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SHORT COMMUNICATION

On explicit solutions of a two-echelon supply chain coordination game

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Abstract A contracting game under asymmetric information specific to two-echelon supply chain coordination between a retailer of unknown type and a supplier is studied. When the parameter which is private information to the retailer (holding cost) is known up to an interval of uncertainty, a uniform discrete approximation for retailer types leads to closed-form solutions where the joint (coordinated) optimal order quantity for a modified holding cost plays a major role. Furthermore, the closed-form solutions result in increasing information rent for higher types under easy-to-verify conditions involving strict lower limits on the total holding costs of retailer and supplier and the difference between uncoordinated optimal costs of consecutive retailer types.

Keywords Two-echelon supply chain coordination · Incentives · Principal-agent problem · Convex optimization

1 Introduction

The purpose of this brief paper is to identify cases of a two-echelon supply chain coordination game under asymmetric information where the problem is solved in closed-form. Departing from the setting of [1] it considers a retailer and a supplier where the retailer has market power to enforce any production quantity on the supplier who does not know the retailer's holding cost (referred to as the retailer type). The supplier, to minimize his costs by inducing different behaviour of the retailer, offers a menu of contracts to the retailer, while ensuring individual rationality (participation incentive) and incentive compatibility (incentive to report the true holding cost) on

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the part of the retailer. The problem is well-studied in various papers under different assumptions (see e.g., continuum of types [1,2], roles of retailer and supplier swapped [3], two-dimensional retailer types [4], supplier setup cost being an additional decision variable [5], two retailer types and no supplier holding cost [6]), and can be situated in the more general literature on contract design, economic incentives and the principal-agent problem; see [7]. In particular, we concentrate on a model where the holding cost parameter of the retailer can be any one of a finite number of types as in [6,8]. The contribution of the present paper is to derive in Proposition 1 and Theorem 1 full (explicit) solutions for multiple types when the retailer types are equally likely, and the holding cost parameters of consecutive retailer types differ by a constant, which would typically be the case when one can only pinpoint an interval of uncertainty on the holding cost of the retailer and defines a discrete uniform distribution on the interval of uncertainty. It is usually difficult to give a closed-form solution in general for this type of model; e.g., [6,8] give a full solution for two retailer types only. Our closed-form results depend on the joint (coordinated) optimal order quantity for a shifted holding cost. They also provide additional insights since the conditions under which they hold are expressed as strict lower bounds on the total holding costs of retailer and supplier, and on the difference between uncoordinated optimal costs of consecutive retailer types. In particular, the results (e.g., Proposition 1 and part of Theorem 1) are valid under the condition (among others) that the total holding cost for the supplier and each assumed retailer type exceeds a (decreasing with type) multiple of the uniform increase in holding cost, and if the difference between uncoordinated optimal costs of consecutive retailer types is larger (up to a problem-specific constant) than a quantity involving the increment in holding cost between consecutive retailer types and the joint (coordinated) optimal order quantity for a shifted holding cost. Furthermore, the (easy to check) conditions guarantee (at least in some parts of our results) increasing information rents to higher types, as is typically the case in economic theory.

2 Two-echelon supply chain coordination

The setting involves a retailer and a supplier. The retailer faces external demand with known rate $d > 0$. Demand must be satisfied. Hence, there is no backlogging. The supply chain is using a pull ordering strategy, i.e., the retailer places orders at the supplier. The retailer follows a cost minimizing optimal policy. The supplier may decrease his costs by convincing the retailer to modify the ordering policy. Every time an order is placed at the supplier, the retailer incurs a fixed ordering cost equal to $f > 0$. Lead times are assumed to be zero, and the retailer is assumed to have a holding cost equal to $h > 0$, which is unknown to the supplier. The retailer who is minimizing costs, places an order if and only if his inventory is depleted. Using the well-known economic order quantity (EOQ) model the per-unit time cost function of the retailer is given as:

$$\phi_R(x) = fd\frac{1}{x} + \frac{1}{2}hx,$$

which admits the minimizer $x_R^* = \sqrt{2df/h}$. The supplier has a similar cost structure with a fixed setup cost equal to $F > 0$, and a holding cost equal to $H > 0$, while production takes place at the constant rate $p > 0$ (production rate p should be greater than or equal demand rate d). To minimize costs, the supplier produces using a just-in-time lot-for-lot policy. The per-unit time cost function of the supplier is

$$\phi_S(x) = Fd\frac{1}{x} + \frac{1}{2}H\frac{d}{p}x,$$

which is minimized at $x_S^* = \sqrt{2pF/H}$. Both the supplier and the retailer act according to their own interests and therefore, their optimal order quantities are most likely sub-optimal for the entire supply chain unless they coordinate to lower the joint total costs. If they cooperated their joint cost function would be

$$\phi_J(x) = d(f + F)\frac{1}{x} + \frac{1}{2}\left(h + H\frac{d}{p}\right)x,$$

which would be minimized at the optimal joint order quantity

$$x_J^* = \sqrt{2d(f + F)/\left(h + H\frac{d}{p}\right)}. \tag{1}$$

It can be shown that x_j^* lies between the individual optimal order quantities x_R^*, x_S^* of the retailer and the supplier, respectively (c.f. [8]).

The retailer is a rational agent minimizing his costs and has market power to dictate any order quantity on the supplier. The retailer chooses his own optimal order quantity (which is equal to the production quantity x_R^*) unless there is an incentive to do otherwise offered by the supplier who wishes to decrease his costs. Such an incentive is usually implemented in the form of a side payment from the supplier to the retailer. In case the incentive scheme is accepted by the retailer, the effect is that the costs decrease for the entire supply chain. Thus, it is assumed that the supplier offers a menu of contracts (x_k, z_k), $k = 1, \ldots, K$, to the retailer, where z_k is a side payment to the retailer in the k-th contract. More precisely, the supplier not knowing precisely the holding cost of the retailer, reckons that the potential holding cost values can be one of $\{h_1, h_2, \ldots, h_K\}$, where $h_1 < h_2 < \cdots < h_K$. We shall refer to $k \in \mathcal{K} = \{1, \ldots, K\}$ as the *type* of the retailer. Hence, the supplier designs a menu of K individually rational contracts to offer to the retailer, applying the general theory of principal-agent problems as in [7], or more precisely the case of a contracting or screening game (hence, the choice of the contract type by the retailer is taken as an indication of his holding cost value). Let ω_k represent the probability (or weight) of type k, and $\phi_R^{k*} = \sqrt{2fdh_k}$ the optimal cost of the retailer if he ordered his EOQ (we refer to this cost as the "uncoordinated optimal cost" of a retailer of type k) .

Let $\phi_R^k(x) = fd\frac{1}{x} + \frac{1}{2}h_k x$ denote the per unit time cost function of the retailer of type k, for $k = 1, \ldots, K$. Invoking the Revelation Principle the supplier faces the following non-convex optimization problem:

$$\min \sum_{k\in\mathcal{K}} \omega_k(\phi_S(x_k) + z_k)$$

subject to

$$\phi_R^k(x_k) - z_k \le \phi_R^{k*}, \quad \forall k \in \mathcal{K} \tag{2}$$
$$\phi_R^k(x_k) - z_k \le \phi_R^k(x_\ell) - z_\ell, \quad \forall k, \quad \ell \in \mathcal{K}$$
$$x_k \ge 0, \quad \forall k \in \mathcal{K}, \tag{3}$$

where he minimizes the expected cost (or weighted total cost) subject to constraints inducing a certain kind of behaviour on the part of the retailer. The constraints of the model are 1. individual rationality constraints (2) that expresses the requirement that the retailer is not worse off compared to his optimal order quantity, and 2. incentive compatibility constraints (3) to induce the retailer not to misreport his type. In [8] this model is studied at length with properties of optimal contracts and explicit solution for two retailer types. In the present section we shall obtain an explicit solution of the problem with multiple retailer types under some conditions on the problem parameters that are quite easy to check. Furthermore, our results are valid for a uniform discrete approximation to the unknown holding cost parameter of the retailer and under some additional conditions.

Let us define the *information rent* variables

$$y_k = z_k - (\phi_R^k(x_k) - \phi_R^{k*}) \ge 0, \quad \forall k \in \mathcal{K}.$$

With this definition we obtain a convex optimization problem, referred to as (2ECH), equivalent to the original non-convex problem above.

$$\min \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + \phi_R^k(x_k) + y_k - \phi_R^{k*}\right)$$

subject to

$$y_\ell - y_k + \frac{1}{2}(h_\ell - h_k)x_\ell \le \phi_R^{\ell*} - \phi_R^{k*}, \quad \forall k, \quad \ell \in \mathcal{K}$$
$$x_k, \quad y_k \ge 0, \quad \forall k \in \mathcal{K}.$$

By Lemmas 3.3 and 3.4 of [8] we know the following facts that are easy to prove: 1. any feasible menu of contracts satisfies $x_1 \ge x_2 \ge \cdots \ge x_K$, and 2. the adjacent incentive compatibility constraints suffice to ensure general incentive compatibility constraints above. Hence we can simplify the problem above to the following problem (2ECHSC):

$$\min \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + \phi_R^k(x_k) + y_k - \phi_R^{k*}\right)$$

subject to

$$y_{k+1} - y_k + \frac{1}{2}(h_{k+1} - h_k)x_{k+1} \le \phi_R^{k+1*} - \phi_R^{k*}, \quad \forall k = 1, \dots, K-1,$$
$$y_k - y_{k+1} + \frac{1}{2}(h_k - h_{k+1})x_k \le \phi_R^{k*} - \phi_R^{k+1*}, \quad \forall k = 1, \dots, K-1,$$
$$y_k \ge 0, \quad \forall k \in \mathcal{K},$$
$$x_1 \ge x_2 \ge \cdots \ge x_K \ge 0.$$

The above is a linearly constrained convex optimization problem solvable using off-the-shelf solvers. It was solved by a cutting plane algorithm in [8]. Its specific separable structure helps in identifying closed-form solutions as we shall see next.

3 Explicit solutions

Now we shall obtain closed-form solutions under certain conditions that are very easy to check. In particular, our results are valid under the assumption that all types have equal weights and all consecutive holding cost values differ by the same constant, provided that some other conditions on the problem parameters are satisfied. These two assumptions (equal weights and uniform difference) could be fulfilled for instance when the supplier is faced with an interval of uncertainty $[\underline{h}, \bar{h}]$ for the retailer holding cost, and having no prior probabilistic information on the true holding cost value of the retailer, adopts a discrete uniform distribution on the interval by choosing K equally spaced values, by Δh, in the interval $[\underline{h}, \bar{h}]$ including the end-points.

We begin the analysis with a simple observation that is useful in solving for the information rent variables when the remaining variables are fixed.

Lemma 1 *If all $a_k \ge 0$ (with $b_k \ge a_k$), $f_k > 0$ for all $k \in \mathcal{K}$, then the following optimization problem*

$$\min_{y_k} \sum_{k=1}^{m} f_k y_k$$

subject to

$$a_k \le y_{k+1} - y_k \le b_k, \quad \forall k = 1, \dots, m-1,$$
$$y_k \ge 0, \quad \forall k = 1, \dots, m,$$

admits an optimal solution of the form $y_1^* = 0$ and $y_k^* = \sum_{i=1}^{k-1} a_i$ for $k = 2, \dots, m$.

Proof By a change of variables $\Delta_i = y_{i+1} - y_i$ for all $i = 1, \dots, m-1$, and observing that we can always set $y_1 = 0$ at optimality, we obtain the equivalent problem

$$\min_{\Delta_i} \sum_{i=2}^{m} f_i \left(\sum_{k=1}^{i-1} \Delta_k \right)$$

subject to

$$c_i \leq \Delta_i \leq c_{i+1}, \quad \forall i = 1, \ldots, m-1,$$

which is solved at $\Delta_i^* = c_i$, $i = 1, \ldots, m-1$. □

Now, under the uniform discrete approximation to the holding cost values if 1. the joint supplier holding cost and the holding cost of the kth retailer type is greater than a positive constant, namely the uniform increment Δh in holding cost times $K-k$, and 2. if the incremental difference between uncoordinated optimal total costs of the consecutive types $k+1$ and k differ by a quantity larger than an easily computable number, then we have a closed-form solution of problem (2ECH). Define for convenience the quantity $x(H, h)$ as the joint (coordinated) optimal order quantity viewed as a function of H and h, [it is in fact x_J^* already defined in (1)]:

$$x(H, h) = \sqrt{2d(f+F)/\left(h + H\frac{d}{p}\right)}. \tag{4}$$

We shall see below that the optimal order quantities for each retailer type k corresponds to a coordinated joint optimal order quantity between the supplier and retailer for a modification of the holding cost. The result is valid if the total holding cost for the supplier and each retailer type exceeds a (decreasing with type) multiple of the uniform increase in holding cost, and if the difference between uncoordinated optimal costs of consecutive retailer types is larger than the product of increment in holding cost with the the joint (coordinated) optimal order quantity for a shifted holding cost. For ease of notation, we define $C_{k+1,k} = \sqrt{2fd(h_k + \Delta h)} - \sqrt{2fdh_k}$ for all $k = 1, \ldots, K-1$. The parameter $C_{k+1,k}$ represents the increase in the "optimal cost value" (in an uncoordinated environment) of retailer type $k+1$ with respect to type k. Notice that $C_{k+1,k}$ is a decreasing function of k.

Proposition 1 *For uniform types ($\omega_k = \omega$ for all $k \in \mathcal{K}$), and equidistant holding costs ($h_{k+1} - h_k = \Delta h$ for all $k = 1, \ldots, K-1$), if*

$$H\frac{d}{p} + h_k > \Delta h(K-k), \quad \forall k = 1, \ldots, K-1, \tag{5}$$

and

$$C_{k+1,k} > \frac{1}{2}x(H, h_k - (K-k)\Delta h)\Delta h, \quad \forall k = 1, \ldots, K-1, \tag{6}$$

then

$$x_k^* = x(H, h_k - (K-k)\Delta h), \quad \forall k = 1, \ldots, K \tag{7}$$

and

$$y_k^* = \sum_{i=1}^{k-1}\left(C_{i+1,i} - \frac{1}{2}x(H, h_i - (K-i)\Delta h)\Delta h\right), \quad \forall k = 1, \ldots, K. \tag{8}$$

solve problem (2ECH).

Proof We omit the non-negativity and monotonicity constraints on the x variables. Our construction will satisfy these automatically. The first-order conditions (that are necessary and sufficient due to convexity[1]) with respect to variables x_k, $k = 1, \ldots, K$ are expressed as follows using non-negative Lagrange multipliers $\lambda_1^L, \lambda_2^U, \lambda_2^L, \ldots, \lambda_{K-1}^U, \lambda_{K-1}^L, \lambda_K^U$:

$$-\frac{d(f+F)}{x_1^2} + \frac{1}{2}\left(h_1 + H\frac{d}{p}\right) - \frac{1}{2}\Delta h \lambda_1^L = 0$$

$$-\frac{d(f+F)}{x_k^2} + \frac{1}{2}\left(h_k + H\frac{d}{p}\right) + \frac{1}{2}\Delta h \lambda_k^U - \frac{1}{2}\Delta h \lambda_k^L = 0 \quad \forall k = 2, \ldots, K-1,$$

$$-\frac{d(f+F)}{x_K^2} + \frac{1}{2}\left(h_K + H\frac{d}{p}\right) + \frac{1}{2}\Delta h \lambda_K^U = 0,$$

and we have the dual feasibility and complementarity conditions:

$$1 + \lambda_1^L - \lambda_2^U \geq 0 \perp y_1 \geq 0,$$

$$1 - \lambda_{k-1}^L + \lambda_k^U + \lambda_k^L - \lambda_{k+1}^U \geq 0 \perp y_k \geq 0, \quad k = 2, \ldots, K-1,$$

$$1 + \lambda_K^U - \lambda_{K-1}^L \geq 0 \perp y_K \geq 0$$

where the notation $\ldots \geq 0 \perp y_k \geq 0$ denotes the complementarity condition between the dual constraint and the variable y_k. Notice that from the pair of non-negative dual multipliers λ_k^U, λ_k^L at most one can assume a positive value at optimality.

Now, set the dual λ multipliers as follows: $\lambda_k^U = 0, \lambda_k^L = K-k$ for $k = 1, \ldots, K-1$, and $\lambda_K^U = 0$. These values are all positive and satisfy the conditions as equalities, i.e., we have

$$1 - \lambda_{k-1}^L + \lambda_k^U + \lambda_k^L - \lambda_{k+1}^U = 0, \quad k = 1, \ldots, K-1,$$

$$1 + \lambda_K^U - \lambda_{K-1}^L = 0,$$

with the exception of

$$1 + \lambda_1^L - \lambda_2^U > 0.$$

The first-order conditions above with respect to the x variables are satisfied by the x_k^* given in the statement of the proposition along with our choice of λ values. They also satisfy non-negativity by (5), and the monotonicity constraints. With the choice of the y_k as in the proposition we have positive (non-negative since $y_1 = 0$) values for all the y variables due to condition (6), and they are in complementarity with the above inequalities. Furthermore, for our choice of x_k^*'s, one simply finds optimal information rents by solving the linear optimization problem [notice that x_k^* satisfy the conditions $a_k > 0, k = 1, \ldots, K-1$ if (6) hold]:

[1] Recall that with linear constraints the Slater condition is not needed, c.f. [9].

$$\min_{y_k} \sum_{k \in \mathcal{K}} \omega_k y_k$$

subject to

$$a_k \le y_{k+1} - y_k \le b_k, \quad \forall k = 1, \ldots, K-1,$$
$$y_k \ge 0, \quad \forall k = 1, \ldots, K,$$

where $a_k = \phi_R^{k+1*} - \phi_R^{k*} - \frac{1}{2}\Delta h x_k^*$ and $b_k = \phi_R^{k+1*} - \phi_R^{k*} - \frac{1}{2}\Delta h x_{k+1}^*$, for all $k = 1, \ldots, K-1$. Since, under the conditions of the proposition we have $a_k < b_k$, and $a_k \ge 0$ for all $k = 1, \ldots, K-1$ then we can invoke Lemma 1 and assert that $y_k^* = \sum_{i=1}^{k-1} a_i$ with $y_1^* = 0$.

Therefore, we have constructed an optimal solution for the problem ignoring non-negativity and monotonicity constraints on the x variables. However, our optimal solution for a relaxation of the original problem is also feasible, and hence optimal for the original problem. □

Under the conditions of the proposition, we obtain that the optimal information rents are increasing in retailer type. However, this is not always the case when we deviate from the conditions of the proposition. Consider an example from [8] with 3 types, $F = 1$, $H = 5$ $f = 6$, and $h_1 = 1$, $h_2 = 2$, $h_3 = 6$, $d = p = 1$ and unit weights. In this case, optimal information rents y_k^* for types $k = 1, 2, 3$ are found to be equal to 0, 0.435 and 0.021, respectively. The difficulty in solving such cases in closed-form stems from the difficulty of solving the information rent problem in parametric form optimally when there exists k such that $a_k = b_k$, and there are negative valued a_ks at an optimal solution. In such cases, Lemma 1 is no longer valid. This situation occurs when two consecutive types are offered the same contract at optimality. This is precisely the case at $k = 1$ in this example, since both types 1 and 2 are offered the same contract at the optimum.

On the other hand, one can easily find examples where the conditions of Proposition 1 are fulfilled. For example consider an interval of uncertainty for the holding cost parameter given as [9/8, 38/8]. Creating $K = 30$ equally likely types with $\Delta h = 1/8$, $d = p = 1$, $f = 9$, $F = 1$, $H = 4$, all conditions hold, and one readily obtains the solution applying the formulae given.

A further explicit result is given below in Theorem 1 where the optimal order quantities are now modifications of the coordinated joint optimal order quantity for an adjusted holding cost value. We also need the additional conditions (9)–(11) which essentially depend on the existence of a *cut-off* or *separator* type k^* which is the smallest value of the type index k such that conditions (5)–(6) are assumed to hold for all types with index larger than or equal to k^* whereas for smaller indices other conditions should apply. When k^* does not exist we can construct an optimal solution under different conditions.

Theorem 1 *Assume uniform types ($\omega_k = \omega$ for all $k \in \mathcal{K}$), and equidistant holding costs ($h_{k+1} - h_k = \Delta h$ for all $k = 1, \ldots, K-1$). Then we have the following:*

1. *If there exists a "separator index value" $k^* \in \mathcal{K}\backslash\{K\}$ such that conditions (5)–(6) hold for $k = k^*, k^*+1, \ldots, K-1$, i.e., k^* is the smallest index such that*

$$x(H, h_{k^*} - (K-k^*)\Delta h) < \frac{2C_{k^*+1,k^*}}{\Delta h}$$

with

$$\frac{2C^2_{k^*,k^*-1}}{\Delta h}\left[h_{k^*-1} + H\frac{d}{p} - (K-k^*+1)\Delta h\right] \leq d(f+F)\Delta h, \tag{9}$$

$$d(f+F)\Delta h \leq 4\frac{C^2_{k,k-1}C^2_{k+1,k}}{C^2_{k,k-1} - C^2_{k+1,k}}, \quad \forall k = 2, \ldots, k^*-1, \tag{10}$$

and,

$$x(H, h_k) < \frac{2C_{k+1,k}}{\Delta h}, \quad \forall k = 1, \ldots, k^*-1, \tag{11}$$

then

$$\begin{aligned} x_k^* &= 2C_{k+1,k}/\Delta h, \quad \forall k = 1, \ldots, k^*-1 \\ x_k^* &= x(H, h_k - (K-k)\Delta h), \quad \forall k = k^*, \ldots, K \end{aligned}$$

with

$$y_k^* = \sum_{i=k^*}^{k-1}\left(C_{i+1,i} - \frac{1}{2}x(H, h_i - (K-i)\Delta h))\Delta h\right), \quad \forall k = k^*+1, \ldots, K,$$

and the remaining information rent variables equal to zero, solve (2ECH).

2. *Under the conditions a, b, and c below:*
   a. *there does not exist $k^* \in \mathcal{K}\backslash\{K\}$ such that conditions (5)–(6) hold*
   b. *there exists $\underline{k}, \bar{k} \in \mathcal{K}$ such that $\exists k : \underline{k} < k < \bar{k}$ and, we have*

$$x(H, h_k) < \frac{2C_{k+1,k}}{\Delta h}, \quad \forall k = 1, \ldots, \underline{k}$$

and

$$\frac{2C_{k+1,k}}{\Delta h} \leq x(H, h_k) \leq \frac{2C_{k,k-1}}{\Delta h}, \quad \forall k = \underline{k}+1, \ldots, \bar{k}-1,$$

$$x(H, h_k) > \frac{2C_{k,k-1}}{\Delta h}, \quad \forall k = \bar{k}, \ldots, K$$

with inequalities

$$d(f+F)\Delta h \geq \frac{2C^2_{\underline{k}+1,\underline{k}}(h_{\underline{k}} - \Delta h + Hd/p)}{\Delta h} \tag{12}$$

$$d(f+F)\Delta h \leq \frac{2C^2_{\bar{k},\bar{k}-1}(h_{\bar{k}} + \Delta h + Hd/p)}{\Delta h} \tag{13}$$

valid,

c.

$$d(f+F)\Delta h \leq 4\frac{C^2_{k,k-1}C^2_{k+1,k}}{C^2_{k,k-1} - C^2_{k+1,k}}, \quad \forall k = 2, \ldots, \underline{k} \text{ and } k = \bar{k}, \ldots, K-1, \tag{14}$$

the following choice of

$$x_k^* = 2C_{k+1,k}/\Delta h \quad \forall k = 1, \ldots, \underline{k}$$
$$x_k^* = x(H, h_k) \quad \forall k = \underline{k}+1, \ldots, \bar{k}-1$$
$$x_k^* = 2C_{k,k-1}/\Delta h \quad \forall k = \bar{k}, \ldots, K$$

with all information rent variables equal to zero solves (2ECH).

Proof We again ignore for the time being the non-negativity and monotonicity conditions. The proof proceeds as the proof of Proposition 1 by constructing a primal-dual pair of solutions satisfying KKT conditions which are sufficient as the problem is convex with linear constraints.

For part 1, let k^* be the index value fulfilling the stated conditions.

The first-order conditions with respect to variables x_k, $k = 1, \ldots, k^* - 1$ are expressed as follows using non-negative Lagrange multipliers $\lambda_1^L, \lambda_2^U, \lambda_2^L, \ldots, \lambda_{k^*-1}^U$, $\lambda_{k^*-1}^L$:

$$-\frac{d(f+F)}{x_1^2} + \frac{1}{2}\left(h_1 + H\frac{d}{p}\right) - \frac{1}{2}\Delta h \lambda_1^L = 0$$

$$-\frac{d(f+F)}{x_2^2} + \frac{1}{2}\left(h_2 + H\frac{d}{p}\right) + \frac{1}{2}\Delta h \lambda_2^U - \frac{1}{2}\Delta h \lambda_2^L = 0,$$

and so on until

$$-\frac{d(f+F)}{x_{k^*-1}^2} + \frac{1}{2}\left(h_{k^*-1} + H\frac{d}{p}\right) + \frac{1}{2}\Delta h \lambda_{k^*-1}^U - \frac{1}{2}\Delta h \lambda_{k^*-1}^L = 0.$$

We set the information variables y_k to zero for $k = 1, \ldots, k^* - 1$, and we choose

$$\lambda_k^L = \frac{2C^2_{k+1,k}(h_k + Hd/p) - d(f+F)\Delta h^2}{2\Delta h C^2_{k+1,k}} > 0, \text{ and } \lambda_k^U = 0, \quad k = 1, \ldots, k^* - 1,$$

(positivity of λ_k^L is a consequence of (11)) and $x_k = 2C_{k+1,k}/\Delta h$, $k = 1, \ldots, k^* - 1$. Now, by simple algebra it is verified that these choices for y, x and λ satisfy the KKT conditions using the conditions (10)–(11). In particular, the fact that the dual constraints

$$1 - \lambda_{k-1}^L + \lambda_k^U + \lambda_k^L - \lambda_{k+1}^U \geq 0, k = 1, \ldots, k^* - 1,$$

(with $\lambda_{k^*}^U = 0$, see below) hold with our choice of λ multipliers is a consequence of condition (10). Furthermore, these are in complementarity with our choice of y variables.

Now, we pass to the part $k = k^*, k^* + 1, \ldots, K$. Here we shall mimic the proof of Proposition 1. Namely, our construction will satisfy the KKT conditions for $k = k^*, k^* + 1, \ldots, K$ as in Proposition 1. More precisely, we consider the first-order conditions with respect to the x variables:

$$-\frac{d(f+F)}{x_k^2} + \frac{1}{2}\left(h_k + H\frac{d}{p}\right) + \frac{1}{2}\Delta h \lambda_k^U - \frac{1}{2}\Delta h \lambda_k^L = 0 \quad \forall k = k^*, \ldots, K-1,$$
$$-\frac{d(f+F)}{x_K^2} + \frac{1}{2}\left(h_K + H\frac{d}{p}\right) + \frac{1}{2}\Delta h \lambda_K^U = 0,$$

and we have the dual feasibility and complementarity conditions:

$$1 - \lambda_{k-1}^L + \lambda_k^U + \lambda_k^L - \lambda_{k+1}^U \geq 0 \perp y_k \geq 0, \quad k = k^*, \ldots, K-1,$$
$$1 + \lambda_K^U - \lambda_{K-1}^L \geq 0 \perp y_K \geq 0.$$

Then we set $y_{k^*} = 0$ and $\lambda_k^L = K - k, \lambda_k^U = 0$ for $k = k^*, k^* + 1, \ldots, K$. Using inequalities (6)–(10), it takes a simple algebraic verification to check that with the x and y variables set as in the statement of the theorem along with the aforementioned choices of dual variables, we have primal feasibility and the KKT conditions satisfied. Notice that for $k = k^* - 1$ condition (6) does not hold; however, our condition (9) ensures that $1 - \lambda_{k^*-1}^L + \lambda_{k^*}^L \geq 0$. Since our optimal point for a relaxed problem satisfies the neglected non-negativity and monotonocity on the x variables, the proof of Part 1 is complete.

For Part 2, there is no index value k^* satisfying (5)–(6) for $k = k^*, k^*+1, \ldots, K-1$. Therefore, we are unable to use (7) for x variables since violation of (5) makes the expression under the square root negative. Furthermore, it is not possible to reuse (8) to set the y variables since this may result in negative values as (6) is violated. However, we shall construct a different solution. In this case, we set all information rent variables y_k to zero, by hypothesis we have "separated" (i.e., $\exists k : \underline{k} < k < \bar{k}$) special types, $\underline{k}$ and $\bar{k}$ such that 1. for $k = 1, \ldots, \underline{k}$ $x(H, h_k) < \frac{2C_{k+1,k}}{\Delta h}$, and 2. for $k = \underline{k}+1, \ldots, \bar{k}-1$, $\frac{2C_{k+1,k}}{\Delta h} \leq x(H, h_k) \leq \frac{2C_{k,k-1}}{\Delta h}$, 3. for $k = \bar{k}, \ldots, K$ $x(H, h_k) > \frac{2C_{k,k-1}}{\Delta h}$.

We note that our construction for x in this part of the theorem (as in the first part) is monotone decreasing since $C_{k+1,k}$ is monotone decreasing. Now, for all indices up to $\underline{k}$ we set the x variables as

$$x_k^* = 2C_{k+1,k}/\Delta h,$$

and, the $\lambda_k^U = 0$, and $\lambda_k^L = \frac{2C_{k+1,k}^2(h_k+Hd/p)-d(f+F)\Delta h^2}{2\Delta h C_{k+1,k}^2} > 0$ for $k = 1, \ldots, \underline{k}$. Due to condition (9) these choices satisfy the KKT conditions (since we set below $\lambda_{\underline{k}+1}^U = 0$). For $k = \underline{k}+1, \ldots, \bar{k}-1$ we set

$$x_k^* = x(H, h_k),$$

and the $\lambda_k^U = 0$, and $\lambda_k^L = 0$. Again, it is easy to check that under the conditions of the theorem, the KKT conditions hold for the aforementioned indices. The only index value where one has to be careful is for $\underline{k}+1$ to make sure that $1-\lambda_{\underline{k}}^L \geq 0$. This holds provided that condition (12) is verified.

For $k = \bar{k}, \ldots, K$ we set

$$x_k^* = 2C_{k,k-1}/\Delta h,$$

$\lambda_k^L = 0$ and $\lambda_k^U = \frac{-2C_{k,k-1}^2(h_k+Hd/p)+d(f+F)\Delta h^2}{2\Delta h C_{k,k-1}^2} > 0$. The KKT conditions are verified to hold for all indices $k = 1, \ldots, \bar{k}-1$ using straightforward algebraic manipulation. Concerning the (border) dual inequality in KKT conditions for $k = \bar{k}-1$ we have to make sure that $\lambda_{\bar{k}}^U \leq 1$, which is ensured due to (13).

If one of the index values $\underline{k}$ or $\bar{k}$ (or both) cannot be found then one extends the arguments above to a larger number of indices, depending on whichever case applies. □

In the terminology of [8], Proposition 1 corresponds to the right-tree pattern. For reasons of symmetry, one would expect a similar result for the left-tree pattern. However, the reader is warned that conditions might have to change in that case. A similar remark holds for Theorem 1. Further research is needed regarding this point.

Note that all one has to do to apply Proposition 1 and Theorem 1 is to compute the quantities $x(H, h_k - (K-k)\Delta h)$, $x(H, h_k)$ and $C_{k,k+1}$ for all k and then check conditions ensuring the validity of the Proposition 1 and Theorem 1. This can be accomplished by a very simple calculation. Hence, one does not need a specialized algorithm or a general-purpose algorithm to solve the special cases studied in the paper. A simple loop will suffice.

As an illustration, for an interval of uncertainty for the holding cost given as $[4/3, 44/3]$ and creating $K = 40$ equally likely types with $\Delta h = 1/3$, $d = p = 1$, $f = 9$, $F = 1$, $H = 4$, we have $k^* = 33$, and the optimal information rent variables values are zero from y_1 to y_{33} while $y_{34} = 0.001$, $y_{35} = 0.004$, $y_{36} = 0.009$, and so on. The optimal menu quantities x_k are found using part 1 of Theorem 1. When we make $\Delta h = 3$ with $h_1 = 4$ in this example, all information rent variables take value zero at optimality, and optimal menu quantities x_k^* are found by Part 2 of Theorem 1. With respect to the indices $\underline{k}, \bar{k}$, in this numerical example $\underline{k} = 6$ and $\bar{k} = 18$. I.e., optimal values of x variables for $k = 1, \ldots, 6$ are obtained by setting to the value

$2C_{k+1,k}/\Delta h$; for those $k = 7, \ldots, 17$ the optimal joint order quantity gives the solution, and for those $k = 18, \ldots, 40$ the value $2C_{k,k-1}/\Delta h$ is used. Indeed, the quantity $d(f + F)\Delta h$ is equal to 30 in this instance and with respect to inequalities (12) and (13), it is sandwiched between 26.377 and 31.296. Note that if one chooses a larger Δh, equal to, say 56, we have $\underline{k} = 0$ while $\bar{k} = 7$. On the other hand, with $h_1 = 4$, $\Delta h = 3$, other parameters being equal, no $\bar{k}$ is observed while $\underline{k} = 6$.

4 Concluding remarks

In this paper, we studied a two-echelon supply chain coordination game between a retailer and a supplier who does not know the true holding cost parameter of the retailer and offers a menu of contracts to reshape the behaviour of the retailer in order to minimize his costs. While the problem is in general difficult to solve in closed form (however, it is easy to solve numerically since it is transformed into a convex optimization problem in [8]), we computed an explicit optimal solution, namely when the supplier addresses the uncertainty about the retailer's holding cost using a uniform discrete approximation, and some strict lower bounds and some inequalities (defined in terms of problem parameters and easy to check) hold.

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