Discrete Optimization 30 (2018) 108–120

Contents lists available at ScienceDirect

Discrete Optimization

www.elsevier.com/locate/disopt

Convex hull results for the warehouse problem

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ARTICLE INFO

Article history:
Received 18 July 2017
Received in revised form 7 June 2018
Accepted 8 June 2018
Available online 22 June 2018

Keywords:
Warehouse problem
Convex hull
Extended formulation
Fourier–Motzkin elimination
Single node flow set
Flow cover inequalities

ABSTRACT

Given an initial stock and a capacitated warehouse, the warehouse problem aims to decide when to sell and purchase to maximize profit. This problem is common in revenue management and energy storage. We extend this problem by incorporating fixed costs and provide convex hull descriptions as well as tight compact extended formulations for several variants. For this purpose, we first derive unit flow formulations based on characterizations of extreme points and then project out the additional variables using Fourier–Motzkin elimination. It turns out that the nontrivial inequalities are flow cover inequalities for some single node flow set relaxations.

1. Introduction

The warehouse problem, introduced by Cahn [1], is to optimally decide on purchasing (or production), storage and sales quantities for a product with a fixed warehouse capacity and a given initial stock. This is a common problem in storage and revenue management in a commodity market. A commodity is a raw material or an agricultural product such as grains, vegetables, coal and natural gas.

The formal definition of the basic problem is as follows. Suppose that the initial stock is S units and the warehouse has a capacity of B units where $0 < S < B$. We are given a planning horizon of n periods. The buying price is c_t and the selling price is p_t in period t. In each period, we can sell at most as much as the inventory from the previous period, i.e., the amount purchased in a period cannot be sold in the same period. The aim of the warehouse problem is to decide on how much to purchase and sell in each period to maximize the total profit.

We define x_t to be the amount purchased and y_t to be the amount sold in period t. For two integers $n_1 \le n_2$, we let $[n_1, n_2] = \{n_1, \dots, n_2\}$. For a vector $a \in \mathbb{R}^n$, we use $a_{ut} = \sum_{i=u}^{t} a_i$ and $a_T = \sum_{i \in T} a_i$ for

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https://doi.org/10.1016/j.disopt.2018.06.002
1572-5286/

$T \subseteq [1, n]$. The basic warehouse problem can be modeled as

$$\max \sum_{t=1}^{n} (p_t y_t - c_t x_t) \tag{1}$$

$$\text{s.t. } s_0 = S, \tag{2}$$

$$s_{t-1} + x_t = y_t + s_t \qquad t \in [1, n], \tag{3}$$

$$0 \le y_t \le s_{t-1} \qquad t \in [1, n], \tag{4}$$

$$0 \le s_t \le B \qquad t \in [1, n], \tag{5}$$

$$x_t \ge 0 \qquad t \in [1, n].$$

Constraint (2) sets the value of the initial stock to zero. Constraints (3) are inventory balance equations. Constraints (4) and (5) ensure that we cannot sell more than what is available in stock from the previous period and that the stock does not exceed the warehouse capacity, respectively. The objective function is equal to the revenue minus purchasing cost.

Charles and Cooper [2] generalize this problem to the case of multiple products and varying prices. Bellman [3] presents a dynamic programming algorithm. Dreyfus [4] shows that the solution can be determined analytically. He shows that there are four policies: sell all the stock, buy up to capacity, sell and buy and do nothing. Consequently, an optimal policy is to do nothing for a number of stages and then alternate between a full and empty warehouse. Eastman [5] models the problem as a shortest path problem. Charles and Cooper [6] use the warehouse model to illustrate how linear programming can be used for allocation of funds in an enterprise. The multistage stochastic warehouse problem is studied by Charnes et al. [7].

Many more complex and mostly stochastic variants of the warehouse problem have been studied in the context of optimal commodity trading and energy storage (see, e.g., Devalkar et al. [8], Harsha and Dahleh [9], Secondi [10,11], Wu et al. [12], Zhou et al. [13]). However, to the best of our knowledge, there is no study on strong formulations of this problem in the presence of fixed costs. In this study, we extend the warehouse problem by including a fixed cost for buying and/or selling and inventory holding costs. We provide convex hull descriptions and tight compact extended formulations.

We study this version of the warehouse problem for several reasons. First it can be viewed as a simple machine on-off model in which there is now an initial intermediate start-up state. Secondly it can be seen as an uncapacitated lot-sizing problem that is not driven by the demands, but in which the costs of production and the bounds on stocks determine the quantities available for sale in each period. In addition one hopes that knowledge of the polyhedral structure of this and related sets can be useful in tackling more complicated versions of the problem.

The approach we take is perhaps of interest for other problems. Specifically it consists of

(1) describing the extreme points of the problem,

(2) using this to construct an automatically integral network flow (or other) formulation involving new auxiliary variables

(3) simplifying this integral formulation by eliminating variables by substitution, adding constraints linking the auxiliary variables to the original variables and showing that the resulting formulation is still integral

(4) eliminating the remaining auxiliary variables using Fourier–Motzkin elimination while using new auxiliary variables to model the choice of two possible terms introduced by this procedure, and finally

(5) eliminating these last auxiliary variables by compactly describing the exponential number of inequalities they induce.

As far as we are aware, most applications of Fourier–Motzkin elimination are very simple. Here it is necessary to use induction and prove explicitly which inequalities are necessary and which are redundant.

1.1. Three variants

We study three variants of the warehouse problem. Let h_t denote the inventory holding cost, f_t and g_t denote the fixed costs for buying and selling, respectively, in period t. In addition to the variables defined above, we define s_t to be the amount of stock at the end of period t and the binary variables z_t and w_t to be 1 if we buy and sell in period t and 0 otherwise, respectively.

- WP1: In the first variant, we include fixed costs only for buying. This variant can be modeled as:

$$\max \sum_{t=1}^{n}(p_t y_t - c_t x_t - f_t z_t - h_t s_t)$$
$$\text{s.t. } (2)\text{-}(5)$$
$$0 \leq x_t \leq B z_t \qquad t \in [1,n], \tag{6}$$
$$z_t \in \{0,1\} \qquad t \in [1,n]. \tag{7}$$

- WP2: In the second variant, we have fixed costs both for buying and selling.

$$\max \sum_{t=1}^{n}(p_t y_t - g_t w_t - c_t x_t - f_t z_t - h_t s_t)$$
$$\text{s.t. } (2)\text{-}(7),$$
$$y_t \leq B w_t \qquad t \in [1,n], \tag{8}$$
$$w_t \in \{0,1\} \qquad t \in [1,n]. \tag{9}$$

- WP3: In the third variant, we have fixed costs both for buying and selling and we do not allow to buy and sell in the same period.

$$\max \sum_{t=1}^{n}(p_t y_t - g_t w_t - c_t x_t - f_t z_t - h_t s_t)$$
$$\text{s.t. } (2)\text{-}(9),$$
$$w_t + z_t \leq 1 \qquad t \in [1,n]. \tag{10}$$

Let X_1', X_2' and X_3' be the feasible sets of problems WP1, WP2 and WP3, respectively. It is possible to model all three problems without the stock variables. For the first variant, the resulting feasible set, denoted X_1, is:

$$y_{1t} \leq S + x_{1,t-1} \qquad t \in [1,n], \tag{11}$$
$$x_{1t} \leq B - S + y_{1t} \qquad t \in [1,n], \tag{12}$$
$$x_t \leq B z_t \qquad t \in [1,n], \tag{13}$$
$$x_t, y_t \geq 0 \qquad t \in [1,n], \tag{14}$$
$$z_t \in \{0,1\} \qquad t \in [1,n]. \tag{15}$$

For the second problem WP2, the feasible set X_2 is given by (11)–(15) plus (8) and (9). Finally, for WP3, the feasible set X_3 is given by (11)–(15) plus (8)–(10). Note that $Proj_{x,y,z,s}X_3 \subseteq Proj_{x,y,z,s}X_2 \subseteq Proj_{x,y,z,s}X_1$.

Our aim is to describe the convex hull of each of the sets X_1, X_2 and X_3 and present tight extended formulations. The main results of the paper are the following theorems.

Theorem 1. *The convex hull of X_1 is given by*

$$y_{1t} \leq S + x_{1,t-1} \qquad t \in [1,n], \tag{16}$$
$$x_{1t} \leq B - S + y_{1t} \qquad t \in [2,n], \tag{17}$$
$$x_t \leq Bz_t \qquad t \in [1,n], \tag{18}$$
$$x_{1t} \leq y_{1t} + \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} \qquad t \in [1,n], \tag{19}$$
$$x_t, y_t \geq 0,\ z_t \leq 1 \qquad t \in [1,n]. \tag{20}$$

Theorem 2. *The convex hull of X_2 is given by*

$$y_{1t} \leq S + x_{1,t-1} \quad t \in [2,n], \tag{21}$$
$$x_{1t} \leq (B-S) + y_{1t} \quad t \in [2,n], \tag{22}$$
$$x_t \leq Bz_t \quad t \in [1,n], \tag{23}$$
$$y_t \leq Bw_t \quad t \in [2,n], \tag{24}$$
$$y_1 \leq Sw_1, \tag{25}$$
$$x_{1t} \leq \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t]} \min\{y_u, Sw_u\} \quad t \in [1,n], \tag{26}$$
$$y_{1t} \leq \sum_{u\in[1,t-1]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t]} \min\{y_u, Sw_u\} \quad t \in [2,n], \tag{27}$$
$$x_t, y_t \geq 0,\ z_t, w_t \leq 1 \quad t \in [1,n]. \tag{28}$$

Theorem 3. *The convex hull of X_3 is given by*

$$y_{1t} \leq S + x_{1,t-1} \quad t \in [2,n], \tag{29}$$
$$x_{1t} \leq B - S + y_{1,t-1} \quad t \in [2,n], \tag{30}$$
$$x_1 \leq (B-S)z_1, \tag{31}$$
$$x_t \leq Bz_t \quad t \in [2,n], \tag{32}$$
$$y_1 \leq Sw_1, \tag{33}$$
$$y_t \leq Bw_t \quad t \in [2,n], \tag{34}$$
$$x_{1t} \leq \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t-1]} \min\{y_u, Sw_u\} \quad t \in [2,n], \tag{35}$$
$$y_{1t} \leq \sum_{u\in[1,t-1]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t]} \min\{y_u, Sw_u\} \quad t \in [2,n], \tag{36}$$
$$z_t + w_t \leq 1 \quad t \in [1,n], \tag{37}$$
$$x_t, y_t \geq 0 \quad t \in [1,n]. \tag{38}$$

Constraints (35) and (36) can be linearized in the space of x, y, z and w as

$$x_{1t} \leq x_{[1,t]\setminus T} + (B-S)z_T + y_{[1,t-1]\setminus V} + Sw_V \quad t \in [2,n], T \subseteq [1,t], V \subseteq [1,t-1], \tag{39}$$
$$y_{1t} \leq x_{[1,t-1]\setminus T} + (B-S)z_T + y_{[1,t]\setminus V} + Sw_V \quad t \in [2,n], T \subseteq [1,t-1], V \subseteq [1,t]. \tag{40}$$

Introducing the variables π_t and ρ_t for $t \in [1,n]$ with

$$\pi_t \le x_t \quad t \in [1,n],$$
$$\pi_t \le (B-S)z_t \quad t \in [1,n],$$
$$\rho_t \le y_t \quad t \in [1,n],$$
$$\rho_t \le Sw_t \quad t \in [1,n],$$

we obtain a polynomial size tight extended formulation for X_3 in which constraints (39) and (40), which are exponential in number, are replaced by

$$x_{1t} \le \pi_{1t} + \rho_{1,t-1} \quad t \in [2,n],$$
$$y_{1t} \le \pi_{1,t-1} + \rho_{1t} \quad t \in [2,n].$$

The same can be done for X_1 and X_2.

In the remaining part of the paper, we prove these results. In Section 2 we provide properties of extreme points and unit flow formulations based on these properties. In Section 3 we present the proof of the convex hull result for X_3 and discuss briefly how to prove the results for X_1 and X_2. We conclude in Section 4 by showing that the nontrivial inequalities are flow cover inequalities for some single node flow set relaxations.

2. Extreme points and unit flow formulations

Though he considered a model without fixed costs, we can interpret the results of Dreyfus [4], as a characterization of the structure of the extreme points in our models. Using our notation

Theorem 4. *At an extreme point of* $\text{conv}(X'_1)$, $\text{conv}(X'_2)$ *and* $\text{conv}(X'_3)$,

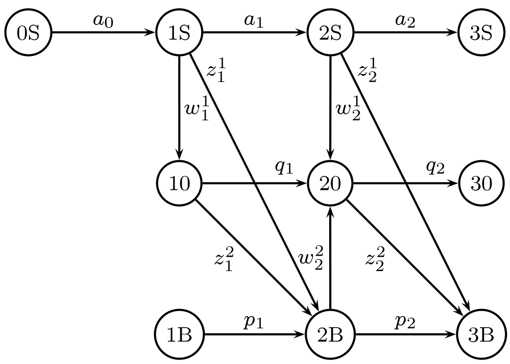
(i) For $t \in [1,n]$, $s_t \in \{0, s_{t-1}, B\}$, $y_t \in \{0, s_{t-1}\}$ *and* $x_t \in \{0, B-S, B\}$, *and if* $x_t = B-S$ *then* $s_{t-1} = S$, $y_t = 0$ *and* $s_t = B$.

(ii) there exists $t' \in [0,n]$ *such that* $s_t = S$ *for* $t \in [1,t']$ *and* $s_t \in \{0,B\}$ *for* $t \in [t'+1,n]$.

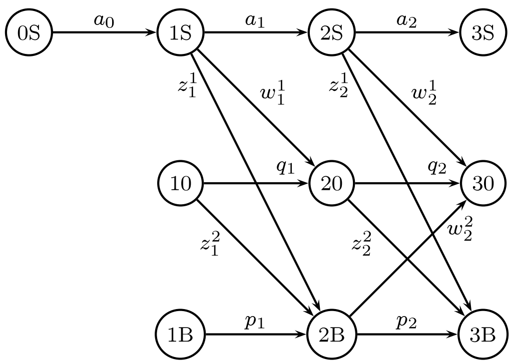
Based on the characterization of the extreme points, we provide unit flow formulations for the three warehouse problems. The networks for two periods are depicted in Fig. 1. In these networks, we have a layer for each period. As the stock can take three values (0, S and B) at an extreme point, we have three nodes for each period other than period 0. The arcs correspond to possible transitions. One unit of flow enters the network at node $0S$, which corresponds to having S units of stock at the end of period 0. The flow variables on different types of arcs are as follows:

- a_t : 1 if $s_{t-1} = s_t = S$ and $x_t = y_t = 0$ and 0 otherwise,
- p_t : 1 if $s_{t-1} = s_t = B$ and $x_t = y_t = 0$ and 0 otherwise,
- q_t : 1 if $s_{t-1} = s_t = 0$ and $x_t = y_t = 0$ and 0 otherwise,
- w_t^1 : 1 if $s_{t-1} = S$ and we sell S units in period t and 0 otherwise,
- w_t^2 : 1 if $s_{t-1} = B$ and we sell B units in period t and 0 otherwise,
- z_t^1 : 1 if we buy $B-S$ units in period t and 0 otherwise,
- z_t^2 : 1 if we buy B units in period t and 0 otherwise.

The unit flow formulations for all three problems are as follows:



(a)



(b)

Fig. 1. (a) Models 1 and 2 (b) Model 3.

- WP1 and WP2:

$$\begin{aligned} a_0 &= 1, \\ a_{t-1} - a_t - z_t^1 - w_t^1 &= 0 \quad t \in [1, n], \\ p_{t-1} + z_{t-1}^1 + z_{t-1}^2 - p_t - w_t^2 &= 0 \quad t \in [2, n], \\ q_{t-1} + w_t^1 + w_t^2 - q_t - z_t^2 &= 0 \quad t \in [1, n], \\ w_1^2 = p_1 = q_0 &= 0, \\ a, z^1, z^2, w^1, w^2, p, q &\geq 0. \end{aligned}$$

- WP3:

$$a_0 = 1, \tag{41}$$
$$a_{t-1} - a_t - z_t^1 - w_t^1 = 0 \quad t \in [1, n], \tag{42}$$
$$p_{t-1} + z_{t-1}^1 + z_{t-1}^2 - p_t - w_t^2 = 0 \quad t \in [2, n], \tag{43}$$
$$q_{t-1} + w_{t-1}^1 + w_{t-1}^2 - q_t - z_t^2 = 0 \quad t \in [2, n], \tag{44}$$
$$w_1^2 = z_1^2 = p_1 = q_1 = 0, \tag{45}$$
$$a, z^1, z^2, w^1, w^2, p, q \geq 0. \tag{46}$$

Both unit flow models are integral, i.e., the flow variables have integer values at the extreme points.

3. Proof of convex hull results

In the sequel we prove the convex hull result for X_3 in several steps. We first construct an extended formulation for conv(X_3) using the unit flow formulation. Then we project out the additional variables using Fourier–Motzkin.

Remark 1. As it models a unit flow problem, the polytope (41)–(46) is integral.

We rewrite the system (41)–(46) in the following equivalent form:

$$
\begin{aligned}
1 &= z^1_{1t} + w^1_{1t} + a_t \quad t \in [1, n],\\
p_t + w^2_{2t} &= z^1_{1,t-1} + z^2_{1,t-1} + p_1 \quad t \in [2, n],\\
q_t + z^2_{2t} &= w^1_{1,t-1} + w^2_{1,t-1} + q_1 \quad t \in [2, n],\\
w^2_1 = z^2_1 = p_1 = q_1 &= 0,\\
a, z^1, z^2, w^1, w^2, p, q &\geq 0.
\end{aligned}
$$

Now we eliminate a_t, p_t, q_t by substitution and remove redundancies giving:

$$z^1_{1n} + w^1_{1n} \leq 1, \tag{47}$$

$$w^2_{2t} \leq z^1_{1,t-1} + z^2_{1,t-1} \quad t \in [2, n], \tag{48}$$

$$z^2_{2t} \leq w^1_{1,t-1} + w^2_{1,t-1} \quad t \in [2, n], \tag{49}$$

$$z^1, z^2, w^1, w^2 \geq 0, \tag{50}$$

$$w^2_1 = z^2_1 = 0. \tag{51}$$

The polytope defined by (47)–(51) is integral since it is a projection of the integral polytope (41)–(46).

Finally we introduce the original variables x_t, y_t, z_t and w_t for $t \in [1, n]$ and add the constraints that relate them to the flow variables of the extended formulation as well as $z_t + w_t \leq 1$ for $t \in [1, n]$:

$$z_t \geq z^1_t + z^2_t \quad t \in [1, n], \tag{52}$$

$$w_t \geq w^1_t + w^2_t \quad t \in [1, n], \tag{53}$$

$$x_t = (B - S)z^1_t + Bz^2_t \quad t \in [1, n], \tag{54}$$

$$y_t = Sw^1_t + Bw^2_t \quad t \in [1, n], \tag{55}$$

$$z_t + w_t \leq 1 \quad t \in [1, n]. \tag{56}$$

Proposition 1. *The polytope (47)– (56) is integral.*

Proof. We use the approach of Lovasz [14] to prove that the polytope is integral. Given a non-zero objective function for which the optimal value is finite, we will show that the set of optimal solutions to the integer program lies on a face defined by one of constraints (47)–(56). Suppose that we are minimizing a linear function over the points in this polytope with integer z^1, z^2, w^1, w^2, z and w. Let c^z_t and c^w_t be the objective function coefficients of variables z_t and w_t, respectively, for $t \in [1, n]$. If all these coefficients are zero, since for each solution of (47)–(51), there exist z and w with (52), (53) and (56) and since the polytope defined by (47)–(51) is integral, all optimal solutions lie on a face defined by one of the constraints (47)–(50). Otherwise, let t be such that c^z_t or c^w_t is nonzero. If $c^z_t > 0$, then all optimal solutions satisfy $z_t = z^1_t + z^2_t$ and similarly if $c^w_t > 0$, then all optimal solutions satisfy $w_t = w^1_t + w^2_t$. If not, then $c^z_t < 0$ or $c^w_t < 0$ and in this case all optimal solutions lie on the face defined by $z_t + w_t \leq 1$. □

We now introduce two relaxations to help in describing the formulations. Let $k \in [1,n]$. First Q_k

$$x_t \le Bz_t \ t \in [k+1,n], \tag{57}$$
$$y_t \le Bw_t \ t \in [k+1,n], \tag{58}$$
$$x_t \ge 0 \ t \in [k+1,n], \tag{59}$$
$$y_t \ge 0 \ t \in [k+1,n], \tag{60}$$
$$x_{1t} \le (B-S) + y_{1,t-1} \ t \in [k+1,n], \tag{61}$$
$$y_{1t} \le S + x_{1,t-1} \ t \in [k+1,n], \tag{62}$$
$$z_t + w_t \le 1 \quad t \in [1,n], \tag{63}$$

and R_k

$$z_1^2 = w_1^2 = 0, \tag{64}$$
$$x_t = (B-S)z_t^1 + Bz_t^2 \ t \in [1,k], \tag{65}$$
$$y_t = Sw_t^1 + Bw_t^2 \ t \in [1,k], \tag{66}$$
$$z_t^1 \ge 0 \ t \in [1,k], \tag{67}$$
$$z_t^2 \ge 0 \ t \in [2,k], \tag{68}$$
$$w_t^1 \ge 0 \ t \in [1,k], \tag{69}$$
$$w_t^2 \ge 0 \ t \in [2,k], \tag{70}$$
$$w_{2t}^2 \le z_{1,t-1}^1 + z_{2,t-1}^2 \ t \in [2,k], \tag{71}$$
$$z_{2t}^2 \le w_{1,t-1}^1 + w_{2,t-1}^2 \ t \in [2,k], \tag{72}$$
$$z_t^1 + z_t^2 \le z_t \ t \in [1,k], \tag{73}$$
$$w_t^1 + w_t^2 \le w_t \ t \in [1,k]. \tag{74}$$

Theorem 5. *After elimination of $z_t^1, w_t^1, z_t^2, w_t^2$ from (47)– (56) for $t \in [k+1,n]$, the resulting polyhedron P_k is given by $Q_k \cap R_k$ plus the constraints:*

$$z_{1k}^1 + w_{1k}^1 \le 1, \tag{75}$$
$$x_{k+1,t} + Sz_{2k}^2 \le \pi_{k+1,t} + \rho_{k+1,t-1} + S(w_{1k}^1 + w_{2k}^2) \quad t \in [k+1,n], \tag{76}$$
$$y_{k+1,t} + (B-S)w_{2k}^2 \le \rho_{k+1,t} + \pi_{k+1,t-1} + (B-S)(z_{1k}^1 + z_{2k}^2) \quad t \in [k+1,n], \tag{77}$$

where $\pi_t = \min\{x_t, (B-S)z_t\}$ and $\rho_t = \min\{y_t, Sw_t\}$ for $t \in [1,n]$.

Proof. The proof is by induction over decreasing values of k. We observe that when $k = n$, P_n is the original system (47)–(56). First we consider the case when $k \ge 1$. The case when $k = 0$ will be discussed separately. The passage from P_k to P_{k-1} consists of a series of eliminations, (i) elimination of z_k^2 and w_k^2 by substitution, (ii) elimination of z_k^1 by Fourier–Motzkin and (iii) elimination of w_k^1 by Fourier–Motzkin and each one will be given and proved in a separate proposition.

Elimination of z_k^2 and w_k^2

Proposition 2. *Elimination of z_k^2 and w_k^2 by substitution gives $Q_k \cap R_{k-1}$ plus the constraints*

$$Sz_k^1 \leq Bz_k - x_k, \tag{78}$$

$$(B-S)w_k^1 \leq Bw_k - y_k, \tag{79}$$

$$(B-S)z_k^1 \leq x_k, \tag{80}$$

$$Sw_k^1 \leq y_k, \tag{81}$$

$$(B-S)z_k^1 \geq x_k + B(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2), \tag{82}$$

$$Sw_k^1 \geq y_k + B(w_{2,k-1}^2 - z_{1,k-1}^1 - z_{2,k-1}^2), \tag{83}$$

$$\begin{aligned} S(B-S)(z_k^1 + w_k^1) \geq\ & B(x_{k+1,t} - \pi_{k+1,t} - \rho_{k+1,t-1}) + S(x_k - y_k) \\ & + BS(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2) \quad t \in [k+1, n], \end{aligned} \tag{84}$$

$$\begin{aligned} S(B-S)(z_k^1 + w_k^1) \geq\ & B(y_{k+1,t} - \rho_{k+1,t} - \pi_{k+1,t-1}) + (B-S)(y_k - x_k) \\ & + B(B-S)(w_{2,k-1}^2 - z_{1,k-1}^1 - z_{2,k-1}^2) \quad t \in [k+1, n], \end{aligned} \tag{85}$$

$$z_k^1 \geq 0, \tag{86}$$

$$w_k^1 \geq 0, \tag{87}$$

$$z_{1k}^1 + w_{1k}^1 \leq 1. \tag{88}$$

Proof. We substitute $z_k^2 = (x_k - (B-S)z_k^1)/B$ and $w_k^2 = (y_k - Sw_k^1)/B$. Inequalities (78) and (79) come from (73) and (74), (80) and (81) come from (68) and (70) for $t = k$, (82) and (83) come from (72) and (71) for $t = k$, (84) and (85) come from (76) and (77). (86) and (87) are the inequalities (67) and (69) for $t = k$. $\square$

Elimination of z_k^1

Proposition 3. *Elimination of z_k^1 by Fourier–Motzkin gives $Q_k \cap R_{k-1}$, the constraints (79), (81), (83) and (87) that are unaffected, plus the constraints*

$$x_k \leq Bz_k, \tag{89}$$

$$x_k \geq 0, \tag{90}$$

$$x_k + Sz_{2,k-1}^2 \leq (B-S)z_k + S(w_{1,k-1}^1 + w_{2,k-1}^2), \tag{91}$$

$$z_{1,k-1}^1 + w_{1,k}^1 \leq 1, \tag{92}$$

$$(B-S)w_k^1 \leq (B-S) + y_{1,k-1} - x_{1k}, \tag{93}$$

$$S(B-S)w_k^1 \geq B(x_{kt} - \pi_{kt} - \rho_{k+1,t-1}) + BS(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2) - Sy_k \quad t \in [k+1, n], \tag{94}$$

$$S(B-S)w_k^1 \geq B(y_{kt} - \rho_{k+1,t} - \pi_{k,t-1}) + B(B-S)(w_{2,k-1}^2 - z_{1,k-1}^1 - z_{2,k-1}^2) - Sy_k \quad t \in [k+1, n]. \tag{95}$$

Proof. Note that z_k^1 appears in constraints (78), (80), (88) with one sign and constraints (82), (84), (85), (86) with the opposite sign.

We start with inequality (86). Inequalities (89), (90), (92) come from combining (86) with (78), (80), (88) respectively.

Next we use inequalities (84). First note that (78) and (80) can be equivalently written as

$$(B-S)Sz_k^1 \le \min\{B(B-s)z_k-(B-S)x_k, Sx_k\} = B\pi_k-(B-S)x_k.$$

Combining this with (84) gives

$$\begin{aligned}B\pi_k-(B-S)x_k \ge\ & B(x_{k+1,t}-\pi_{k+1,t}-\rho_{k+1,t-1})+S(x_k-y_k)\\ &+BS(z_{2,k-1}^2-w_{1,k-1}^1-w_{2,k-1}^2)-S(B-S)w_k^1 \quad t\in[k+1,n],\end{aligned}$$

which is the same as (94).

Combining (84) for $t\in[k+1,n]$ with (88) gives

$$\begin{aligned}S(B-S)(1-z_{1,k-1}^1-w_{1,k-1}^1) \ge\ & B(x_{k+1,t}-\pi_{k+1,t}-\rho_{k+1,t-1})+S(x_k-y_k)\\ &+BS(z_{2,k-1}^2-w_{1,k-1}^1-w_{2,k-1}^2),\end{aligned}$$

which simplifies to

$$Sx_{1k}+Bx_{k+1,t} \le S(B-S)+B\pi_{k+1,t}+Sy_{1k}+B\rho_{k+1,t-1},$$

using $x_u=(B-S)z_u^1+Bz_u^2$ and $y_u=Sw_u^1+Bw_u^2$ for $u\in[1,k-1]$ and $z_1^2=w_1^2=0$.

We consider an instance when $x_u>(B-S)y_u$ for $u\in T\subseteq[k+1,t]$ and $y_u>Sw_u$ for $u\in V\subseteq[k+1,t-1]$. The inequality takes the form:

$$Sx_{1k}+Bx_T \le S(B-S)+B(B-S)z_T+Sy_{1k}+By_{[k+1,t-1]\setminus V}+BSw_V.$$

This is dominated by taking $x_{1t}-y_{1,t-1}\le B-S$ with weight S, $x_u-Bz_u\le 0$ with weight $B-S$ for $u\in T$, $x_u\ge 0$ for $u\in[k+1,t]\setminus T$ with weight S, $y_u-By_u\le 0$ with weight S for $u\in V$ and $y_u\ge 0$ for $u\in[k+1,t-1]\setminus V$ with weight $B-S$, as in

$$\begin{aligned}Sx_{1k}+Sx_{k+1,t} &\le S(B-S)+Sy_{1k}+Sy_{k+1,t-1},\\ (B-S)x_T &\le (B-S)Bz_T,\\ -Sx_{[k+1,t]\setminus T} &\le 0,\\ 0 &\le BSw_V-Sy_V,\\ 0 &\le (B-S)y_{[k+1,t-1]\setminus V}.\end{aligned}$$

Now we use inequalities (85). Combining (78) and (80) in the form $(B-S)Sz_k^1\le B\pi_k-(B-S)x_k$ with (85) gives (95).

The inequality obtained from (85) with (88) is

$$(B-S)y_{1k}+By_{k+1,t} \le S(B-S)+B\rho_{k+1,t}+(B-S)x_{1,k}+B\pi_{k+1,t-1},$$

which is the same as

$$(B-S)y_{1k}+By_T \le S(B-S)+BSw_T+(B-S)x_{1k}+Bx_{[k+1,t-1]\setminus V}+B(B-S)z_V$$

for $T\subseteq[k+1,t]$ and $V\subseteq[k+1,t-1]$. This is the sum of $B-S$ times $y_{1t}\le S+x_{1,t-1}$, S times $y_u-By_u\le 0$ for $u\in T$ and $B-S$ times $y_u\ge 0$ for $u\in[k+1,t]\setminus T$, $B-S$ times $x_u-Bz_u\le 0$ for $u\in V$ and S times $x_u\ge 0$ for $u\in[k+1,t-1]\setminus V$, and hence is dominated.

Finally we use (82). Inequalities (91) and (93) come from combining (82) with (78) and (88), respectively. Combining (82) with (80) gives $z_{2,k-1}^2\le w_{1,k-1}^1+w_{2,k-1}^2$ that is dominated by (72) for $t=k-1$ as $w_{k-1}^1, w_{k-1}^2\ge 0$. This concludes the proof of Proposition 3. □

Elimination of w_k^1

Proposition 4. *Elimination of w_k^1 by Fourier–Motzkin gives P_{k-1}.*

Proof. The variable w_k^1 does not appear in constraints (89), (90) and (91). Constraint (91) is the same as (76) for $t = k$. w_k^1 appears in the constraints (83), (87), (94) and (95) with one sign and in (79), (81), (92) and(93) with opposite sign.

Observe that (83) is the same as (95) for $t = k$. We treat it together with (95).

We start with constraint (87). Constraints (58), (60), (75) and (61) for $t = k$ come from combining (87) with (79), (81), (92) and (93), respectively.

Next we combine (95) for $t \in [k, n]$. Combining (79) and (81) with (95) gives (77) for $t \in [k, n]$. Inequalities (95) and (92) give

$$(B-S)y_{1k} + B(y_{k+1,t} - \rho_{k+1,t}) \le S(B-S) + (B-S)x_{1,k-1} + B\pi_{k,t-1} \quad t \in [k, n]. \tag{96}$$

Consider an instance with $y_u > Sw_u$ for $u \in T \subseteq [k+1, t]$ and $x_u > (B-S)z_u$ for $u \in V \subseteq [k, t-1]$. Taking $y_{1t} - x_{1,t-1} \le S$ with weight $(B-S)$, $y_u \le Bw_u$ with weight S for $u \in T$, $x_u \le Bz_u$ with weight $(B-S)$ for $u \in V$, $0 \le y_u$ for $u \in [k+1, t] \setminus V$ with weight $B-S$ and $0 \le x_u$ for $u \in [k, t-1]$ with weight S, we see that the inequality is dominated.

Inequalities (95) and (93) give

$$\begin{aligned}&(B-S)y_k + B(y_{k+1,t} - \rho_{k+1,t}) + Sx_{1k} + B(B-S)w^2_{2,k-1}\\ &\le S(B-S) + Sy_{1,k-1} + B(B-S)(z^1_{1,k-1} + z^2_{2,k-1}) + B\pi_{k,t-1},\end{aligned}$$

that one can rewrite as

$$\begin{aligned}&(B-S)y_{1k} + B(y_{k+1,t} - \rho_{k+1,t}) + Sx_k + BSz^2_{2,k-1}\\ &\le S(B-S) + (B-S)x_{1,k-1} + BS(w^1_{1,k-1} + w^2_{2,k-1}) + B\pi_{k,t-1}.\end{aligned} \tag{97}$$

Suppose that $y_u > Sw_u$ for $u \in T \subseteq [k+1, t]$ and $x_u > (B-S)z_u$ for $u \in V \subseteq [k, t-1]$.

If $k \in V$, then we take $y_{1,t} - x_{1,t-1} \le S$ with weight $(B-S)$, $y_u - Bw_u \le 0$ with weight S for $u \in T$, $x_k + Sz^2_{2,k-1} - (B-S)z_k - S(w^1_{1,k-1} + w^2_{2,k-1}) \le 0$ with weight B, $x_u - Bz_u \le 0$ with weight $(B-S)$ for $u \in V \setminus \{k\}$, $0 \le x_u$ with weight S for $u \in [k+1, t] \setminus V$ and $-y_u \le 0$ with weight $(B-S)$ for $u \in [k+1, t] \setminus T$. This gives (97).

When $k \notin V$, we replace the third inequality by $z^2_{2,k-1} \le w^1_{1,k-1} + w^2_{2,k-1}$ with weight BS to show that the inequality is dominated.

Finally we combine inequalities (94) for $t \in [k+1, n]$. (79) and (81) are equivalent to

$$(B-S)Sw_k^1 \le B\rho_k - Sy_k.$$

This combined with (94) give (76) for $t \in [k+1, n]$. The inequalities (94) and (92) give

$$Sx_{1,k-1} + B(x_{kt} - \pi_{kt}) \le S(B-S) + Sy_{1k} + B\rho_{k+1,t-1}.$$

Inequalities (94) and (93) give

$$\begin{aligned}&Sx_{1,k} + B(x_{kt} - \pi_{kt}) + BSz^2_{2,k-1}\\ &\le S(B-S) + Sy_{1k} + BSw^2_{2,k-1} + B\rho_{k+1,t-1}.\end{aligned}$$

The proof of dominance for these last two families of inequalities is similar to the cases above with x, y and $S, (B-S)$ interchanged. This concludes the proof of Proposition 4. $\square$

Elimination of z_1^1, w_1^1 from P_1

Finally to complete the proof of Theorem 3, we consider the elimination of z_1^1, w_1^1 from P_1 as $z_1^2 = w_1^2 = 0$. We obtain $x_1 = (B-S)z_1^1 \leq (B-S)z_1$ and similarly $y_1 \leq Sw_1$. The constraints (76) take the form $x_{1t} - \pi_{1t} \leq \rho_{1,t-1}$ and we note that $\rho_1 = y_1$ as $y_1 \leq Sw_1$. Similarly the constraints (77) take the form $y_{1t} - \rho_{1t} \leq \pi_{1,t-1}$ with $\pi_1 = x_1$ as $x_1 \leq (B-S)z_1$. This concludes the proof of Theorem 3. □

Proofs of Theorems 1 and 2

The proof of Theorem 2 is almost identical to that of Theorem 3. We just note the following changes. We replace (56) with $z_t, w_t \leq 1$ for $t \in [1,n]$. We drop $z_1^2 = 0$ and add $z_1^2 \geq 0$ in every step. Consequently, all the terms of the form sum $z_{2\tau}^2$ are replaced by $z_{1\tau}^2$. In addition, constraint (49) becomes

$$z_{1t}^2 \leq w_{1t}^1 + w_{1t}^2 \quad t \in [1,n],$$

denoted (49^n).

The changes in the description of P_k are:

(61^n): $x_{1t} \leq (B-S) + y_{1t} \quad t \in [k+1,n]$

(72^n): $z_{1t}^2 \leq w_{1t}^1 + w_{2t}^2 \quad t \in [1,k]$

(76^n): $x_{k+1,t} + Sz_{1k}^2 \leq \pi_{k+1,t} + \rho_{k+1,t} + S(w_{1k}^1 + w_{2k}^2) \quad t \in [k+1,n]$

The changes in the system that we obtain after eliminating z_k^2 and w_k^2 are:

(82^n): $(B-S)(z_k^1 + w_k^1) \geq x_k - y_k + B(z_{1,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2)$

(84^n): $S(B-S)(z_k^1 + w_k^1) \geq B(x_{k+1,t} - \pi_{k+1,t} - \rho_{k+1,t}) + S(x_k - y_k) + BS(z_{1,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2) \quad t \in [k+1,n]$

In the elimination of z_k^1, when combining (82^n), the arguments are similar to those above leading to the new version (94^n):

$$S(B-S)w_k^1 \geq B(x_{kt} - \pi_{kt} - \rho_{k+1,t}) + BS(z_{1,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2) - Sy_k.$$

When we combine (82^n) with (78), we obtain (91^n):

$$Bx_k + BSz_{1,k-1}^2 \leq B(B-S)z_k + BS(w_{1,k-1}^1 + w_{2,k-1}^2) + S(y_k + (B-S)w_k^1).$$

Combining (82^n) with (88) gives $x_{1k} \leq B - S + y_{1k}$, which is ($61^n$) for $t = k$ and (93) is dropped. Finally, combining (82^n) with (80) gives $Bz_{1,k-1}^2 + Sw_k^1 \leq B(w_{1k}^1 + w_{2,k-1}^2) + y_k$ that is a combination of $z_{1,k-1}^2 \leq w_{1,k-1}^1 + w_{2,k-1}^2$, which is ($72^n$) for $t = k-1$, $w_k^1 \geq 0$ and $y_k \geq Sw_k^1$.

In the elimination of w_k^1, the variable appears in the constraints (83), (87), (91^n), (94^n) and (95) with one sign and in (79), (81) and (92) with opposite sign.

Combining (87) with (79), (81), and (92) give (58), (60) and (75), respectively.

As there are no changes in (95), combining this inequality with (79) and (81) give (77) for $t \in [k,n]$ and combining them with (92) gives dominated inequalities.

Combining (94^n) for $t \in [k+1,n]$ with (79) and (81) give (76) for $t \in [k+1,n]$. Combining (94^n) with (92) give

$$Sx_{1,k-1} + B(x_{kt} - \pi_{kt}) \leq S(B-S) + Sy_{1k} + B\rho_{k+1,t} \tag{98}$$

that are dominated.

Combining (91^n) with (79) and (81) give (76) for $t = k$. Combining (91^n) with (92) gives $S(B-S) + Sy_{1k} + (B-S)Bz_k \geq Sx_{1k} + (B-S)x_k$ which is a combination of $B - S + y_{1k} \geq x_{1k}$ and $Bz_k \geq x_k$.

After eliminating z_1^1 using Proposition 3, we use $w_1^1 = y_1/S$ to obtain the result.

To prove Theorem 1, it suffices to project out the variables w_t from the polyhedron describing $\mathrm{conv}(X_2)$. Constraints (24), (25) disappear, (27) reduces to $0 \leq \sum_{u\in[1,t-1]} \min\{x_u, (B-S)z_u\}$ and is dominated and (26) reduces to $x_{1t} \leq \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} + y_{1t}$.

4. Final remarks

We terminate with a couple of brief observations relating the model and results to other work. The following claim for the constant capacity single node flow set is well-known and cited in Atamtürk et al. [15].

When b is not a multiple of C, the convex hull of the single node flow set

$$\sum_{j \in N^+} x_j - \sum_{j \in N^-} x_j \leq b, x_j \leq C z_j \; j \in N, x \in R_+^{|N|}, z \in \{0,1\}^{|N|},$$

with $N = N^+ \cup N^-, N^+ \cap N^- = \emptyset$ is obtained by adding the constraints

$$x_T - (C - \lambda) z_T \leq b - \left\lceil \frac{b}{C} \right\rceil (C - \lambda) + x_{N^- \setminus L} + \lambda z_L$$

where $T \subseteq N^+$, $|T| \geq \lceil \frac{b}{C} \rceil$, $L \subseteq N^-$ and $\lambda = \lceil \frac{b}{C} \rceil C - b$.

A proof for the case when $N^- = \emptyset$ is given in Padberg et al. [16] and for the case in which the z variables are integers in Atamtürk [17], but it is an open question/conjecture for the 0-1 case.

With $b = B - S$ and $C = B$, we see that inequalities (35) for fixed t are precisely the flow cover inequalities for the single node flow set consisting of (12) for t, (8)–(9) and (13)–(15). Similarly the inequalities (36) are obtained from (11) for fixed t, (8)–(9) and (13)–(15). Thus the convex hull of X_3 is obtained as the intersection of these convex hulls for each fixed t.

As mentioned in the Introduction, it is also natural to view the warehouse model in the lot-sizing context with x_t, z_t as production and set-up variables and y_t, w_t as sales with fixed costs, where there are constant bounds on the stocks, but without fixed demands. Here one might also wish to look at constant capacity production and time-varying bounds on the stock levels.

References

[1] A.S. Cahn, The warehouse problem, Bull. Amer. Math. Soc. 54 (11) (1948) 1073.
[2] A. Charnes, W.W. Cooper, Generalizations of the warehousing model, Oper. Res. 6 (4) (1955) 131–172.
[3] R. Bellman, On the theory of dynamic programming-a warehousing problem, Manage. Sci. 2 (3) (1956) 272–275.
[4] S.E. Dreyfus, An analytic solution of the warehouse problem, Manage. Sci. 4 (1) (1957) 99–104.
[5] W.L. Eastman, A note on the multi-commodity warehouse problem, Manage. Sci. 5 (3) (1959) 327–331.
[6] A. Charnes, W.W. Cooper, M.H. Miller, Application of linear programming to financial budgeting and the costing of funds, J. Bus. 32 (1) (1959) 20–46.
[7] A. Charnes, J. Dreze, M. Miller, Decision and horizon rules for stochastic planning problems: A linear example, Econometrica 34 (2) (1966) 307–330.
[8] S.K. Devalkar, R. Anupindi, A. Sinha, Integrated optimization of procurement, processing, and trade of commodities, Oper. Res. 59 (6) (2011) 1369–1381.
[9] P. Harsha, M. Dahleh, Optimal management and sizing of energy storage under dynamic pricing for the efficient integration of renewable energy, IEEE Trans. Power Syst. 30 (3) (2015) 1164–1181.
[10] N. Secomandi, An analysis of the control-algorithm re-solving issue in inventory and revenue management, Manuf. Serv. Oper. Manag. 10 (3) (2008) 468–483.
[11] N. Secomandi, Optimal commodity trading with a capacitated storage asset, Manage. Sci. 56 (3) (2010) 449–467.
[12] O.Q. Wu, D.D. Wang, Z. Qin, Seasonal energy storage operations with limited flexibility: The price-adjusted rolling intrinsic policy, Manuf. Serv. Oper. Manag. 14 (3) (2012) 455–471.
[13] Y. Zhou, A. Scheller-Wolf, N. Secomandi, S. Smith, Electricity trading and negative prices: storage vs. disposal, Manage. Sci. 62 (3) (2015) 880–898.
[14] L. Lovasz, Graph theory and integer programming, Ann. Discrete Math. 4 (1979) 141–158.
[15] A. Atamturk, A. Gomez, S. Kucukyavuz, Three-partition inequalities for constant capacity fixed-charge network flow problems, Networks 67 (2016) 299–315.
[16] M.W. Padberg, T.J. Van Roy, L.A. Wolsey, Valid inequalities for fixed charge problems, Math. Program. 33 (1985) 842–861.
[17] A. Atamtürk, On capacitated network design cutset polyhedra, Math. Program. Ser. B 92 (2002) 425–437.