fixed k,x,α be a random process with trajectories in Skorokhod space $\mathcal{D}_\infty^r$, and let $x(t)$, $t \ge 0$ be a right-continuous *SMP* in X independent of $\mathcal{F}_k$, $k \ge 0$, S_0 be an initial value. Denote by $0 = t_0 < t_1 < \ldots$ the epochs of sequential jumps for $x(\cdot)$, $x_k = x(t_k)$, $k \ge 0$. We construct a process with semi-Markov switches (or in a semi-Markov environment) as follows: put $S_{k+1} = S_k + \xi_k$, where $\xi_k = \zeta_k(\tau_k, x_k, S_k)$, $\tau_k = t_{k+1} - t_k$, and denote

$$\zeta(t) = S_k + \zeta_k(t - t_k, x_k, S_k) \quad \text{as} \quad t_k \le t < t_{k+1},\quad t \ge 0. \tag{5.7}$$

Then a two-component process $(x(t), \zeta(t))$, $t \ge 0$ is called a Process with Semi-Markov Switches (*PSMS*). Let us introduce also an imbedded process

$$S(t) = S_k \quad \text{as} \quad t_k \le t < t_{k+1}. \tag{5.8}$$

Then two-component process $(x(t), S(t))$ forms a *RPSM* with additional Markov switches.

Suppose that $\{\zeta(t,x),\ t \ge 0\}$ is a family of *MP* and $\zeta(t,x,\alpha)$ denotes the process $\zeta(t,x)$ with initial value α. In that case the process $(x(t), \zeta(t))$ forms a Markov random evolution (when the process $x(t)$ is *MP*) or semi-Markov one (when the process $x(t)$ is *SMP*) (see Hersh, 1974; Kurtz, 1973; Kertz, 1978b; Pinsky, 1975; Korolyuk and Swishchuk, 1994).

2.6 SWITCHING PROCESSES

Now we give a general construction of a Switching Process (*SP*). Let

$$\mathcal{F}_k = \{(\zeta_k(t,x,\alpha), \tau_k(x,\alpha), \beta_k(x,\alpha)), t \ge 0, x \in X, \alpha \in \mathcal{R}^r\},\quad k \ge 0$$

be jointly independent parametric families where $\zeta_k(t, x, \alpha)$ at each fixed k, x, α be a random process in Skorokhod space $\mathcal{D}_\infty{}^r$ and $\tau_k(x, \alpha), \beta_k(x, \alpha)$ be possibly dependent on $\zeta_k(\cdot, x, \alpha)$ random variables, $\tau_k(\cdot) > 0$, $\beta_k(\cdot) \in X$. Let also (x_0, S_0) be independent of $\mathcal{F}_k, k \geq 0$, initial value. We put

$$t_0 = 0, \quad t_{k+1} = t_k + \tau_k(x_k, S_k), \quad S_{k+1} = S_k + \xi_k(x_k, S_k),$$

$$x_{k+1} = \beta_k(x_k, S_k), \quad k \geq 0, \tag{5.9}$$

where $\xi_k(x, \alpha) = \zeta_k(\tau_k(x, \alpha), x, \alpha)$, and set

$$\zeta(t) = S_k + \zeta_k(t - t_k, x_k, S_k), \ x(t) = x_k, \ \text{ as } \ t_k \leq t < t_{k+1}, \ t \geq 0. \tag{5.10}$$

Then a two-component process $(x(t), \zeta(t))$, $t \geq 0$ is called a *SP* (see Anisimov, 1977, 1978). In concrete applications the component $x(\cdot)$ usually means some random environment, and $S(\cdot)$ means the trajectory of the system. We also mention that the general construction of a *SP* allows the dependence (feedback) between both components $x(\cdot)$ and $S(\cdot)$.

2.7 EXAMPLES OF SWITCHING PROCESSES

Now we consider some models of *SP*'s as examples.

PII with semi-Markov switches. Let $x(t), t \geq 0$ be a *SMP* with state space X and let independent of it and jointly independent families of homogeneous processes with independent increments (*PII*) $\{\xi_k(t, x), t \geq 0, x \in X\}, k \geq 0$ and random variables $\{\gamma_k(x), x \in X\}, k \geq 1$ with values in $\mathcal{R}^r$ be given. Suppose for simplicity that distributions of variables introduced do not depend on the index k.

We construct a two-component process $(x(t), \zeta(t)), t \geq 0$ with values in $(X, \mathcal{R}^r)$ as follows. Let (x_0, ζ_0) be some initial value. Denote the epochs of sequential jumps for $x(t)$ by $0 = t_0 < t_1 < t_2$ and let $x_k = x(t_k), k \geq 0$ be the embedded *MP*. Then we put $\zeta(0) = \zeta_0$ and

$$\zeta(t) = \zeta(t_k) + \xi_k(t, x_k) - \xi_k(t_k, x_k), \ \ t_k \leq t < t_{k+1},$$

$$\zeta(t_{k+1}) = \zeta(t_{k+1} - 0) + \gamma_{(k+1)}(x_{k+1}), \ \ k \geq 0.$$

By the construction on the fixed trajectory of $x(t)$ the process $\zeta(t)$ is operating like a non-homogeneous process with independent increments and with additional jumps in the epochs $t_k, k > 0$ of the sizes $\gamma_k(x_k)$. It is called *PII* with semi-Markov switches (see Anisimov, 1973).

We remark that if the process $x(\cdot)$ is a *MP*, then the pair $(x(t), \zeta(t))$, $t \geq 0$ forms a *PII* with Markov switches (or *MP* homogeneous in the 2nd component (see Ežov and Skorokhod, 1969).

In this way we can construct a Poisson process with semi-Markov switches. Let the family of non-negative functions $\lambda(x)$, $x \in X$ and *SMP* $x(t), t \geq 0$ with values in X be given. Denote by $\Pi_{\lambda(\cdot)}(t)$ a Poisson process with instantaneous value of parameter $\lambda(x(t))$ at time t. Then a two-component process $(x(t), \Pi_{\lambda(\cdot)}(t))$, $t \geq 0$ is a *PSMS*. In particular if $x(t)$ is a *MP* then the process $\Pi_{\lambda(\cdot)}(t)$ is a Markov modulated Poisson process. Processes of this type arise at study of input flows at queueing models in a random environment.

Random movements with SMP switches. Let $\{v(i, \alpha), \alpha \in \mathcal{R}^r\}$, $i = 1, 2, \ldots, m$ be a family of continuous vector-valued functions in $\mathcal{R}^r$, and $x(t), t \geq 0$ be a *SMP* with finite number of states $X = \{1, 2, \ldots, d\}$. We put $\zeta_k(t, i, \alpha) = tv(i, \alpha)$, $t \geq 0$, $i = \overline{1, d}$. Denote by $0 = t_0 < t_1 < \ldots$ times of sequential jumps for $x(t)$, $x_k = x(t_k)$. Then *PSMS* $(x(t), \zeta(t)), t \geq 0$ constructed by the family of processes $\{\zeta_k(t, i, \alpha),\ t \geq 0,\ \overline{1, m}\}$ and switching times $t_k, k \geq 0$ forms a random movement in R^r with semi-Markov switches. If we denote $\nu(t) = \max\{k : k \geq 0, t_k < t\}, \zeta_k = \zeta(t_k)$, then :

$$\zeta(t) = \zeta(0) + \sum_{k=0}^{\nu(t)-1} (t_{k+1} - t_k) v(x_k, \zeta_k) + (t - t_{\nu(t)}) v(x_{\nu(t)}, \zeta_{\nu(t)}).$$

Dynamical systems in semi-Markov environment. Let $\{f(x, \alpha),\ \alpha \in \mathcal{R}^r\}$, $x \in X$ be a family of deterministic functions with values in $\mathcal{R}^r$, $\Gamma_k = \{\gamma_k(x, \alpha), x \in X, \alpha \in \mathcal{R}^r\}$, $k \geq 0$, be jointly independent families of random variables with values in $\mathcal{R}^r$ and $x(t), t \geq 0$ be a *SMP* in X independent of introduced families Γ_k. Put $x_k = x(t_k)$ and denote by $0 = t_0 < t_1 < \ldots$ sequential times of jumps for the process $x(t)$. We introduce the process $\zeta(t)$ as follows : $\zeta(0) = \zeta_0$ and

$$d\zeta(t) = f(x_k, \zeta(t))dt, \quad t_k \leq t < t_{k+1},$$

$$\zeta(t_{k+1} + 0) = \zeta(t_{k+1} - 0) + \gamma_k(x_k, \zeta(t_{k+1} - 0)), \quad k \geq 0.$$

Then the process $\zeta(t)$ forms a dynamical system with semi-Markov switches.

Stochastic differential equations with semi-Markov switches. Let $\{c(x, a), b(x, a), x \in X, \alpha \in R\}$ be deterministic vector and matrix-valued functions of dimensions r and $r \times r$ respectively, $w(t), t \geq 0$ be a standard Wiener process in $\mathcal{R}^r$ and let $x(t), t \geq 0$ be a *SMP* independent of $w(\cdot)$. We introduce the process $\zeta(t)$ as a solution of the following stochastic differential equation :

$$\zeta(0) = \zeta_0, \quad \mathrm{d}\zeta(t) = c(x(t), \zeta(t))\mathrm{d}t + b(x(t), \zeta(t))\mathrm{d}w(t),$$

where at each $x \in X$ the coefficients $c(x, a)$ and $b(x, a)$ satisfy the conditions of the existence and uniqueness theorem. Then the pair $(x(t), \zeta(t)),\ t \geq 0$ forms a *PSMS*. It is also possible to describe a feedback between components. Another example can be Markov continuous time space-dependent branching processes (see Anisimov, 1996b).

Switching state-dependent queueing models. A class of *SP*'s gives a possibility to describe various classes of stochastic queueing models such as state-dependent queueing systems and networks $SM_Q/M_Q/m/\infty$, $M_{SM,Q}/M_{SM,Q}/l/k$, $(M_{SM,Q}/M_{SM,Q}/m_i/k_i)^r$, with batch Markov or semi-Markov input, finite number of nodes, different types of calls, impatient calls and possibly of a random size (volume of information or necessary job), batch state-dependent service, which are switched by some external semi-Markov environment and current values of queues, also retrial queueing models, etc.
For these models switching times are usually times of any changes in the system (Markov models), times of jumps of the environment (in case of external semi-Markov environment), times of exit from some regions for the process generated by queue, waiting times, etc.

3. AVERAGING PRINCIPLE AND DIFFUSION APPROXIMATION FOR RPSM

We study limit theorems for *RPSM* in the triangular scheme for the case of fast switches. This means that we consider the process on the interval $[0, nT]$, $n \to \infty$ and characteristics of the process depend on the parameter n in such a way that the number of switches on each interval $[na, nb]$, $0 < a < b < T$ tends, by probability, to infinity. Then, under natural assumptions, the normed trajectory of $S_n(nt)$ uniformly converges by probability to some function which is the solution of an ordinary differential equation, and normed difference between trajectory and this solution weakly converges in Skorokhod space $\mathcal{D}_T$ to some diffusion process.

Let us consider *AP* and *DA* type theorems for simple *RPSM*, because these results have various applications in queueing models (see Anisimov, 1995, 1996a; Anisimov and Lebedev, 1992).

Let for each $n = 1, 2, \ldots$ $\mathcal{F}_{nk} = \left\{ (\xi_{nk}(\alpha)), \tau_{nk}(\alpha)), \alpha \in \mathcal{R}^r \right\}$, $k \geq 0$ be jointly independent families of random variables taking values in $\mathcal{R}^r \times [0, \infty)$, with distributions do not depend on index k, and let S_{n0} be independent of $\mathcal{F}_{nk}, k \geq 0$ initial value in $\mathcal{R}^r$. Put

$$t_{n0} = 0, \quad t_{nk+1} = t_{nk} + \tau_{nk}(S_{nk}), \quad S_{nk+1} = S_{nk} + \xi_{nk}(S_{nk}), \; k \geq 0, \tag{5.11}$$

$$S_n(t) = S_{nk} \;\text{ as }\; t_{nk} \leq t < t_{nk+1}, \; t \geq 0.$$

Assume that there exist functions $m_n(\alpha) = E\tau_{n1}(n\alpha)$, $b_n(\alpha) = E\xi_{n1}(n\alpha)$.

Theorem 1 *(Averaging principle)*
Suppose that for any $N > 0$

$$\lim_{L\to\infty} \limsup_{n\to\infty} \sup_{|\alpha|<N} \left\{ E\tau_{n1}(n\alpha)\chi(\tau_{n1}(n\alpha) > L) + E|\xi_{n1}(n\alpha)|\chi(|\xi_{n1}(n\alpha)| > L) \right\} = 0, \tag{5.12}$$

as $\max(|\alpha_1|, |\alpha_2|) < N$,

$$|m_n(\alpha_1) - m_n(\alpha_2)| + |b_n(\alpha_1) - b_n(\alpha_2)| < C_N|\alpha_1 - \alpha_2| + \alpha_n(N), \tag{5.13}$$

where C_N are some bounded constants, $\alpha_n(N) \to 0$ uniformly in $|\alpha_1| < N, |\alpha_2| < N$, and there exist functions $m(a) > 0$, $b(a)$ and a proper random variable s_0 such that as $n \to \infty$, $n^{-1}S_{n0} \xrightarrow{\text{P}} s_0$, and for any $\alpha \in \mathcal{R}^r$

$$m_n(\alpha) \to m(\alpha) > 0, \; b_n(\alpha) \to b(\alpha). \tag{5.14}$$

Then

$$\sup_{0\leq t\leq T} |n^{-1}S_n(nt) - s(t)| \xrightarrow{\text{P}} 0, \tag{5.15}$$

where

$$s(0) = s_0, \;\; \mathrm{d}s(t) = m(s(t))^{-1}b(s(t))\mathrm{d}t, \tag{5.16}$$

and T is any positive number such that $y(+\infty) > T$ with probability one, where

$$y(t) = \int_0^t m(\eta(u))\mathrm{d}u, \tag{5.17}$$

$$\eta(0) = s_0, \; \mathrm{d}\eta(u) = b(\eta(u))\mathrm{d}u \tag{5.18}$$

(it is supposed that a solution of equation (5.18) exists on each interval and is unique).

Now we consider a convergence of the process $\gamma_n(t) = n^{-1/2}(S_n(nt) - ns(t))$, $t \in [0, T]$ to some diffusion process. Denote

$$\tilde{b}_n(\alpha) = m_n(\alpha)^{-1} b_n(\alpha), \quad \tilde{b}(\alpha) = m(\alpha)^{-1} b(\alpha),$$

$$\rho_n(\alpha) = \xi_{n1}(n\alpha) - b_n(\alpha) - \tilde{b}(\alpha)(\tau_{n1}(n\alpha) - m_n(\alpha)),$$

$$q_n(\alpha, z) = \sqrt{n}\Big(\tilde{b}_n(\alpha + \frac{1}{\sqrt{n}} z) - \tilde{b}(\alpha)\Big), \quad D_n^2(\alpha) = E\rho_n(\alpha)\rho_n(\alpha)^*$$

(we denote the conjugate vector by the symbol *).

Theorem 2 *(Diffusion approximation)*
Let conditions (5.13)-(5.14) be satisfied where in (5.13) $\sqrt{n}\alpha_n(N) \to 0$, there exist continuous vector-valued function $q(\alpha, z)$ and matrix-valued function $D^2(\alpha)$ such that in any domain $|\alpha| < N$ $|q(\alpha, z)| < C_N(1 + |z|)$, and uniformly in $|\alpha| < N$ at each fixed z

$$\sqrt{n}\Big(\tilde{b}_n(\alpha + n^{-1/2} z) - \tilde{b}(\alpha)\Big) \to q(\alpha, z), \tag{5.19}$$

$$D_n^2(\alpha) \to D^2(\alpha), \tag{5.20}$$

$\gamma_n(0) \overset{w}{\Rightarrow} \gamma_0$, *and for any $N > 0$*

$$\lim_{L\to\infty} \limsup_{n\to\infty} \sup_{|\alpha|<Nn} \Big\{E\tau_{n1}^2(\alpha)\chi(\tau_{n1}(\alpha) > L) + E|\xi_{n1}(\alpha)|^2 \chi(|\xi_{n1}(\alpha)| > L)\Big\} = 0. \tag{5.21}$$

Then the sequence of the processes $\gamma_n(t)$ J-converges on any interval [0,T] such that $y(+\infty) > T$ to the diffusion process $\gamma(t)$ which satisfies the following stochastic differential equation solution of which exists and is unique: $\gamma(0) = \gamma_0$,

$$\mathrm{d}\gamma(t) = q(s(t), \gamma(t))\mathrm{d}t + D(s(t))m(s(t))^{-1/2}\mathrm{d}w(t), \tag{5.22}$$

where $s(\cdot)$ satisfies equation (5.16) (J-convergence denotes a weak convergence of measures in Skorokhod space D_T.)

Proof of Theorems 1, 2. Let us introduce sequences $\eta_{nk} = n^{-1} S_{nk}$, $y_{nk} = n^{-1} t_{nk}, k \geq 0$ and processes $\eta_n(u) = \eta_{nk}$, $y(u) = y_{nk}$ as $n^{-1}k \leq u < n^{-1}(k+1), u \geq 0$. Put $\nu_n(t) = \min\{k : k > 0,\ t_{nk+1} > nt\}$, $\mu_n(t) = \inf\{u : u > 0,\ y_n(u) > t\}$. By definition, $y_n(n^{-1}\nu_n(t)) \leq t <$

$y_n(n^{-1}\nu_n(t)+1))$ and $\mu_n(t) = n^{-1}(\nu_n(t)+1)$. As far as $S_n(nt) = S_{n\nu_n(t)}$, we have a representation

$$n^{-1}S_n(nt) = \eta_n(n^{-1}\nu_n(t)) = \eta_n(\mu_n(t) - 1/n).$$

Thus, *RPSM* $n^{-1}S_n(nt)$ is constructed as a superposition of two processes: $\eta_n(t)$ and $\mu_n(t)$. First we'll study the behaviour of the processes $\eta_n(t)$ and $y_n(t)$, then $\mu_n(t)$ and their superposition. According to (5.11), we can write the relations $\eta_{nk+1} = \eta_{nk} + n^{-1}b_n(\eta_{nk}) + \varphi_{nk}$, $y_{nk+1} = y_{nk} + n^{-1}m_n(\eta_{nk}) + \psi_{nk}$, $k \geq 0$, where $\varphi_{nk} = n^{-1}(\xi_{nk}(n\eta_{nk}) - b_n(\eta_{nk}))$, $\psi_{nk} = n^{-1}(\tau_{nk}(n\eta_{nk}) - m_n(\eta_{nk}))$.

Sequences φ_{nk} and $\psi_{nk}, k \geq 0$ are martingale differences with respect to the sequence of σ-algebras σ_{nk} generated by variables $\{\eta_{ni}, i \leq k\}$. Assume that condition (5.12) holds uniformly in $\alpha \in \mathcal{R}^r$. Then, using the result of Grigelionis (1973), it's not difficult to prove that for any $t > 0$, $\max_{m \leq nt} |\sum_{k=0}^{m} \varphi_{nk}| \xrightarrow{P} 0$. Further applying results of Gikhman and Skorokhod (1978) and using relation (5.14) we obtain

$$\sup_{u \leq t} |\eta_n(u) - \eta(u)| \xrightarrow{P} 0, \quad \sup_{u \leq t} |y_n(u) - y(u)| \xrightarrow{P} 0 \tag{5.23}$$

(see (5.17)(5.18)). As far as $m(a) > 0$, the process $y(t)$ increases strictly monotonically. Thus, the process $y^{-1}(t) = \mu(t)$ exists for such t that $y(+\infty) > t$ with probability one, is continuous and

$$\sup_{u \leq t} |\mu_n(u) - \mu(u)| \xrightarrow{P} 0. \tag{5.24}$$

Using the result of Billingsley (1977) about U-convergence of a superposition of random functions and relation (5.23), we obtain (5.15). Finally, we remark that $\Pr\Big\{\sup_{u \leq t} |s(t)| > N\Big\} \xrightarrow{P} 0$ as $N \to \infty$. Thus it is sufficient to check all conditions in each bounded region $|\alpha| \leq N$. Theorem 1 is proved.

Further denote $v_{nk} = \gamma_n(y_{nk})$, $\tilde{s}_{nk} = s(y_{nk}), k \geq 0$, and suppose for simplicity that s_0 is a nonrandom variable. As far as relation (5.15) holds, the trajectory η_{nk} , $k = 0, 1, \ldots, nT$ belongs to some bounded region with probability close to one. Thus, it is enough to check all conditions only in each bounded region. We have by the construction

$$v_{nk+1} = v_{nk} + n^{-1/2}\Big(\xi_{nk}(n\eta_{nk}) - n(\tilde{s}_{nk+1} - \tilde{s}_{nk})\Big).$$

Using Lagrange formula and relation (5.11), we obtain that

$$\tilde{s}_{nk+1} - \tilde{s}_{nk} = n^{-1}\tilde{b}(\tilde{s}_{nk})\tau_{nk} + \delta_{nk}^{(1)} = n^{-1}\tilde{b}(\tilde{s}_{nk})m_n(\eta_{nk}) + \delta_{nk}^{(1)} + \delta_{nk}^{(2)},$$

where $\tau_{nk} = \tau_{nk}(n\eta_{nk})$, $|\delta_{nk}^{(1)}| \le Cn^{-2}\tau_{nk}^2$ and $E|\delta_{nk}^{(2)}|^2 \le Cn^{-2}$.

After transformation we obtain that

$$v_{nk+1} = v_{nk} + n^{-1}m_n(\eta_{nk})q_n(\tilde{s}_{nk}, v_{nk}) + n^{-1/2}\alpha_{nk} + \delta_{nk}^{(3)}, \tag{5.25}$$

where $\alpha_{nk} = \xi_{nk}(n\eta_{nk}) - b_n(\eta_{nk}) - \tilde{b}(\tilde{s}_{nk})(\tau_{nk} - m_n(\eta_{nk}))$, $\mathbf{E}|\delta_{nk}^{(3)}|^2 \le n^{-3/2}C$. It is not difficult to prove that $\max_{k\le nT} |\sum_{i=o}^{k} \delta_{ni}^{(3)}| \xrightarrow{P} 0$. If $n \to \infty$, $k/n \to t$ and $v_{nk} = z$, then according to Theorem 1 $\eta_{n,[nt]} \xrightarrow{P} \eta(t)$, and $\tilde{s}_{n,[nt]} \to s(y(t)) = \eta(\mu(y(t))) = \eta(t)$. It means that a coefficient at $1/n$ in the right-hand side of (5.25) tends in probability to the value $m(\eta(t))q(\eta(t), z)$. Further, $\mathbf{E}[\alpha_{nk}/\eta_{nk}] = 0$, $\mathbf{E}[\alpha_{nk}\alpha_{nk}^*/\eta_{nk} = \alpha] \to D(\alpha)^2$ and, according to (5.21) variables $|\alpha_{nk}|^2$ are uniformly integrable in each bounded region. Let us introduce a random process $v_n(t) = v_{nk}$ as $k/n \le u < (k+1)/n$, $u \ge 0$. Then from representation (5.25) and results of Gikhman and Skorokhod (1975), it follows that the sequence of processes $v_n(u)$ J-converges on the interval $[0, T]$ to a diffusion process $v(u)$ satisfying the following stochastic differential equation: $v(0) = \gamma_0$,

$$\mathrm{d}v(u) = m(\eta(u))q(\eta(u), v(u))\mathrm{d}u + D(\eta(u))\mathrm{d}w(u). \tag{5.26}$$

We remark that at $\frac{1}{n}t_{nk} \le t < \frac{1}{n}t_{nk+1}$

$$|s(t) - s(\frac{1}{n}t_{nk})| \le \frac{1}{n}\tau_{nk} \sup_{\frac{1}{n}t_{nk}\le u\le \frac{1}{n}t_{nk+1}} |\tilde{b}(s(u))|.$$

Thus as $\mu_n(T) < \mu(T) + \varepsilon$,

$$\sup_{0\le t\le T} |\gamma_n(t) - v_n(\mu_n(t) - \frac{1}{n})| \le \frac{1}{\sqrt{n}} C_T \max_{k\le n(\mu(T)+\varepsilon)} \tau_{nk}, \tag{5.27}$$

where $C_T = \sup_{u\le\mu(T)+\varepsilon} |\tilde{b}(s(u))|$. It is not so hard to prove (see Anisimov, 1995) that for any $C > 0$ $\max_{k\le nC}\ n^{-1/2}\tau_{nk} \xrightarrow{P} 0$. Finally, we obtain that

$$\sup_{0\le t\le T} |\gamma_n(t) - v_n(\mu_n(t) - 1/n)| \xrightarrow{P} 0.$$

But the sequence of processes $v_n(\mu_n(t) - 1/n)$ J-converges to the process $v(\mu(t)) = \gamma(t)$. As far as $\mu'(t) = m(s(t))^{-1}$, we calculate the stochastic differential for process $\gamma(t)$ using the formula $\mathrm{d}w(\mu(t)) \sim \sqrt{\mu'(t)}\mathrm{d}w(t)$ and obtain equation (5.22). Theorem 2 is proved.

In conclusion of this section let us consider an important case when process $S_n(t)$ is a homogeneous *MP*. Suppose that $S_n(t)$ is a regular step-wise process and there exist intensities of transition probabilities

$q_n(\alpha, A), \alpha \in \mathcal{R}^r, A \in B^r_{\mathcal{R}}, \alpha \neq A$ such that $q_n(\alpha) = q_n(\alpha, \mathcal{R}^r \backslash \{\alpha\}) < \infty$ for any $\alpha \in \mathcal{R}^r$. We introduce independent families of random variables $\{\xi_{nk}(\alpha), \alpha \in \mathcal{R}^r\}, k \geq 0$ and $\{\tau_{nk}(\alpha), \alpha \in R\}, k \geq 0$ with values in $\mathcal{R}^r$ and $[0, \infty)$ respectively and such that $\tau_{nk}(n\alpha)$ has exponential distribution with parameter $q_n(\alpha)$ and $\Pr\Big\{\xi_{nk}(n\alpha) \in A\Big\} = q_n(\alpha)^{-1} q_n(\alpha, A+\alpha), \alpha \neq A$, where $A+\alpha = \{z : z-\alpha \in A\}$. It is clear that *RPSM* which is defined by families $(\zeta_{nk}(\alpha), \tau_{nk}(\alpha))$ is equivalent to our *MP* $S_n(t)$. Denote $m_n(\alpha) = q_n(\alpha)^{-1}, D^2_n(\alpha) = \mathbf{E}\xi_{n1}(n\alpha)\xi_{n1}(n\alpha)^*$ and keep other notations.

Corollary 1 *If conditions of Theorems 1, 2 hold, then the relation (5.15) takes place and the sequence of processes $\gamma_n(t)$ weakly converges to the diffusion process $\gamma(t)$ satisfying equation (5.22).*

We remark that in this case conditions (5.12) and (5.21) for variables $\tau_{n1}(\alpha)$ are automatically satisfied.

4. PROCESSES WITH SEMI-MARKOV SWITCHES

Consider now *AP* and *DA* type theorems for *PSMS*. Let for each $n > 0$, $\mathcal{F}_{nk} = \{\zeta_{nk}(t, x, \alpha),\ t \geq 0,\ x \in X,\ \alpha \in \mathcal{R}^r\}$, $k \geq 0$ be jointly independent families of random processes in D^r_∞, $x_n(t)$, $t \geq 0$ be a *SMP* in X independent of $\mathcal{F}_{nk}$, S_{n0} be an initial value. Let also $0 = t_{n0} < t_{n1} < \cdots$ be the epochs of sequential jumps of $x_n(\cdot)$, $x_{nk} = x_n(t_{nk})$, $k \geq 0$. We construct a *PSMS* according to formula (5.7): put $S_{nk+1} = S_{nk} + \xi_{nk}$, where $\xi_{nk} = \zeta_{nk}(\tau_{nk}, x_{nk}, S_{nk})$, $\tau_{nk} = t_{nk+1} - t_{nk}$, and denote

$$\zeta_n(t) = S_{nk} + \zeta_{nk}(t - t_{nk}, x_{nk}, S_{nk}) \quad \text{as} \quad t_{nk} \leq t < t_{nk+1},\ t \geq 0. \quad (5.28)$$

Then the process $(x_n(t),\ \zeta_n(t))$, $t \geq 0$ is a *PSMS*.

At first we study an *AP* for the switched component $\zeta_n(\cdot)$. Consider for simplicity a homogeneous case (distributions of processes $\zeta_{nk}(\cdot)$ do not depend on the index $k \geq 0$). Let $\tau_n(x)$ be a sojourn time in the state x for *SMP* $x_n(\cdot)$. Denote for each $x \in X$, $\alpha \in \mathcal{R}^r$

$$\xi_n(x, \alpha) = \zeta_{n1}(\tau_n(x), x, \alpha), \quad g_n(x, \alpha) = \sup_{t < \tau_n(x)} |\zeta_{n1}(t, x, \alpha)|.$$

4.1 ASYMPTOTICALLY MIXING ENVIRONMENT

Suppose that MP x_{nk}, $k \geq 0$ has at each $n \geq 0$ a stationary measure $\pi_n(A)$, $A \in \mathcal{B}_X$ and denote $m_n(x) = \mathbf{E}\,\tau_n(x)$, $b_n(x,\alpha) = \mathbf{E}\,\xi_n(x,n\alpha)$,

$$m_n = \int_X m_n(x)\pi_n(\mathrm{d}x), \quad b_n(\alpha) = \int_X b_n(x,\alpha)\pi_n(\mathrm{d}x),$$

$$\alpha_n(k) = \sup_{A,B\in\mathcal{B}_X, i\geq 0} |\mathbf{P}\{x_{ni} \in A, x_{ni+k} \in B\} - \mathbf{P}\{x_{ni} \in A\}\,\mathbf{P}\{x_{ni+k} \in B\}|.$$

Theorem 3 *Suppose that* $n^{-1}S_{n0} \xrightarrow{\mathrm{P}} s_0$, *there exists a sequence of integers* r_n *such that*

$$n^{-1}r_n \to O, \ \sup_{k\geq r_n} \alpha_n(k) \to O, \tag{5.29}$$

for any $N > 0, \varepsilon > 0$

$$\lim_{n\to\infty} \sup_{|\alpha|<N} \sup_x n\mathbf{P}\{n^{-1}g_n(x,\alpha) > \epsilon\} = 0, \tag{5.30}$$

$$\lim_{L\to\infty} \limsup_{n\to\infty} \sup_{|\alpha|<N} \sup_x \{\mathbf{E}\tau_{n1}(x)\chi(\tau_{n1}(x) > L) + \mathbf{E}\,|\xi_{n1}(x,n\alpha)|\chi(|\xi(x,n\alpha)| > L)\} = 0,$$

for any x *as* $\max(|\alpha_1|,|\alpha_2|) < N$ $|b_n(x,\alpha_1) - b_n(x,\alpha_2)| < C_N|\alpha_1 - \alpha_2| + \alpha_n(N)$, *where* C_N *are some constants,* $\alpha_n(N) \to 0$ *uniformly on* $|\alpha_1| < N$, $|\alpha_2| < N$, *also there exists a function* $b(\alpha)$ *and a constant* m *such that for any* $\alpha \in \mathcal{R}^r$ $b_n(\alpha) \to b(\alpha)$, $m_n \to m > 0$. *Then for any* $T > 0$

$$\sup_{0\leq t\leq T} |n^{-1}\zeta_n(nt) - s(t)| \xrightarrow{\mathrm{P}} 0, \tag{5.31}$$

where

$$s(0) = s_0, \quad \mathrm{d}s(t) = m^{-1}b(s(t))\,\mathrm{d}t \tag{5.32}$$

(it is supposed that a solution of the equation (5.32) exists on each interval and is unique).

Remark 1 *Condition (5.29) covers also more general situations than only the case when the process* x_{nk} *is ergodic in the limit. For instance a state space can form n-S-set (see Anisimov, 1973, 1996a).*

Consider a *DA* for the sequence of processes $\gamma_n(t) = n^{-1/2}(\zeta_n(nt) - ns(t))$. Introduce a uniformly strong mixing coefficient for the process x_{nk}: $\varphi_n(r) = \sup_{x,y,A} |\mathbf{P}\{x_{nr} \in A/x_{no} = x\} - P\{x_{nr} \in A/x_{no} = y\}|$, $r >$

0. Put $\tilde{b}_n(\alpha) = b_n(\alpha)m_n^{-1}$, $\tilde{b}(\alpha) = b(\alpha)m^{-1}$, $\rho_{nk}(x,\alpha) = \xi_{nk}(x,n\alpha) - b_n(x,\alpha) - \tilde{b}(\alpha)(\tau_{nk}(x) - m_n(x))$, $D_n(x,\alpha)^2 = \mathbf{E}\rho_{n1}(x,\alpha)\rho_{n1}(x,\alpha)^*$, and $\gamma_n(x,\alpha) = b_n(x,\alpha) - b_n(\alpha) - \tilde{b}(\alpha)(m_n(x) - m_n)$.

Theorem 4 *Suppose that $\gamma_n(0) \overset{\mathrm{w}}{\Rightarrow} \gamma_0$, there exist fixed $r > 0$ and $q \in [0,1)$ such that $\varphi_n(r) \le q$, $n > 0$, conditions of Theorem 3 hold where $\sqrt{n}\alpha_n(N) \to 0$, and for any $N > 0$ the following conditions are satisfied:*

$$\lim_{n\to\infty} \sup_{|\alpha|<N} \sup_x n\mathbf{P}\left\{n^{-1/2}g_n(x,\alpha) > \varepsilon\right\} = 0, \ \forall\varepsilon > 0; \tag{5.33}$$

$$\lim_{L\to\infty} \lim_{n\to\infty} \sup_{|\alpha|<N} \sup_x \left\{\mathbf{E}\tau_{n1}(x)^2\chi(\tau_{n1}(x) > L) + \mathbf{E}|\xi_{n1}(x,n\alpha)|^2\chi(|\xi_{n1}(x,n\alpha)| > L)\right\} = 0;$$

$|D_n(x,\alpha_1)^2 - D_n(x,\alpha_2)^2| \le C_N|\alpha_1 - \alpha_2| + \alpha_n(N)$, as $\max(|\alpha_1|,|\alpha_2|) < N$, *where $\alpha_n(N) \to 0$ uniformly in $|\alpha_1| < N$, $|\alpha_2| < N$;*
there exist continuous vector-valued function $q(\alpha,z)$ and matrix-valued functions $D(\alpha)$ and $B(\alpha)$ such that in any domain $|\alpha| < N$ $|q(\alpha,z)| < C_N(1+|z|)$, uniformly in $|\alpha| < N$ at each fixed z

$$\sqrt{n}\left(\tilde{b}_n(\alpha + n^{-1/2}z) - \tilde{b}(\alpha)\right) \to q(\alpha,z);$$

at any $\alpha \in \mathcal{R}^m$

$$D_n(\alpha)^2 = \int_X D_n(x,\alpha)^2\pi_n(\mathrm{d}x) \to D(\alpha)^2,$$

$$B_n^{(1)}(\alpha)^2 + B_n^{(2)}(\alpha)^2 + (B_n^{(2)}(\alpha)^*)^2 \to B(\alpha)^2,$$

where $B_n^{(1)}(\alpha)^2 = \int_X \gamma_n(x,\alpha)\gamma_n(x,\alpha)^*\pi_n(\mathrm{d}x)$, *and*

$$B_n^{(2)}(\alpha)^2 = \sum_{k\ge1} \mathbf{E}\,\gamma_n(x_{n0},\alpha)\gamma_n(x_{nk},\alpha)^*, \tag{5.34}$$

where $\mathbf{P}\{x_{n0} \in A\} = \pi_n(A)$, $A \in \mathcal{B}_X$. Then the sequence of processes $\gamma_n(t)$ J-converges to the diffusion process $\gamma(t)$: $\gamma(0) = \gamma_0$,

$$\mathrm{d}\gamma(t) = q(s(t),\gamma(t))\,\mathrm{d}t + m^{-\frac{1}{2}}(D(s(t))^2 + B(s(t))^2)^{\frac{1}{2}}\mathrm{d}w(t), \tag{5.35}$$

where $w(t)$ is a standard Wiener process in $\mathcal{R}^r$ and the solution of (5.35) exists and is unique.

The proof of Theorems 3, 4 follows the same scheme as the proof of Theorems 1, 2 and uses the results about the convergence of stochastic recurrent sequences in Markov environment to solutions of stochastic differential equations (see Anisimov and Yarachkovskiy, 1986). More details can be found in Anisimov (1994).
These results also can be extended on non-homogeneous in time models (see Anisimov, 1995).

4.2 ASYMPTOTICALLY CONSOLIDATED ENVIRONMENT

Now we consider the case when condition (5.29) is not true. It means that states of the environment do not asymptotically communicate. Suppose for simplicity that MP x_{nk} has a finite state space $X = \{1, 2, \dots, d\}$. We keep the previous notations. Let the following representation holds:

$$X = \bigcup j \in Y X_j, \quad \text{where } X_{j_1} \bigcap X_{j_2} = \emptyset \text{ at } j_1 \neq j_2, \tag{5.36}$$

and also one-step transition probabilities $p_n(i,l) = \Pr\{x_{n1} = l/x_{n0} = i\}$ are represented in the form

$$p_n(i,l) = p_n^{(0)}(i,l) + n^{-1} h_n(i,l), \quad i,l = \overline{1,d}, \tag{5.37}$$

where $\limsup_{n\to\infty} \max_{i,l} |h_n(i,l)| < C$, and for any $j \in Y$ $p_n^{(0)}(i,l) \equiv 0$ at $i \in X_j$, $l \notin X_j$.

For each $j \in Y$ denote by $x_{nk}^{(j)}, k \geq 0$ an auxiliary MP with state space X_j and transition probabilities $p_n^{(0)}(i,l)$, $i, l \in X_j$. Suppose that at each j the process $x_{nk}^{(j)}$ satisfies condition (5.29) and denote by $\pi_n^{(j)}(i), i \in X_j$ its stationary distribution. Further for any $j \in Y$, $m \in Y$, $j \neq m$ we put

$$\lambda_n(j,m) = \sum_{i\in X_j} \pi_n^{(j)}(i) \sum_{l \in X_m} h_n(i,l),$$

$$\widehat{m}_n(j) = \sum_{i \in X_j} \pi_n^{(j)}(i) m_n(i), \quad \widehat{b}_n(j,\alpha) = \sum_{i\in X_j} \pi_n^{(j)}(i) b_n(i,\alpha).$$

Suppose that there exist values $\lambda(j,m), \widehat{m}(j)$ and continuous functions $\widehat{b}(j,\alpha)$ such that for any $\alpha \in \mathcal{R}^r, j, m \in Y$, $j \neq m$

$$\lambda_n(j,m) \to \lambda(j,m), \ \widehat{m}_n(j) \to \widehat{m}(j) > 0, \ \widehat{b}_n(j,\alpha) \to \widehat{b}(j,\alpha).$$

Denote by $y(t, j_0)$ a MP with values in Y, intensities of transition probabilities $\lambda(j,m)/\widehat{m}(j)$, $j, m \in Y, j \neq m$ and the initial value j_0. Denote also by $z(t, j_0, s_0)$ a solution of differential equation: $z(0, j_0, s_0) = s_0$,

$$\mathrm{d}z(t, j_0, s_0) = \widehat{m}(y(t,j_0))^{-1}\, \widehat{b}(y(t,j_0), z(t,j_0,s_0))\mathrm{d}t. \tag{5.38}$$

Let us introduce a consolidated process $\widehat{x}_n(t) = j$ as $x_n(t) \in X_j$, $t \geq 0$.

Theorem 5 *Suppose that at our assumptions* $\Pr(x_n(0) \in X_{j_0}) \to 1$ *as* $n \to \infty$, *relations (5.36),(5.37) are true and corresponding conditions of* *regularity for variables* $\tau_n(\cdot), \xi_n(\cdot), g_n(\cdot)$ *given in Theorem 3 hold. Then the sequence of processes* $(\widehat{x}_n(t), n^{-1}\zeta_n(t))$ *J-converges on each interval* $[0,T]$ *to the process* $(y(t,j_0), z(t,j_0,s_0))$.

The proof is based on limit theorems for *SP*'s in the case of rare switches (see Anisimov, 1978, 1988ab). The main steps are as follows. The process $(\widehat{x}_n(t), n^{-1}\zeta_n(t)$ is represented as a *SP* for which switching times are the times of sequential jumps between regions X_j. Then on the interval between two jumps the process $n^{-1}\zeta_n(t)$ behaves as a process in asymptotically quasi-ergodic Markov environment and on the base of results of Theorem 3 it converges to a solution of differential equation with coefficients averaged by stationary measure in corresponding region. Further an interval of time between two sequential switches asymptotically has an exponential distribution with parameter which is obtained by averaging in stationary measure of normed transition probabilities from a region (see Anisimov, 1973, 1988a). Thus the limiting process can be described as a solution of a differential equation with Markov switches. In the case that $\widehat{b}_0(j,\alpha) \equiv 0$, it is also possible to prove a *DA* for $\zeta_n(t)$. We mention that in this case a class of limiting processes belongs to the class of dynamical systems or diffusion processes with Markov switches (see section 2.7).

5. APPLICATIONS

5.1 RANDOM MOVEMENTS

Consider *AP* and *DA* for a random movement with semi-Markov switches described in the section 2.7. Suppose that sojourn times of *SMP* $x(t)$ depend on parameter n in such a way that $\tau_n(i) = n^{-1}\tau(i)$. Assume that 2nd moments exist and denote $\mathbf{E}\tau(i) = m(i), \mathbf{Var}\,\tau(i) = \sigma^2(i), i = \overline{1,d}$.

1) At first consider an ergodic case. Suppose that the embedded *MP* x_k doesn't depend on parameter n and is irreducible. Denote by $\pi_i, i = \overline{1,d}$ its stationary distribution. Let $m = \sum_{i=1}^{d} m(i)\pi_i > 0$, $b(\alpha) = \sum_{i=1}^{d} v(i,\alpha)m(i)\pi_i$. At stationary conditions $(\mathbf{P}\{x_0 = i\} = \pi_i, i = \overline{1,d})$ we denote $B^{(2)}(\alpha)^2 = \sum_{k\geq 1} \mathbf{E}\, m(x_0)m(x_k)(v(x_0,\alpha)-m^{-1}b(\alpha))(v(x_k,\alpha)-m^{-1}b(\alpha))^*$, $B(\alpha)^2 = \sum_{i=i}^{m} \pi_i m(i)^2(v(i,\alpha)-m^{-1}b(\alpha))(v(i,\alpha)-m^{-1}b(\alpha))^* + B^{(2)}(\alpha)^2 + (B^{(2)}(\alpha)^*)^2$, $D(\alpha)^2 = \sum_{i=i}^{d} \pi_i(v(i,\alpha) - m^{-1}b(\alpha))(v(i,\alpha) - m^{-1}b(\alpha))^*\sigma(i)^2$.

Statement 1 *Let functions $v(i,\alpha)$ be locally Lipschitz and have no more than linear growth. Then for any $T > 0$,*

$$\sup_{0\le t\le T} |\zeta_n(t) - s(t)| \xrightarrow{\mathrm{P}} 0,$$

where $s(\cdot)$ satisfies equation (5.32), and the sequence $\sqrt{n}(\zeta_n(t)-s(t))$ J-converges to the diffusion process satisfying equation (5.35) with $q(\alpha,z) = m^{-1}b'(\alpha)z$.

The proof directly follows from the results of Theorems 3, 4.

2) Further suppose that the embedded *MP* also depends on the parameter n in such a way that conditions (5.36),(5.37) hold. For simplicity suppose that each region X_j forms in a limit one essential class. Let $x_k^{(j)}$ be an auxiliary *MP* in X_j with limiting transition probabilities and stationary distribution $\pi^{(j)}(i), i \in X_j$. At any $j \in Y$ denote

$$\widehat{m}(j) = \sum_{i\in X_j} m(i)\pi^{(j)}(i), \quad \widehat{b}(j,\alpha) = \sum_{i\in X_j} v(i,\alpha)m(i)\pi^{(j)}(i). \qquad (5.39)$$

Let $y(t,j_0)$ be the *MP* introduced in Theorem 5.

Statement 2 *Suppose that at our assumptions $\Pr(x_n(0) \in X_{j_0}) \to 1$ as $n \to \infty$, at any $j \in Y$, $\widehat{m}(j) > 0$ and functions $\widehat{b}(j,\alpha)$ are locally Lipschitz and have no more than linear growth. Then the sequence $\zeta_n(t)$ J-converges on each interval $[0,T]$ to the process $z(t,j_0,s_0)$ (see (5.38)).*

3) Consider now the case when in (5.39) $\widehat{b}(j,\alpha) \equiv 0$. For each region X_j put $\widehat{D}(j)^2 = \sum_{i\in X_j} v(i,0)v(i,0)^*\sigma(i)^2\pi^{(j)}(i)$, $\widehat{B}^{(1)}(j)^2 = \sum_{i\in X_j} m(i)^2 \times v(i,0)v(i,0)^*\pi^{(j)}(i)$, in stationary conditions ($\Pr(x_0^{(j)} = i) = \pi^{(j)}(i), i \in X_j$) define

$$\widehat{B}^{(2)}(j)^2 = \sum_{k\ge 1} \mathbf{E}\, m(x_0^{(j)})m(x_k^{(j)})v(x_0^{(j)},0)v(x_k^{(j)},0)^*,$$

and denote $\widehat{C}(j)^2 = \widehat{D}(j)^2 + \widehat{B}^{(1)}(j)^2 + \widehat{B}^{(2)}(j)^2 + (\widehat{B}^{(2)}(j)^*)^2$.

Statement 3 *At conditions of Statement 2 the sequence $\sqrt{n}\zeta_n(t)$ J-converges to the process $\gamma(t,j_0,s_0)$ which can be represented as follows :*

$$\gamma(t,j_0,s_0) = \int_0^t \widehat{m}(y(t,j_0))^{-1/2}\widehat{C}(y(t,j_0))\,dw(t).$$

This is the Wiener process with Markov switches.

5.2 SEMI-MARKOV STATE-DEPENDENT QUEUEING MODELS

The results obtained can be effectively applied to the analysis of overloading state-dependent semi-Markov queueing models. Consider as an example a queueing system $SM/M_{SM,Q}/1/\infty$. Let $x(t)$, $t \geq 0$ be a *SMP* with values in X. Denote by $\tau(x)$ a sojourn time in the state x. Let non-negative function $\mu(x,\alpha)$, $x \in X$, $\alpha \geq 0$, be given. There is one server and infinitely many places for waiting. At first consider the model when calls enter the system one at a time at the epochs of jumps $t_1 < t_2 < \ldots$ of the process $x(t)$. Put $x_k = x(t_k + 0)$. If a call enters the system at time t_k and the number of calls in the system becomes equal to Q, then the intensity of service on the interval $[t_k, t_{k+1})$ is $\mu(x_k, n^{-1}Q)$. After service the call leaves the system. Let Q_{n0} be an initial number of calls, and $Q_n(t)$ be a number of calls in the system at time t.

1) At first consider the case when the embedded *MP* x_k, $k \geq 0$ doesn't depend on parameter n and is uniformly ergodic with stationary measure $\pi(A)$, $A \in \mathcal{B}_X$. We put $m(x) = \mathbf{E}\tau(x)$, $m = \int_X m(x)\pi(\mathrm{d}x)$, $c(\alpha) = \int_X \mu(x,\alpha)m(x)\pi(\mathrm{d}x)$, $b(\alpha) = (1 - c(\alpha))m^{-1}$, $g(x,\alpha) = 1 - m(x)(1 - c(\alpha) + \mu(x,\alpha)m)m^{-1}$, $G(\alpha) = c'(\alpha)$, $d^2(x) = \mathbf{Var}\,\tau(x)$, $d^2 = \int_X d^2(x)\pi(\mathrm{d}x)$, $e_1(\alpha) = \int_X \mu^2(x,\alpha)d^2(x)\pi(\mathrm{d}x)$, $e_2(\alpha) = \int_X \mu(x,\alpha)d^2(x)$ $\pi(\mathrm{d}x)$ and $D^2(\alpha) = c(\alpha)+e_1(\alpha)+2(1-c(\alpha))m^{-1}e_2(\alpha)+(1-c(\alpha))^2m^{-2}d^2$.

Statement 4 *Suppose that $m > 0$, the function $\mu(x,\alpha)$ is locally Lipschitz with respect to α uniformly in $x \in X$, the function $c(\alpha)$ has no more then linear growth and $n^{-1}Q_n(0)\xrightarrow{\mathrm{P}}s_0 > 0$. Then the relation (5.31) holds with $\zeta_n(nt) = Q_n(nt)$, where $\mathrm{d}s(t) = m^{-1}(1 - c(s(t))\mathrm{d}t$, $s(0) = s_0$, and T is any positive value such that $s(t) > 0$, $t \in [0,T]$. Suppose in addition that variables $\tau(x)^2$ are uniformly integrable, the function $c(\alpha)$ is continuously differentiable, $n^{-1/2}(Q_n(0) - s_0)\overset{\mathrm{W}}{\Rightarrow}\gamma_0$, and*

$$B^2(a) = \mathbf{E}\Big(g(x_0,\alpha)^2 + 2\sum_{k=1}^{\infty} g(x_0,\alpha)g(x_k,\alpha)\Big),$$

where $P\{x_0 \in A\} = \pi(A), A \in \mathcal{B}_X$. Then the sequence of processes $\gamma_n(t) = n^{-1/2}(Q_n(nt) - ns(t))$ J-converges on the interval $[0,T]$ to the diffusion process $\gamma(t) : \gamma(0) = \gamma_0$,

$$\mathrm{d}\gamma(t) = -m^{-1}G(s(t))\gamma(t)\mathrm{d}t + m^{-1/2}\left(D^2(s(t)) + B^2(s(t))\right)^{1/2}\mathrm{d}w(t).$$

Proof. At first we represent a queue in the system as a *PSMS.* In our case epochs t_k are switching times and variable $\xi_{nk}(x, n\alpha)$ can be represented in a form : $\xi_{n1}(x, n\alpha) = 1 - \Pi_{\mu(x,\alpha)}(\tau(x))$, where $\Pi_\lambda(t)$ is a Poisson process with parameter λ. It is easy to see that $\mathbf{E}\xi_1(x, n\alpha) = 1 - \mu(x, \alpha)m(x)$ and using result of Theorem 3 it is not difficult to obtain *AP*. Further we can simply calculate another characteristics and obtain *DA* using result of Theorem 4. We mention also that the process of changing queue is monotone on each interval $[t_k, t_{k+1})$. Thus *U*-convergence of embedded *RPSM* to a limit process automatically implies *U*-convergence of *PSMS* that is conditions (5.30), (5.33) are automatically satisfied. This finally proves Statement 4.

We remark that condition $s(t) > 0,\ t \in [0, T]$ is in fact a heavy traffic condition. For instance it is always true if $c(\alpha) < 1,\ \alpha > 0$.

2) Now suppose that the embedded *MP* $x_k,\ k \geq 0$ also depends on parameter n in such a way that conditions (5.36), (5.37) hold. For simplicity we consider the case of a finite state space X. Suppose that each region X_j forms in a limit one essential class and denote by $\pi^{(j)}(i), i \in X_j$ its stationary distribution. At any $j \in Y$ denote

$$\widehat{m}(j) = \sum_{i \in X_j} m(i)\pi^{(j)}(i), \quad \widehat{c}(j, \alpha) = \sum_{i \in X_j} \mu(i, \alpha)m(i)\pi^{(j)}(i). \tag{5.40}$$

Let $y(t, j_0)$ be a *MP* introduced in Theorem 5.

Statement 5 *If at our assumptions* $\Pr(x_n(0) \in X_{j_0}) \to 1$ *as* $n \to \infty$, *at any* $j \in Y$ $\widehat{m}(j) > 0$ *and functions* $\widehat{c}(j, \alpha)$ *are locally Lipschitz and have no more than linear growth, then the sequence* $n^{-1}Q_n(nt)$ *J-converges on the interval* $[0, T]$ *to the process* $q(t, j_0, s_0)$ *such that* $q(0, j_0, s_0) = s_0$ *and*

$$\mathrm{d}q(t, j_0, s_0) = \widehat{m}(y(t, j_0))^{-1}\Big(1 - \widehat{c}(y(t, j_0), q(t, j_0, s_0))\Big)\mathrm{d}t,$$

and T *is any positive value such that* $q(t, j_0, s_0) > 0$ *for all* $t \in [0, T]$ *with probability one.*

5.3 MARKOV MODELS WITH SEMI-MARKOV SWITCHES

Consider now a queueing system $M_{SM,Q}/M_{SM,Q}/1/\infty$. Let $x(t),\ t \geq 0$ be a *SMP* with values in $X = \{1, 2, \ldots, d\}$ and sojourn times $\tau(i)$. Let the family of non-negative functions $\{\lambda(i, \alpha),\ \mu(i, \alpha),\ \alpha \geq 0\},\ i \in X$ be given. There is one server and infinitely many places for waiting. The instantaneous rates of input flow and service depend on the state of $x(\cdot)$, value of the queue and parameter n in the following way: if at time t, $x(t) = i$ and $Q_n(t) = Q$, then an input rate is $\lambda(i, n^{-1}Q)$ and service rate is $\mu(i, n^{-1}Q)$. Calls enter the system one at a time. We mention that here times t_k are also switching times but at these times we have no additional jumps of input flow and finishing service.

1) At first consider the case when the embedded *MP* $x_k,\ k \geq 0$ doesn't depend on parameter n and is irreducible with stationary distribution $\pi_i,\ i \in X$. We keep the previous notations for values $m(i)$ and m and put $b(\alpha) = \sum_i (\lambda(i,\alpha) - \mu(i,\alpha))m(i)\pi_i$.

Statement 6 *Suppose that functions $\lambda(i,\alpha), \mu(i,\alpha)$ are locally Lipschitz with respect to α, $m > 0$, the function $b(\alpha)$ has no more then linear growth and $n^{-1}Q_n(0) \xrightarrow{P} s_0 > 0$. Then the relation (5.31) holds with $\zeta_n(nt) = Q_n(nt)$, where T is any positive value such that $s(t) > 0$ on the interval $[0,T]$.*

2) Now suppose that the embedded *MP* $x_k,\ k \geq 0$ also depends on parameter n in such a way that conditions (5.36), (5.37) hold. Suppose that each region X_j forms in a limit one essential class and denote by $\pi^{(j)}(i), i \in X_j$ its stationary distribution. At any $j \in Y$ denote $\widehat{m}(j) = \sum_{i \in X_j} m(i)\pi^{(j)}(i)$, $\widehat{b}(j,\alpha) = \sum_{i \in X_j} (\lambda(i,\alpha) - \mu(i,\alpha))m(i)\pi^{(j)}(i)$. Let $y(t, j_0)$ be a *MP* introduced in Theorem 5.

Statement 7 *If at our assumptions conditions of Statement 5 are valid (also for functions $\lambda(i,\alpha)$), then the sequence $n^{-1}Q_n(nt)$ J-converges on the interval $[0,T]$ to the process $q(t, j_0, s_0)$ such that $q(0, j_0, s_0) = s_0$ and*

$$dq(t, j_0, s_0) = \widehat{m}(y(t,j_0))^{-1}\widehat{b}(y(t,j_0), q(t,j_0,s_0))dt.$$

Using the same technique we can apply these results to retrial queues and queueing networks $(SM/M_{SM,Q}/1/\infty)^r$, $(M_{SM,Q}/M_{SM,Q}/1/\infty)^r$ of a semi-Markov type with input and service depending on the state of some *SMP* and current values of queues in the nodes, different types of customers, impatient customers, etc.

Some non-Markov queueing models $G_Q/M_Q/1/\infty$, $SM_Q/M_Q/1/\infty$ and $(G_Q/M_Q/1/\infty)^r$ are considered in (Anisimov, 1992, 1995, 1996a).

Another direction of applications can be branching processes and dynamical systems with stochastic perturbations. For near-critical branching processes with semi-Markov switches and large number of particles an *AP* is proved by Anisimov (1996b), and for dynamical systems with quick semi-Markov perturbations *AP* and *DA* are given in (Anisimov, 1994, 1995).

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