

LOCALLY FINITE DERIVATIONS AND MODULAR COINVARIANTS

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Abstract

We consider a finite-dimensional $\mathbb{k}G$ -module V of a p -group G over a field \mathbb{k} of characteristic p . We describe a generating set for the corresponding Hilbert Ideal. In case G is cyclic, this yields that the algebra $\mathbb{k}[V]_G$ of coinvariants is a free module over its subalgebra generated by $\mathbb{k}G$ -module generators of V^* . This subalgebra is a quotient of a polynomial ring by pure powers of its variables. The coinvariant ring was known to have this property only when G was cyclic of prime order [M. Sezer, Decomposing modular coinvariants, *J. Algebra* **423** (2015), 87–92]. In addition, we show that if G is the Klein 4-group and V does not contain an indecomposable summand isomorphic to the regular module, then the Hilbert Ideal is a complete intersection, extending a result of the second author and Shank [M. Sezer and R. J. Shank, Rings of invariants for modular representations of the Klein four group, *Trans. Amer. Math. Soc.* **368** (2016), 5655–5673].

1. Introduction

Let \mathbb{k} be a field of positive characteristic p and V a finite-dimensional \mathbb{k} -vector space, and $G \leq \mathrm{GL}(V)$ a finite group. Then the induced action on V^* extends to the symmetric algebra $\mathbb{k}[V] := S(V^*)$ by the formula $\sigma(f) = f \circ \sigma^{-1}$ for $\sigma \in G$ and $f \in \mathbb{k}[V]$. The ring of fixed points $\mathbb{k}[V]^G$ is called the *ring of invariants*, and is the central object of study in invariant theory. Another object which is often studied is the *Hilbert Ideal*, \mathcal{H} , which is defined to be the ideal of $\mathbb{k}[V]$ generated by invariants of positive degree, in other words

$$\mathcal{H} = \mathbb{k}[V]_+^G \mathbb{k}[V].$$

In this article, we study the quotient $\mathbb{k}[V]_G := \mathbb{k}[V]/\mathcal{H}$ which is called the algebra of *coinvariants*. An equivalent definition is $\mathbb{k}[V]_G := \mathbb{k}[V] \otimes_{\mathbb{k}[V]^G} \mathbb{k}$, which shows that this object is, in a sense, dual to $\mathbb{k}[V]^G$.

As $\mathbb{k}[V]_G$ is a finite-dimensional $\mathbb{k}G$ -module, it is generally easier to handle than the ring of invariants. On the other hand, much information about $\mathbb{k}[V]^G$ is encoded in $\mathbb{k}[V]_G$. For example, Steinberg [13] famously showed that $\dim(\mathbb{C}[V]_G) = |G|$ if and only if (G, V) is a complex reflection group. Combined with the theorem of Chevalley [2], Shephard and Todd [11], this shows that

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$\dim(\mathbb{C}[V]_G) = |G|$ if and only if $\mathbb{C}[V]^G$ is a polynomial ring. Smith [12] later generalized this by showing that $\dim(\mathbb{k}[V]_G) = |G|$ if and only if G is a (pseudo)-reflection group, where \mathbb{k} is any field. Further, the polynomial property of $\mathbb{C}[V]^G$ is equivalent to the Poincaré duality property of $\mathbb{C}[V]_G$, by Kane [6] and Steinberg [13].

Before we continue, we fix some terminology. Let x_0, \dots, x_n be a basis for V^* . We will say x_i is a *terminal variable* if the vector space spanned by the other variables is a $\mathbb{k}G$ -submodule of V^* . Note that if G is a p -group, then $V^G \neq 0$ and there is a choice of a basis for V that contains a fixed point. Then the dual element corresponding to the fixed point is a terminal variable in the basis consisting of dual elements of this basis. For any $f \in \mathbb{k}[V]$, we define the norm

$$N^G(f) = \prod_{h \in G \cdot f} h.$$

For every terminal variable x_i , we choose a polynomial $N(x_i)$ in $\mathbb{k}[V]^G$ which, when viewed as a polynomial in x_i is monic of minimal positive degree. While $N(x_i)$ is not unique in general, its degree is well defined. Since $N^G(x_i)$ is monic of degree $[G : G_{x_i}]$, the degree of $N(x_i)$ is bounded above by this number. By ‘degree of x_i ’, we understand degree of $N(x_i)$ as a polynomial in x_i and denote it by $\deg(x_i)$. We will show that the degree of a terminal variable is always a p -power.

The algebras of modular coinvariants for cyclic groups of order p were studied by the second author [8], and previously by the second author and Shank [9]. Note that there is a choice of basis such that an indecomposable representation of a p -group is afforded by an upper triangular matrix with 1s on the diagonal and the bottom variable is a terminal variable. In [8], the following was proven.

PROPOSITION 1.1. *Let G be a cyclic group of order p and V a $\mathbb{k}G$ -module that contains $k + 1$ non-trivial summands. Choose a basis x_0, x_1, \dots, x_n in which the variables x_0, x_1, \dots, x_k are the bottom variables of the respective Jordan blocks, and let A be the $\mathbb{k}G$ -subalgebra of $\mathbb{k}[V]$ generated by x_{k+1}, \dots, x_n . Denote the image of x_i in $\mathbb{k}[V]_G$ by X_i . Then*

- (1) *The Hilbert Ideal of $\mathbb{k}[V]^G$ is generated by $N^G(x_0), N^G(x_1), \dots, N^G(x_k)$, and polynomials in A .*
- (2) *$\mathbb{k}[V]_G$ has dimension divisible by p^{k+1} .*
- (3) *$\mathbb{k}[V]_G$ is free as a module over its subalgebra \mathcal{T} generated by X_0, X_1, \dots, X_k .*
- (4) *$\mathcal{T} \cong \mathbb{k}[t_0, \dots, t_k] / (t_0^p, \dots, t_k^p)$, where t_0, \dots, t_k are independent variables.*

The goal of this article is to generalize the above, as far as possible, to the case of all finite p -groups. In particular, we show in section two:

THEOREM 1.2. *Let G be a finite p -group and V a $\mathbb{k}G$ -module that contains $k + 1$ non-trivial summands. Choose a basis x_0, x_1, \dots, x_n in which the variables x_0, x_1, \dots, x_k coming from each summand are terminal variables. Let d_i denote $\deg(x_i)$ for $0 \leq i \leq k$. Retain the notation in the proposition above, then*

- (1) *There is a choice for polynomials $N(x_0), N(x_1), \dots, N(x_k)$ such that the Hilbert Ideal of $\mathbb{k}[V]^G$ is generated by $N(x_0), N(x_1), \dots, N(x_k)$, and polynomials in A .*
- (2) *$\mathbb{k}[V]_G$ has dimension divisible by $\prod_{i=0}^k d_i$.*

Suppose in addition that one has $d_i = \deg(N^G(x_i))$ for $0 \leq i \leq k$. Then we have:

- (3) *$\mathbb{k}[V]_G$ is free as a module over its subalgebra \mathcal{T} generated by X_0, X_1, \dots, X_k .*
- (4) *$\mathcal{T} \cong \mathbb{k}[t_0, \dots, t_k] / (t_0^{d_0}, \dots, t_k^{d_k})$, where t_0, \dots, t_k are independent variables.*

In Section 3, we describe the situation for a p -group, where the complete intersection property of the Hilbert Ideal corresponding to a module is inherited from the Hilbert Ideal of the indecomposable summands of the module. The final section is devoted to applications of our main results to cyclic p -groups and the Klein 4-group. It turns out that for a cyclic p -group, the bottom variables x_i of Jordan blocks satisfy $\deg(x_i) = \deg(N^G(x_i))$. Consequently, (3) and (4) above hold for a cyclic p -group. Additionally, for the Klein 4-group, we show that the Hilbert Ideal corresponding to a module is a complete intersection as long as the module does not contain the regular module as a summand. This generalizes a result of the second author and Shank [10], where the complete intersection property was established for indecomposable modules only.

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2. Main results

Throughout this section, we let G be a finite p -group, \mathbb{k} a field of characteristic p and V a $\mathbb{k}G$ -module, which may be decomposable. As trivial summands do not contribute to the coinvariants, we assume no direct summand of V is trivial. Let $x_0, x_1, x_2, \dots, x_n$ be a basis of V^* and assume that x_0 is a terminal variable. Then x_1, x_2, \dots, x_n generate a G -subalgebra which we denote by A . We can define a nonlinear action of $(\mathbb{k}, +)$ on $\mathbb{k}[V]$ as follows:

$$t \cdot x_0 = x_0 + t; \quad (2.1)$$

$$t \cdot x_i = x_i \quad \text{for any } i > 0. \quad (2.2)$$

The terminality of x_0 ensures this commutes with the action of G . It is well known that any action of the additive group of an infinite field of prime characteristic is determined by a *locally finite iterative higher derivation*. This is a family of \mathbb{k} -linear maps $\Delta^i : \mathbb{k}[V] \rightarrow \mathbb{k}[V]$, $i \geq 0$ satisfying the following properties:

- (1) $\Delta^0 = \text{id}_{\mathbb{k}[V]}$.
- (2) For all $i > 0$ and $a, b \in \mathbb{k}[V]$, one has $\Delta^i(ab) = \sum_{j+k=i} \Delta^j(a) \Delta^k(b)$.
- (3) For all $b \in \mathbb{k}[V]$, there exists $i \geq 0$ such that $\Delta^i(b) = 0$.
- (4) For all i, j , one has $\Delta^j \circ \Delta^i = \binom{i+j}{j} \Delta^{i+j}$.

The equivalence of the group action and the l.f.i.h.d. is given by the formula

$$t \cdot b = \sum_{i \geq 0} t^i \Delta^i(b). \quad (2.3)$$

See [3, 14], for more details on l.f.i.h.d.'s.

Let $f \in \mathbb{k}[V]^G$ be homogeneous of degree d in x_0 . We write

$$f = f_d x_0^d + f_{d-1} x_0^{d-1} + \dots + f_0,$$

where $f_i \in A$. We have

$$t \cdot f = f_d (x_0 + t)^d + f_{d-1} (x_0 + t)^{d-1} + \cdots + f_0 = \sum_{i \geq 0} t^i \Delta^i(f). \quad (2.4)$$

That is to say that $\Delta^i(f)$ is the coefficient of t^i in the above expression. As the action of G commutes with the action of \mathbb{k} , we see that $\Delta^i(f) \in \mathbb{k}[V]^G$ for all $i \geq 0$.

REMARK 2.1.

- (1) Clearly $\Delta^1 = \frac{\partial}{\partial x_0}$. So the previous paragraph generalizes [8, Lemma 1].
- (2) Equation (2.4) gives that $\Delta^j(x_0^i) = \binom{i}{j} x_0^{i-j}$ provided $i \geq j$. Then, from Lucas's theorem [5] on binomial coefficients in characteristic p , we see that we can think of Δ^{p^j} as 'Differentiation by $x_0^{p^j}$ ': if the coefficient of p^j in the base p expansion of m is a , then we have

$$\Delta^{p^j}(x_0^m) = \begin{cases} ax_0^{m-p^j} & a > 0; \\ 0 & a = 0. \end{cases}$$

For later use, we also note the following consequence: for a homogeneous $f \in \mathbb{k}[V]$, $\Delta^j(f)$ contains a non-zero constant if and only if the monomial x_0^j appears in f .

- (3) In [4], a G -equivariant map is constructed from polynomials whose x_0 -degree is at most ep^r ($0 < e < p$) to polynomials whose x_0 -degree is at most p^r . This map turns out to be a non-zero scalar multiple of $\Delta^{(e-1)p^r}$.

We have the following statement generalizing [8, Lemma 2]:

LEMMA 2.2. *Let $f \in \mathbb{k}[V]$ be a homogeneous polynomial of degree d in x_0 . Write $f = f_d x_0^d + f_{d-1} x_0^{d-1} + \cdots + f_0$, where $f_i \in A$. Then we have*

$$\sum_{i=0}^d (-1)^i x_0^i \Delta^i(f) = f_0.$$

Proof. Write $f = f(x_0, x_1, x_2, \dots, x_n)$. For any $t \in \mathbb{k}$, we have

$$t \cdot f = f(t \cdot x_0, t \cdot x_1, \dots, t \cdot x_n) = f(x_0 + t, x_1, x_2, \dots, x_n).$$

As this holds for all t it also holds when t is replaced by $(-x_0)$, and hence by Equation (2.3), we have $\sum_{i=0}^d (-1)^i x_0^i \Delta^i(f) = (-x_0) \cdot f = f(0, x_1, x_2, \dots, x_n) = f_0$ as required. \square

We also note that the degree of a terminal variable is a p -power.

LEMMA 2.3. *For any terminal variable $x_0 \in V^*$, $\deg(x_0)$ is a power of p .*

Proof. Let d denote the degree of x_0 and suppose $f \in \mathbb{k}[V]^G$ is monic as a polynomial in x_0 of degree $d = d_r p^r + d_{r-1} p^{r-1} + \cdots + d_0$ with $0 \leq d_i < p$ and $d_r \neq 0$. If $d_j \neq 0$ for some $j < r$, then $\Delta^{p^j}(f) \in \mathbb{k}[V]^G$ has degree $d - p^j > 0$ as a polynomial in x_0 and its leading coefficient is

in \mathbb{k} . Similarly, if $d_j = 0$ for $j < r$ and $d_r > 1$, then $\Delta^{p^r}(f) \in \mathbb{k}[V]^G$ has degree $d - p^r > 0$ in x_0 and its leading coefficient is in \mathbb{k} . Both cases violate the minimality of d . \square

LEMMA 2.4. *Let d denote the degree of x_0 . Then $\Delta^j(\mathcal{H}) \subseteq \mathcal{H}$ for $j < d$.*

Proof. Let $f \in \mathbb{k}[V]$. From the second assertion of Remark 2.1, we get that $\Delta^j(f)$ contains a non-zero constant if and only if the monomial x_0^j appears in f . Therefore, by the minimality of d , we have $\Delta^j(\mathbb{k}[V]_+^G) \subseteq \mathcal{H}$ for $j < d$. Now the result follows from property (2) of l.f.i.h.d.'s. \square

From this point on, we adopt the notation of the introduction. This means that $x_0, x_1, x_2, \dots, x_k$ are terminal variables coming from different summands, and $A = \mathbb{k}[x_{k+1}, x_{k+2}, \dots, x_n]$. For each $i = 0, \dots, k$ let $d_i = p^{r_i}$ be the degree of x_i . Since setting variables outside of a summand to zero sends invariants to invariants of the summand, we may also assume that $N(x_i)$ depends only on variables that come from the summand that contains x_i . We denote by Δ_i the l.f.i.h.d. associated to x_i . We use reverse lexicographic order with $x_i > x_j$ whenever $0 \leq i \leq k$ and $k+1 \leq j \leq n$.

THEOREM 2.5. *\mathcal{H} is generated by $N(x_0), \dots, N(x_k)$ and polynomials in A . Moreover, the lead term ideal of \mathcal{H} is generated by $x_0^{p^{r_0}}, x_1^{p^{r_1}}, \dots, x_k^{p^{r_k}}$ and monomials in A .*

Proof. Let $f \in \mathbb{k}[V]^G$. Since $N(x_0)$ is monic in x_0 , we may perform polynomial division and write $f = qN(x_0) + r$ where r has x_0 -degree $< p^{r_0}$, and it is easily shown that $q, r \in \mathbb{k}[V]^G$. Then dividing r by $N(x_1)$ yields another invariant remainder r' that has x_1 -degree $< p^{r_1}$. Since x_0 -degree of $N(x_1)$ is zero, it follows that x_0 -degree of r' is still $< p^{r_0}$. Thus, by repeating the process with each terminal variable, and replacing f with the final remainder we assume that x_i -degree of f is $< p^{r_i}$ for $0 \leq i \leq k$.

Let i be minimal such that f has non-zero degree $d < p^{r_i}$ in the terminal variable x_i . We apply Lemma 2.2 with $\Delta = \Delta_i$ to see that

$$f = f_0 - \left(\sum_{j=1}^d (-1)^j x_i^j \Delta_i^j(f) \right),$$

where f_0 is the ‘constant term’ of f , that is, $f_0 \in \mathbb{k}[x_{i+1}, \dots, x_n]$. So from the previous lemma, we get that $f_0 \in \mathcal{H}$ since $d < p^{r_i}$. Moreover, since Δ_i decreases x_i -degrees and does not increase degrees in any other variable, the x_i -degree of each $\Delta_i^j(f)$ in the expression above is strictly less than d , and the x_l -degree for every $i < l \leq k$ remains strictly less than p^{r_l} . Thus, by induction on degree, f can be expressed as a $\mathbb{k}[V]$ -combination of elements of \mathcal{H} whose degrees in the terminal variables x_0, \dots, x_i are all zero, and degrees in the remaining terminal variables x_l for $i < l \leq k$ are strictly less than p^{r_l} , respectively. Repeating the same argument with the remaining terminal variables gives us that f can be written as a $\mathbb{k}[V]$ -combination of elements of $\mathcal{H} \cap A$ together with $N(x_1), \dots, N(x_k)$ as required. The first assertion of the theorem follows.

Note that the leading monomial of $N(x_i)$ is $x_i^{p^{r_i}}$ for $0 \leq i \leq k$. So it remains to show that all other monomials in the lead term ideal of \mathcal{H} lie in A . Recall that by Buchberger’s algorithm a Gröbner basis is obtained by reduction of S -polynomials of a generating set by polynomial division, see [1, Section 1.7]. By the first part, \mathcal{H} has a generating set consisting of $N(x_i)$ for $0 \leq i \leq k$ and polynomials in A . But the S -polynomial of two polynomials in A is also in A , and via polynomials in A , it also reduces to a polynomial in A . Finally, the S -polynomial of $N(x_i)$ and

a polynomial in A and the S -polynomial of a pair $N(x_i)$ and $N(x_j)$ with $0 \leq i \neq j \leq k$ reduce to zero since their leading monomials are pairwise relatively prime. \square

COROLLARY 2.6. *The vector space dimension of $\mathbb{k}[V]_G$ is divisible by $\prod_{0 \leq i \leq k} d_i = p^{\sum_{i=0}^k r_i}$.*

Proof. The set of monomials that are not in the lead term ideal of \mathcal{H} form a vector space basis for $\mathbb{k}[V]_G$. Let Λ denote this set of monomials. By the previous theorem, a monomial $M \in A$ lies in Λ if and only if $Mx_0^{a_0}, \dots, x_k^{a_k}$ lies in Λ for $0 \leq a_i < p^{r_i}$ and $0 \leq i \leq k$. It follows that the size of the set Λ is divisible by $p^{\sum_{i=0}^k r_i}$. \square

The following generalizes the content of [8, Theorem 5] partially for a p -group.

THEOREM 2.7. *Let x_i be a terminal variable of degree d , and write $N(x_i) = x_i^d + \sum_{j=0}^{d-1} f_j x_i^j$, where x_i -degree of f_j is zero for $0 \leq j \leq d-1$. Then $x_i^d + f_0 \in \mathcal{H}$.*

Proof. Consider $\bar{N} = N(x_i) - x_i^d$. This is a polynomial of degree $e < d$ in x_i . By Lemma 2.2,

$$\sum_{j=0}^e (-1)^j x_i^j \Delta_i^j(\bar{N}) = f_0,$$

since f_0 is the constant term of \bar{N} . Now recall that $\Delta_i^j(x_i^d)$ is the coefficient of t^j in $(x_i + t)^d = x_i^d + t^d$ (note that d is a p -power by Lemma 2.3). Thus, $\Delta_i^j(x_i^d) = 0$ for all $0 < j < d$. As Δ_i^j is a linear map for all j it follows that $\Delta_i^j(N(x_i)) = \Delta_i^j(\bar{N})$ for all $0 < j < d$. Therefore,

$$\sum_{j=1}^e (-1)^j x_i^j \Delta_i^j(N(x_i)) = f_0 - \bar{N}.$$

As $\Delta_i^j(N(x_i)) \in \mathcal{H}$ for all $j < d$ by Lemma 2.4, we get that $f_0 - \bar{N} \in \mathcal{H}$. Therefore $x_i^d + f_0 = N(x_i) - \bar{N} + f_0 \in \mathcal{H}$ as required. \square

LEMMA 2.8. *Suppose that for each $i = 0, \dots, k$ we have $x_i^{d_i} \in \mathcal{H}$. Then $\mathbb{k}[V]_G$ is free as a module over its subalgebra \mathcal{T} generated by X_0, X_1, \dots, X_k , and $\mathcal{T} \cong \mathbb{k}[t_0, \dots, t_k] / (t_0^{d_0}, \dots, t_k^{d_k})$, where t_0, \dots, t_k are independent variables.*

Proof. The hypothesis on the x_i is equivalent to $X_i^{d_i} = 0$ in $\mathbb{k}[V]_G$. Let t_0, \dots, t_k be independent variables and consider the natural surjective ring homomorphism from $\mathbb{k}[t_0, \dots, t_k]$ to $\mathbb{k}[X_0, \dots, X_k]$. Since $X_i^{d_i} = 0$, the kernel of this map contains $(t_0^{d_0}, \dots, t_k^{d_k})$. If this ideal is not all the kernel, then \mathcal{H} must contain a polynomial in x_0, \dots, x_k such that no monomial in this polynomial is divisible by $x_i^{d_i}$ for $0 \leq i \leq k$. This is a contradiction with the description of the lead term ideal in Theorem 2.5.

Secondly, let Λ denote the set of monomials in $\mathbb{k}[V]$ that are not in the lead term ideal of \mathcal{H} . Then the set of images of monomials in $\Lambda' = \Lambda \cup A$ generate $\mathbb{k}[V]_G$ over \mathcal{T} . Further, they generate freely because $Mx_0^{a_0}, \dots, x_k^{a_k} \in \Lambda$ for all $M \in \Lambda'$ and $0 \leq a_i < d_i$ and $0 \leq i \leq k$, and the images of monomials in Λ form a vector space basis for $\mathbb{k}[V]_G$. \square

Proof of Theorem 1.2. The first two assertions of the theorem are contained in Theorem 2.5 and its corollary. Next assume that $d_i = \deg(N^G(x_i))$ for $0 \leq i \leq k$. So we can take $N(x_i) = N^G(x_i)$. Then from Theorem 2.7, it follows that $x_i^{d_i} \in \mathcal{H}$ for $0 \leq i \leq k$ since the constant term of $N^G(x_i)$ (as a polynomial in x_i) is zero. Now the third and the fourth assertions follow from Lemma 2.8. \square

3. Complete intersection property of \mathcal{H}

In this section, we show that if the Hilbert Ideals of two modules are generated by fixed points and powers of terminal variables, then so is the Hilbert Ideal of the direct sum. As an incidental result, we prove that the degree of a terminal variable does not change after taking direct sums. We continue with the notation and the convention of the previous section. Let V_1 and V_2 be arbitrary $\mathbb{k}G$ -modules. We choose a basis $x_{1,1}, \dots, x_{n_1,1}, y_{1,1}, \dots, y_{m_1,1}$ for V_1^* and $x_{1,2}, \dots, x_{n_2,2}, y_{1,2}, \dots, y_{m_2,2}$ for V_2^* such that $x_{1,1}, \dots, x_{n_1,1}, x_{1,2}, \dots, x_{n_2,2}$ are fixed points. Note that both $\mathbb{k}[V_1]$ and $\mathbb{k}[V_2]$ are subrings of $\mathbb{k}[V_1 \oplus V_2]$, and we identify

$$\mathbb{k}[V_1 \oplus V_2] = \mathbb{k}[x_{1,1}, \dots, x_{n_1,1}, x_{1,2}, \dots, x_{n_2,2}, y_{1,1}, \dots, y_{m_1,1}, y_{1,2}, \dots, y_{m_2,2}].$$

Note that if $y_{i,j}$ is a terminal variable in V_j^* for some $1 \leq i \leq m_j, 1 \leq j \leq 2$, then it is also a terminal variable in $V_1^* \oplus V_2^*$.

LEMMA 3.1. *Assume the notation of the previous paragraph. Let $y_{i,j} \in V_j^*$ be a terminal variable. Then the degrees of $y_{i,j}$ in V_j^* and $V_1^* \oplus V_2^*$ are equal.*

Proof. Since $\mathbb{k}[V_j]^G \subseteq \mathbb{k}[V_1 \oplus V_2]^G$, we have that the degree of $y_{i,j}$ in V_j^* is bigger than its degree in $V_1^* \oplus V_2^*$. On the other hand, the restriction map $\mathbb{k}[V_1 \oplus V_2]^G \rightarrow \mathbb{k}[V_j]^G$ given $f \rightarrow f|_{V_j}$ preserves any power of the form $y_{i,j}^d$. This gives the reverse inequality. \square

We denote the Hilbert Ideals $\mathbb{k}[V_1 \oplus V_2]_+^G \mathbb{k}[V_1 \oplus V_2]$, $\mathbb{k}[V_1]_+^G \mathbb{k}[V_1]$ and $\mathbb{k}[V_2]_+^G \mathbb{k}[V_2]$ with \mathcal{H} , \mathcal{H}_1 and \mathcal{H}_2 , respectively.

THEOREM 3.2. *Assume that \mathcal{H}_1 and \mathcal{H}_2 are generated by the powers of the variables in V_1^* and V_2^* , respectively, and that the variables $y_{1,1}, \dots, y_{m_1,1}, y_{1,2}, \dots, y_{m_2,2}$ are terminal variables. Then \mathcal{H} is generated by the union of the generating sets for \mathcal{H}_1 and \mathcal{H}_2 .*

Proof. Assume that \mathcal{H}_1 is generated by $x_{1,1}, \dots, x_{n_1,1}, y_{1,1}^{d_{1,1}}, \dots, y_{m_1,1}^{d_{m_1,1}}$ and \mathcal{H}_2 is generated by $x_{1,2}, \dots, x_{n_2,2}, y_{1,2}^{d_{1,2}}, \dots, y_{m_2,2}^{d_{m_2,2}}$. We show that $d_{i,j}$ is equal to the degree of the variable $y_{i,j}$ for $1 \leq i \leq m_j$ and $1 \leq j \leq 2$. For simplicity, we set $i = j = 1$ and denote the degree of $y_{1,1}$ with d . Since \mathcal{H}_1 is generated by monomials, each monomial in a polynomial in \mathcal{H}_1 is divisible by one of its monomial generators. So we get $d_{1,1} \leq d$. On the other hand, since $y_{1,1}^{d_{1,1}}$ is a member of \mathcal{H}_1 there is a positive degree invariant with a monomial that divides $y_{1,1}^{d_{1,1}}$. So by the minimality of d , we get $d \leq d_{1,1}$ as well. By Lemma 3.1, $d_{i,j}$ is also equal to the degree of $y_{i,j}$ in $\mathbb{k}[V_1 \oplus V_2]^G$. We claim that the union of the generating sets for \mathcal{H}_1 and \mathcal{H}_2 generate \mathcal{H} . Otherwise, there exists a polynomial f in \mathcal{H} that contains a non-constant monomial $\prod_{1 \leq i \leq m_j, 1 \leq j \leq 2} y_{i,j}^{e_{i,j}}$ with $0 \leq e_{i,j} < d_{i,j}$. Let $\Delta_{i,j}$ denote the derivation with respect to the terminal variable $y_{i,j}$. Then applying $\Delta_{i,j}^{e_{i,j}}$ successively to f for $1 \leq i \leq m_j, 1 \leq j \leq 2$ yields an invariant with a non-zero constant. This is a contradiction by Lemma 2.4 since $e_{i,j} < d_{i,j}$. \square

We end this section with an example which shows that the degree of a terminal variable may be strictly less than the degree of its norm:

EXAMPLE 3.3. Let $H = \langle \sigma, \tau \rangle$ be the Klein 4-group, \mathbb{k} a field of characteristic 2 and $m \geq 2$. Let $\Omega^{-m}(\mathbb{k})$ be a vector space of dimension $m = 2n + 1$ over \mathbb{k} . Choose a basis $\{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_{m+1}\}$ of V^* . One can define an action of H on V in such a way that its action on V^* is given

by $\sigma(y_j) = y_j + x_j$, $\sigma(x_j) = x_j$, $\tau(y_j) = y_j + x_{j-1}$, $\tau(x_j) = x_j$ using the convention that $x_0 = x_{m+1} = 0$.

The variables y_1, y_2, \dots, y_{m+1} are terminal. One can readily check that

$$y_2^2 + x_2y_2 + x_1y_2 + x_2y_1 + x_1y_3 + y_1x_3$$

is invariant under H (note the last term is zero if $m = 2$), so y_2 has degree 2. On the other hand, y_2 is not fixed by either σ or τ , which means $N^H(y_2)$ has degree 4. It is interesting to note that $x_1y_2 + x_2y_1 + x_1y_3 + y_1x_3 \in \mathcal{H}$, so we still have $y_2^2 \in \mathcal{H}$.

4. Cyclic p -groups and the Klein 4-group

In this section, we apply the results of the previous sections to cyclic p -groups and the Klein 4-group. Let $G = Z_{p^r}$ denote a cyclic group of order p^r . Fix a generator σ of G . There are p^r indecomposable $\mathbb{k}G$ -modules V_1, \dots, V_{p^r} over \mathbb{k} , and each indecomposable module V_i is afforded by σ^{-1} acting via a Jordan block of dimension i with ones on the diagonal. For an arbitrary $\mathbb{k}G$ -module V , we write

$$V = \bigoplus_{i=0}^k V_{n_i} \quad (\text{with } 1 \leq n_i \leq p^r \text{ for all } i),$$

where each V_{n_i} is spanned as a vector space by $e_{1,i}, \dots, e_{n_i,i}$. Then the action of σ^{-1} is given by $\sigma^{-1}(e_{j,i}) = e_{j,i} + e_{j+1,i}$ for $1 \leq j < n_i$ and $\sigma^{-1}(e_{n_i,i}) = e_{n_i,i}$. Note that the fixed point space V^G is \mathbb{k} -linearly spanned by $e_{n_i,0}, \dots, e_{n_i,k}$. The dual $V_{n_i}^*$ is isomorphic to V_{n_i} . Let $x_{1,i}, \dots, x_{n_i,i}$ denote the corresponding dual basis, then we have

$$\mathbb{k}[V] = \mathbb{k}[x_{j,i} \mid 1 \leq j \leq n_i, 0 \leq i \leq k],$$

and the action of σ is given by $\sigma(x_{j,i}) = x_{j,i} + x_{j-1,i}$ for $1 < j \leq n_i$ and $\sigma(x_{1,i}) = x_{1,i}$ for $0 \leq i \leq k$. Notice that the variables $x_{n_i,i}$ for $0 \leq i \leq k$ are terminal variables. We follow the notation of Section 2 and denote $x_{n_i,i}$ with x_i . We show that Theorem 1.2 applies completely to G by computing $\deg(x_i)$ explicitly for $0 \leq i \leq k$. For each $0 \leq i \leq k$, let a_i denote the largest integer such that $n_i > p^{a_i-1}$.

LEMMA 4.1. *We have $\deg(x_i) = p^{a_i}$. In particular, we may take $N(x_i) = N^G(x_i)$.*

Proof. From [7, Lemma 3], we get that $\deg(x_i)$ is at least p^{a_i} . On the other hand, since $p^{a_i} \geq n_i > p^{a_i-1}$, a Jordan block of size n_i has order p^{a_i} . That is, this block affords a faithful module of the subgroup of G of size p^{a_i} . It follows that the orbit of x_i has p^{a_i} elements and so that the orbit product $N^G(x_i)$ is a monic polynomial that is of degree p^{a_i} in x_i . \square

Applying Theorem 1.2, we obtain the following.

PROPOSITION 4.2. Assume the notation of Theorem 1.2 with specialization $G = Z_{p^r}$. We have an isomorphism

$$\mathbb{k}[X_0, \dots, X_k] \cong \mathbb{k}[t_0, \dots, t_k] / (t_0^{p^{a_0}}, \dots, t_k^{p^{a_k}}).$$

Moreover, $\mathbb{k}[V]_G$ is free as a module over $\mathbb{k}[X_0, \dots, X_k]$. □

Now let H denote the Klein 4-group and $p = 2$. For each indecomposable $\mathbb{k}H$ -module V , there exists a basis of V^* with one of the terminal variables x_i satisfying $\deg(x_i) = [H : H_{x_i}]$, see [10]. In this source, it is also proven that with the exception of the regular module, each basis consists of fixed points and the terminal variables, and the Hilbert Ideal of every such module is generated by fixed points and the powers of the terminal variables. So we have by Theorems 1.2 and 3.2:

PROPOSITION 4.3. Let V be a $\mathbb{k}H$ -module containing $k + 1$ indecomposable summands. There is a basis $\{x_0, x_1, \dots, x_n\}$ of V^* in which x_0, x_1, \dots, x_k are terminal variables, each coming from one summand, such that $\mathbb{k}[V]_H$ is free as a module over its subalgebra \mathcal{T} generated by the images X_0, X_1, \dots, X_k of the terminal variables. Moreover, $\mathcal{T} \cong \mathbb{k}[t_0, \dots, t_k] / (t_0^{a_0}, \dots, t_k^{a_k})$, where t_0, \dots, t_k are independent variables, and for each i , we have $a_i = 2$ or 4 .

PROPOSITION 4.4. Let V be a $\mathbb{k}H$ -module such that V does not contain the regular module $\mathbb{k}H$ as a summand. Then there exists a basis of V^* such that $\mathbb{k}[V]_+^H \mathbb{k}[V]$ is generated by powers of basis elements. In particular, $\mathbb{k}[V]_+^H \mathbb{k}[V]$ is a complete intersection.

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