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On vector invariants of the symmetric group*

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Abstract — The purpose of this paper is to give a proof of the results announced by the author [7] in 1982 on the algebraic independence over a field k of any non-degenerate system of mn distinct basis invariants in the ring $k[x_{11}, \dots, x_{1n}; \dots; x_{m1}, \dots, x_{mn}]$ with respect to the symmetric group $G = S_n$. The result of this paper can be extended to the case of an arbitrary finite group.
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1. INTRODUCTION

Let m, n be positive integers, k be a field of characteristic 0 and let

$$A_{mn} = k[x_{11}, \dots, x_{1n}; \dots; x_{m1}, \dots, x_{mn}]$$

be the algebra of polynomials in mn indeterminates x_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$. The symmetric group $G = S_n$ operates on the algebra A_{mn} as a group of k-automorphisms by the rule $gx_{ij} = x_{i,g(j)}$, $g \in G$. Denote by A_{mn}^G the subalgebra of invariants of the algebra A_{mn} with respect to the group G and define elementary symmetric polynomials $u_{r_1,\dots,r_m} \in A_{mn}^G$ of vectors $(x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn})$ by means of the formal identity

$$\prod_{j=1}^{n}(1 + x_{1j}z_1 + \dots + x_{mj}z_m) = 1 + \sum_{1 \le r_1 + \dots + r_m \le n} u_{r_1,\dots,r_m} z_1^{r_1} \dots z_m^{r_m}. \tag{1}$$

The polynomials $u_{r_1,\dots,r_m}$ form a basis of the algebra A_{mn}^G (see [5], [1, A. IV, p. 62], [2, p. 9], and [8, p. 37]). In other words, each element F of the algebra A_{mn}^G may be represented as a polynomial in $u_{r_1,\dots,r_m}$ with coefficients from k. That basis contains $\binom{m+n}{m} - 1$ elements connected with one another by different algebraic relations.

On the other hand by the Noether normalization theorem (see [5] and [2, p. 17]), there exist mn algebraically independent basis invariants $u_1, \dots, u_{mn}$ among $u_{r_1,\dots,r_m}$ such that A_{mn}^G is finitely generated over $k[u_1, \dots, u_{mn}]$ as a module. This means that the transcendence degree of A_{mn}^G over k is equal to mn (see also [3, p. 68]). Since any invariant $u \in A_{mn}^G$ is algebraically dependent on $u_1, \dots, u_{mn}$, the set $\{u_1, \dots, u_{mn}\}$ can be also considered as a basis of another type in the algebra A_{mn}^G. The theorem of Noether provides only the

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existence of such a basis. The purpose of this paper is to give an effective version of the Noether result.

We say that a system of distinct basis invariants $u_{r_1,\dots,r_m}$ is non-degenerate if for every positive integer $\mu \le m$ and any integral sequence $(i_1, \dots, i_\mu)$, $1 \le i_1 < \dots < i_\mu \le m$, it contains at most μn elements $u_{r_1,\dots,r_m}$ with the condition that for every $i \in \{i_1, \dots, i_\mu\}$ there exists at least one $r_i \ge 1$ and for any $i \notin \{i_1, \dots, i_\mu\}$ all the corresponding r_i are zeros. Note that any subsystem of a non-degenerate system is also non-degenerate.

Theorem 1. *Any $s \le mn$ elements $u_{r_1,\dots,r_m}$ of a non-degenerate system of distinct basis invariants are algebraically independent over the field k.*

In addition, this paper contains a generalization of the well-known Waring's formulae (see [1, A. IV, p. 99])

$$\sum_{j=1}^{n} x_j^\sigma = \sum_{l_1+2l_2+\dots+nl_n=\sigma} c(l_1, \dots, l_n) u_1^{l_1} \dots u_n^{l_n}, \tag{2}$$

where $u_1, \dots, u_n$ are the elementary symmetric polynomials of the vector $(x_1, \dots, x_n)$,

$$c(l_1, \dots, l_n) = (-1)^{l_2+2l_3+\dots+(n-1)l_n} \frac{\sigma(l_1 + \dots + l_n - 1)!}{l_1! \dots l_n!},$$

and the sum on the right-hand side is over all non-negative integers $l_1, \dots, l_n$ satisfying the condition $l_1 + 2l_2 + \dots + nl_n = \sigma$, to the case of any number $m \ge 1$ of vectors $(x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn})$.

Theorem 2. *Let $\sigma, \sigma_1, \dots, \sigma_m$ be fixed, and let $l_1, \dots, l_n; r_{1\mu}, \dots, r_{m\mu}; s_{1\nu}, \dots, s_{m\nu}$ be arbitrary non-negative integers satisfying the relations*

$$\begin{aligned} \sigma_1 + \sigma_2 + \dots + \sigma_n &= l_1 + 2l_2 + \dots + nl_n = \sigma, \\ r_{1\mu} + r_{2\mu} + \dots + r_{m\mu} &= \nu, \qquad 1 \le \mu \le l_\nu, \\ s_{1\nu} + s_{2\nu} + \dots + s_{m\nu} &= \nu l_\nu, \qquad 1 \le \nu \le n, \end{aligned}$$

and let

$$v_{s_{1\nu},\dots,s_{m\nu}} = \sum \prod_{\mu=1}^{l_\nu} u_{r_{1\mu},\dots,r_{m\mu}},$$

where the sum is over all $r_{11}, \dots, r_{1l_\nu}; \dots; r_{m1}, \dots, r_{ml_\nu}$ such that $r_{i1} + \dots + r_{il_\nu} = s_{i\nu}$, $1 \le i \le m$. Then

$$\sum_{j=1}^{n} x_{1j}^{\sigma_1} \dots x_{mj}^{\sigma_m} = \frac{\sigma_1! \dots \sigma_m!}{\sigma!} \sum c(l_1, \dots, l_n) \sum \prod_{\nu=1}^{n} v_{s_{1\nu},\dots,s_{m\nu}},$$

where the internal sum is over all $s_{11}, \dots, s_{1n}; \dots; s_{m1}, \dots, s_{mn}$ such that $s_{i1} + \dots + s_{in} = \sigma_i$, $1 \le i \le m$, and the external sum is over all non-negative integers $l_1, \dots, l_n$, satisfying the condition $l_1 + 2l_2 + \dots + nl_n = \sigma$.

These results allow us to pass from mn indeterminates $x_{11}, \ldots, x_{1n}; \ldots; x_{m1}, \ldots, x_{mn}$ to mn new indeterminates $u_{r_1,\ldots,r_m}$, which can be chosen among the basis invariants $u_{r_1,\ldots,r_m}$ in an arbitrary way. Note also that the results can be extended easily to the case of positive characteristic $p > c(m,n)$.

2. NOTATION AND LEMMAS

Let $\sigma = (\sigma_{11}, \ldots, \sigma_{1n}; \ldots; \sigma_{m1}, \ldots, \sigma_{mn})$ be a binary sequence of length mn and $j_1, \ldots, j_\mu$ be the numbers of all those subsequences $\sigma = (\sigma_{1j}, \ldots, \sigma_{mj})$, $1 \le j \le n$, each of which has at least one non-zero component. Put s_i, $1 \le i \le m$, equal to the number of j, $1 \le j \le n$, such that $\sigma_{ij} = 1$, and define the weight $w(\sigma)$ of the sequence σ as

$$w(\sigma) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{ij}.$$

For any subset $\{l_1, \ldots, l_s\}$ of the set $\{1, 2, \ldots, n\}$ we denote by $u_{r_1,\ldots,r_m}^{(l_1,\ldots,l_s)}$ the elementary symmetric polynomials of the vectors $x_i = (x_{ij})_{1 \le j \le n,\, j \notin \{l_1,\ldots,l_s\}}$, $1 \le i \le m$, and put

$$u_{r_1,\ldots,r_m}^{(l_1,\ldots,l_s)} = \begin{cases} 1 & \text{if } (r_1, \ldots, r_m) = (0, \ldots, 0), \\ 0 & \text{if } r_i < 0 \text{ at least for one } i = 1, 2, \ldots, m. \end{cases}$$

Lemma 1. *Let $\sigma = (\sigma_{11}, \ldots, \sigma_{1n}; \ldots; \sigma_{m1}, \ldots, \sigma_{mn})$ be a binary sequence of length mn and let $w(\sigma) = v$.*

(i) *If at least one subsequence $\sigma_j = (\sigma_{1j}, \ldots, \sigma_{mj})$, $1 \le j \le n$, contains at least two components $\sigma_{i_1 j} = 1$ and $\sigma_{i_2 j} = 1$, then*

$$\frac{\partial^{w(\sigma)} u_{r_1,\ldots,r_m}}{\partial x_{11}^{\sigma_{11}} \ldots \partial x_{1n}^{\sigma_{1n}} \ldots \partial x_{m1}^{\sigma_{m1}} \ldots \partial x_{mn}^{\sigma_{mn}}} = 0.$$

(ii) *If every subsequence $\sigma_j = (\sigma_{1j}, \ldots, \sigma_{mj})$, $1 \le j \le n$, contains at most one component $\sigma_{ij} = 1$, then*

$$\frac{\partial^{w(\sigma)} u_{r_1,\ldots,r_m}}{\partial x_{11}^{\sigma_{11}} \ldots \partial x_{1n}^{\sigma_{1n}} \ldots \partial x_{m1}^{\sigma_{m1}} \ldots \partial x_{mn}^{\sigma_{mn}}} = u_{r_1-s_1,\ldots,r_m-s_m}^{(j_1,\ldots,j_v)}.$$

Proof. The first statement trivially follows from relation (1). To prove the second statement, we use induction on the weight $w(\sigma) \le n$ of σ. For $w(\sigma) = 0$ the statement is obvious. We assume that the statement is true for $w(\sigma) = v$, $0 \le v < n$, and prove its validity for $w(\sigma) = v + 1 \le n$.

Let $\sigma = (\sigma_{11}, \ldots, \sigma_{1n}; \ldots; \sigma_{m1}, \ldots, \sigma_{mn})$ be a binary sequence with the above property and let $\{j_1, \ldots, j_v\}, w(\sigma), \{s_1, \ldots, s_m\}$ be the corresponding parameters defined by this sequence. If $w(\sigma) = v$, by the induction assumption and by definition of $u_{r_1-s_1,\ldots,r_m-s_m}^{j_1,\ldots,j_v}$ we have

$$\frac{\partial^{w(\sigma)} u_{r_1,\ldots,r_m}}{\partial x_{11}^{\sigma_{11}} \ldots \partial x_{1n}^{\sigma_{1n}} \ldots \partial x_{m1}^{\sigma_{m1}} \ldots \partial x_{mn}^{\sigma_{mn}}} = u^{(j_1,\ldots,j_v)}_{r_1-s_1,\ldots,r_m-s_m}, \tag{3}$$

$$\prod_{\substack{i=1 \\ j\notin\{j_1,\ldots,j_v\}}}^{n} (1 + x_{1j}z_1 + \ldots + x_{mj}z_m) = \sum u^{(j_1,\ldots,j_v)}_{r_1-s_1,\ldots,r_m-s_m} z_1^{r_1-s_1} \ldots z_m^{r_m-s_m}, \tag{4}$$

where the sum is over all $r_1, \ldots, r_m$, such that $0 \le (r_1 - s_1) + \ldots + (r_m - s_m) \le n - v$. Denote by j_{v+1} one of the numbers $j \in \{1, 2, \ldots, n\}$ for which $\sigma_j = (\sigma_{1j}, \ldots, \sigma_{mj}) = (0, \ldots, 0)$, and apply the operator $\partial/\partial x_{ij_{v+1}}$ to both sides of identity (4). As a result we obtain the relation

$$z_i \prod_{\substack{j=1 \\ j\notin\{j_1,\ldots,j_{v+1}\}}}^{n} (1 + x_{1j}z_1 + \ldots + x_{mj}z_m)$$

$$= \sum_{0\le(r_1-s_1)+\ldots+(r_m-s_m)\le n-v} \left(\frac{\partial}{\partial x_{ij_{v+1}}} u^{(j_1,\ldots,j_m)}_{r_1-s_1,\ldots,r_m-s_m} \right) z_1^{r_1-s_1} \ldots x_m^{r_m-s_m}.$$

Now we use the identity

$$z_i \prod_{\substack{j=1 \\ j\notin\{j_1,\ldots,j_{v+1}\}}}^{n} (1 + x_{1j}z_1 + \ldots + x_{mj}z_m)$$

$$= z_i \left(\sum_{0\le(r_1-s_1)+\ldots+(r_m-s_m)\le n-v-1} u^{(j_1,\ldots,j_{v+1})}_{r_1-s_1,\ldots,r_m-s_m} z_1^{\,r_1-s_1} \ldots z_m^{\,r_m-s_m} \right)$$

$$= \sum_{0\le(r_1-s_1)+\ldots+(r_m-s_m)\le n-v} u^{(j_1,\ldots,j_{v+1})}_{r_1-s_1,\ldots,r_i-s_i-1,\ldots,r_m-s_m} z_1^{r_1-s_1} \ldots z_m^{r_m-s_m},$$

and then compare it with the previous relation. As a result we get

$$\frac{\partial u^{(j_1,\ldots,j_v)}_{r_1-s_1,\ldots,r_m-s_m}}{\partial x_{ij_{v+1}}} = u^{(j_1,\ldots,j_{v+1})}_{r_1-s_1,\ldots,r_i-s_i-1,\ldots,r_m-s_m},$$

which implies, in view of (3), that

$$\frac{\partial^{w(\sigma)+1} u_{r_1,\ldots,r_m}}{\partial x_{11}^{\sigma_{11}} \ldots \partial x_{1n}^{\sigma_{1n}} \ldots \partial x_{ij_{v+1}} \ldots \partial x_{m1}^{\sigma_{m1}} \ldots \partial x_{mn}^{\sigma_{mn}}} = u^{(j_1,\ldots,j_{v+1})}_{r_1-s_1,\ldots,r_i-s_i-1,\ldots,r_m-s_m}.$$

This proves the lemma.

Denote by R_{mn} the set of all sequences $r = (r_1, \ldots, r_m)$ of non-negative integers $r_1, \ldots, r_m$, satisfying the condition $1 \le r_1 + \ldots + r_m \le n$, and consider a subset $\mathcal{M}_s$ of the set R_{mn} of cardinality s. Let us index the elements of the set $\mathcal{M}_s$ by the numbers $1, 2, \ldots, s$, setting $r = r(k)$, $1 \le k \le s$, and for a given set $\mathcal{N}_s = \{(i_1, j_1), \ldots, (i_s, j_s)\}$ of distinct integral pairs (i_l, j_l), where $1 \le i_l \le m$, $1 \le j_l \le n$, $1 \le l \le s$, let us introduce into the consideration the determinant

$$\Delta_s = \det \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{(i,j)\in \mathcal{N}_s,\ 1\le k\le s}.$$

Let $\tau = (\tau_{11}, \ldots, \tau_{1n}; \ldots; \tau_{m1}, \ldots, \tau_{mn})$ be an integral sequence of length mn with non-negative components τ_{ij} and let

$$w(\tau) = \sum_{i=1}^{m} \sum_{j=1}^{n} \tau_{ij}$$

be the weight of this sequence.

For each integral pair (i,j), $1 \le i \le m$, $1 \le j \le n$, we consider the set of all sequences $k_{ij} = (k_{ij}^{(1)}, \ldots, k_{ij}^{(\tau_{ij})})$ with mutually distinct components $k_{ij}^{(1)}, \ldots, k_{ij}^{(\tau_{ij})} \in \{1, 2, \ldots, s\}$, and put $k_{ij} = \varnothing$ for $\tau_{ij} = 0$. Set

$$\sigma_{ij}(k) = \begin{cases} 1 & \text{if } k \in \{k_{ij}^{(1)}, \ldots, k_{ij}^{(\tau_{ij})}\}, \\ 0 & \text{if } k \notin \{k_{ij}^{(1)}, \ldots, k_{ij}^{(\tau_{ij})}\}, \end{cases}$$

$$\sigma(k) = (\sigma_{11}(k), \ldots, \sigma_{1n}(k); \ldots; \sigma_{m1}(k), \ldots, \sigma_{mn}(k)),$$

$$w(\sigma(k)) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{ij}(k)$$

and denote by $\Delta_s(k_{11}, \ldots, k_{1n}; \ldots; k_{m1}, \ldots, k_{mn})$ the determinant which is obtained from Δ_s if we replace the elements

$$\frac{\partial u_{r(k)}}{\partial x_{ij}}, \qquad (i,j) \in \mathcal{N}_s,$$

of the kth column by the elements

$$\frac{\partial^{w(\sigma(k))+1} u_{r(k)}}{\partial x_{11}^{\sigma_{11}(k)} \ldots \partial x_{1n}^{\sigma_{1n}(k)} \ldots \partial x_{m1}^{\sigma_{m1}(k)} \ldots \partial x_{mn}^{\sigma_{mn}(k)} \partial x_{ij}}, \qquad (i,j) \in \mathcal{N}_s.$$

Applying to Δ_s the well-known differentiation rule of determinants with functional elements and noting that

$$\frac{\partial^{\sigma_{ij}} u_{r(k)}}{\partial x_{ij}^{\sigma_{ij}}} = 0$$

for all $k = 1, \ldots, s$ and for all (i,j), $1 \le i \le m$, $1 \le j \le n$, such that $\sigma_{ij} > 1$, we obtain the following result.

Lemma 2. *For a given integral sequence* $\tau = (\tau_{11}, \ldots, \tau_{1n}; \ldots; \tau_{m1}, \ldots, \tau_{mn})$ *with non-negative components,*

$$\frac{\partial^{w(\tau)} \Delta_s}{\partial x_{11}^{\tau_{11}} \ldots \partial x_{1n}^{\tau_{1n}} \ldots \partial x_{m1}^{\tau_{m1}} \ldots \partial x_{mn}^{\tau_{mn}}} = \sum_{(k_{11}, \ldots, k_{mn})} \Delta_s(k_{11}, \ldots, k_{1n}; \ldots; k_{m1}, \ldots, k_{mn}).$$

For a given sequence

$$\mathcal{M}_s = \{r(k) \in R_{mn} \mid 1 \le k \le s\}, \qquad 1 \le s \le mn,$$

let us consider the corresponding system

$$U_s = \{u_{r(k)} \mid r(k) \in \mathcal{M}_s,\ 1 \le k \le s\}$$

of vector invariants $u_{r(k)}$ and denote by

$$J_s = \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{1\le i\le m,\ 1\le j\le n,\ 1\le k\le s}$$

the Jacobian of the system U_s.

Lemma 3. *For a given non-degenerate system*

$$U_s = \{u_{r(k)} \mid r(k) \in \mathcal{M}_s, 1 \le k \le s\}, \qquad 1 \le s \le mn,$$

of distinct vector invariants u_r, it is possible to find an integral sequence $\tau = (\tau_{11}, \dots, \tau_{1n}; \dots; \tau_{m1}, \dots, \tau_{mn})$, $1 \le \tau_{ij} \le s$, and a set $\mathcal{N}_s = \{(i_1,j_1), \dots, (i_s,j_s)\}$ of distinct integral pairs (i_l,j_l), $1 \le i_l \le m$, $1 \le j_l \le n$, $1 \le l \le s$, such that

$$\frac{\partial^{w(\tau)}}{\partial x_{11}^{\tau_{11}} \dots \partial x_{1n}^{\tau_{1n}} \dots \partial x_{m1}^{\tau_{m1}} \dots \partial x_{mn}^{\tau_{mn}}} \det \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{\substack{(i,j)\in \mathcal{N}_s, \\ 1\le k\le s}} = \tau_{11}! \dots \tau_{1n}! \dots \tau_{m1}! \dots \tau_{mn}! \det A_s, \tag{5}$$

where A_s is a binary $s \times s$ matrix of the rank s, which may be reduced by elementary row operations to the unit $s \times s$ matrix

$$I_s = \left\| \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \right\|.$$

Proof. We prove the lemma by induction on s, $1 \le s \le mn$. Let $u_r = u_{r_1,\dots,r_m}$ be an arbitrary elementary symmetric polynomial in m vector variables

$$(x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn})$$

and let $w(r) = r_1 + \dots + r_m$ be the weight of the sequence $r = (r_1, \dots, r_m)$.

By Lemma 1 we have

$$\frac{\partial^{w(r)} u_r}{\partial x_{11} \dots \partial x_{1r_1} \partial x_{2,r_1+1} \dots \partial x_{2,r_1+r_2} \dots \partial x_{m,r_1+\dots+r_{m-1}+1} \dots \partial x_{m,r_1+\dots+r_m}} = u_{0,\dots,0}^{(l_1,\dots,l_v)} = 1,$$

and hence for $s = 1$ the assertion is true.

We assume that the assertion holds for $s-1 \geq 1$ and prove its validity for $s \geq 2$. Let $\mathcal{M}_s = \{r(k) \in R_{mn} \mid 1 \leq k \leq s\}$ be an ordered set of cardinality s, consisting of elements $r = (r_1, \dots, r_m) \in R_{mn}$, let $U_s = \{u_{r(k)} \mid r(k) \in \mathcal{M}_s\}$ be the corresponding non-degenerate system of distinct vector invariants $u_{r(k)}$, $1 \leq k \leq s$, and let

$$J_s = \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq s}$$

be the Jacobian of the system U_s. We assume that the set $\mathcal{M}_s$ is ordered in such a way that $w(r(1)) \leq w(r(2)) \leq \dots \leq w(r(s))$. The case where all elements of the system $\mathcal{M}_s$ have weight 1 is trivial, hence without loss of generality we can assume that at least one sequence $r(k) = (r_1(k), \dots, r_m(k))$, $1 \leq k \leq s$, has weight $w(r(k)) \geq 2$. Thus we can assume that $v = w(r(s)) \geq 2$.

By the induction assumption we can find an integral sequence

$$\tau' = (\tau'_{11}, \dots, \tau'_{1n}; \dots; \tau'_{m1}, \dots, \tau'_{mn}), \qquad 1 \leq \tau'_{ij} \leq s-1,$$

and an $(s-1) \times (s-1)$ matrix

$$H'_{s-1} = \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{(i,j) \in \mathcal{N}'_{s-1},\ 1 \leq k \leq s-1}, \qquad \mathcal{N}'_{s-1} = \{(i_1, j_1), \dots, (i_{s-1}, j_{s-1})\}, \tag{6}$$

consisting of the first $s-1$ columns and $s-1$ rows of the Jacobian J_s, such that the determinant $\Delta'_{s-1} = \det H'_{s-1}$ of the matrix H'_{s-1} satisfies the relation

$$\frac{\partial^{w(\tau')} \Delta'_{s-1}}{\partial x_{11}^{\tau'_{11}} \dots \partial x_{1n}^{\tau'_{1n}} \dots \partial x_{m1}^{\tau'_{m1}} \dots \partial x_{mn}^{\tau'_{mn}}} = \tau'_{11}! \dots \tau'_{1n}! \dots \tau'_{m1}! \dots \tau'_{mn}! \det A'_{s-1}, \tag{7}$$

where A'_{s-1} is a non-singular binary $(s-1) \times (s-1)$ matrix, which may be reduced by elementary row operations to the unit matrix I_{s-1}.

Now we fix the set $\mathcal{N}'_{s-1} = \{(i_1, j_1), \dots, (i_{s-1}, j_{s-1})\}$ and the sequence $\tau' = (\tau'_{11}, \dots, \tau'_{1n}; \dots; \tau'_{m1}, \dots, \tau'_{mn})$ with the above mentioned properties, and show that there exists an integral pair (i_s, j_s), $1 \leq i_s \leq m$, $1 \leq j_s \leq n$, and an integral sequence $\tau = (\tau_{11}, \dots, \tau_{1n}; \dots; \tau_{m1}, \dots, \tau_{mn})$, $1 \leq \tau_{ij} \leq s$, such that equality (5) is valid for the determinant

$$\Delta_s = \det \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{(i,j) \in \mathcal{N}_s,\ 1 \leq k \leq s} \qquad \mathcal{N}_s = \{(i_1, j_1), \dots, (i_s, j_s)\}.$$

Next we describe in detail the induction process necessary to construct the required pair (i_s, j_s) and the required sequence $\tau = (\tau_{11}, \dots, \tau_{1n}; \dots; \tau_{m1}, \dots, \tau_{mn})$. Let

$$\mathcal{N}'_{s-1} = \{(i_1, j_1), \dots, (i_{s-1}, j_{s-1})\}, \qquad \tau' = (\tau'_{11}, \dots, \tau'_{1n}; \dots; \tau'_{m1}, \dots, \tau'_{mn})$$

be such that the determinant Δ'_{s-1} of matrix (6) satisfies relation (7). Let

$$\sigma = (\sigma_{11}, \dots, \sigma_{1n}; \dots; \sigma_{m1}, \dots, \sigma_{mn})$$

be a binary sequence with parameters

$$\nu = w(\sigma) = w(r(s)) = r_1(s) + \ldots + r_m(s), \quad (s_1, \ldots s_m) = (r_1(s), \ldots, r_m(s)),$$

which has the property that any its subsequences of the form $\sigma_\rho = (\sigma_{1\rho}, \ldots, \sigma_{m\rho})$, $1 \le \rho \le n$, contains at most one component $\sigma_{i\rho} = 1$. Let $\rho_1, \ldots, \rho_\nu$ be the numbers of all those sequences $\sigma_\rho = (\sigma_{1\rho}, \ldots, \sigma_{m\rho})$, $1 \le \rho \le n$, which contain exactly one component $\sigma_{i\rho} = 1$ and assume that $j_s \in \{\rho_1, \ldots, \rho_s\}$. Then by Lemma 1 we have

$$\frac{\partial^{w(\sigma)} u_{r(s)}}{\partial x_{11}^{\sigma_{11}} \ldots \partial x_{1n}^{\sigma_{1n}} \ldots \partial x_{m1}^{\sigma_{m1}} \ldots \partial x_{mn}^{\sigma_{mn}}} = u_{0,\ldots,0}^{(\rho_1,\ldots,\rho_\nu)} = 1. \tag{8}$$

Denote by $\omega = (\omega_{11}, \ldots, \omega_{1n}; \ldots; \omega_{m1}, \ldots, \omega_{mn})$ the binary sequence which is obtained from $\sigma = (\sigma_{11}, \ldots, \sigma_{1n}; \ldots; \sigma_{m1}, \ldots, \sigma_{mn})$ if we replace a subsequence $\sigma_{j'} = (0, \ldots, 0, 1, 0, \ldots, 0)$ beginning with i' zeros by the zero sequence $(0, \ldots, 0)$. There are

$$L_s = \frac{n!}{r_1(s)! \ldots r_m(s)!(n - r_1(s) - \ldots - r_m(s))!}$$

different sequences $\sigma = (\sigma_{11}, \ldots, \sigma_{1n}; \ldots; \sigma_{m1}, \ldots, \sigma_{mn})$, satisfying relation (8), and these sequences generate

$$M_s = \sum_{i=1}^{m} \frac{n!}{r_1(s)! \ldots (r_i(s) - 1)! \ldots r_m(s)!(n - r_1(s) - \ldots - r_m(s) + 1)!}$$

binary sequences $\alpha = (\alpha_{11}, \ldots, \alpha_{1n}; \ldots; \alpha_{m1}, \ldots, \alpha_{mn})$, with components

$$\alpha_{ij} = \frac{\partial}{\partial x_{ij}} \left(\frac{\partial^{w(\omega)} u_{r(s)}}{\partial x_{11}^{\omega_{11}} \ldots \partial x_{1n}^{\omega_{1n}} \ldots \partial x_{m1}^{\omega_{m1}} \ldots \partial x_{mn}^{\omega_{mn}}} \right)$$

which corresponds to all possible sequences $\sigma = (\sigma_{11}, \ldots, \sigma_{1n}; \ldots; \sigma_{m1}, \ldots, \sigma_{mn})$. The sequences α are not necessarily distinct, but for any given i, $1 \le i \le m$, they do contain exactly

$$N_s = \binom{n}{r_1(s) + \ldots + r_m(s) - 1} = \binom{n}{\nu - 1}$$

distinct sequences of the form $\alpha = (0, \ldots, 0; \ldots; \alpha_{i1}, \ldots, \alpha_{in}; \ldots; 0, \ldots, 0)$, where each sequence $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$ has exactly $r_1(s) + \ldots + r_m(s) - 1$ zeros and $n - r_1(s) - \ldots - r_m(s) + 1$ units. This provides a lot of essentially different candidates for the last column of the matrix A_s which correspond to various integral sequences $\tau = (\tau_{11}, \ldots, \tau_{1n}; \ldots; \tau_{m1}, \ldots, \tau_{mn})$ of the form

$$\tau = (\tau'_{11} + \omega_{11}, \ldots, \tau'_{1n} + \omega_{1n}; \ldots; \tau'_{m1} + \omega_{m1}, \ldots, \tau'_{mn} + \omega_{mn}). \tag{9}$$

We will use this fact later on, and now we establish the validity of relation (5). Namely, we prove by induction on s that for any given integral pair $(i_s, j_s) \notin \mathcal{N}'_{s-1}$, $1 \le$

$i_s \le m$, $1 \le j_s \le n$, and any given integral sequence $\tau = (\tau_{11}, \ldots, \tau_{1n}; \ldots; \tau_{m1}, \ldots, \tau_{mn})$ of form (9) relation (5) holds with a unique binary $s \times s$ matrix A_s. The matrix A_s can be constructed as follows. Expanding the determinant Δ_s according to the last column, we obtain

$$\Delta_s = \sum_{(i,j)\in \mathcal{N}_s,\ (i,j)=(i_l,j_l)} (-1)^{s+l} \frac{\partial u_{r(s)}}{\partial x_{ij}} \Delta_{i,j,s-1}.$$

By the induction assumption we have

$$\frac{\partial^{w(\tau')}\Delta_{i,j,s-1}}{\partial x_{11}^{\tau'_{11}} \ldots \partial x_{1n}^{\tau'_{1n}} \ldots \partial x_{m1}^{\tau'_{m1}} \ldots \partial x_{mn}^{\tau'_{mn}}} = \tau'_{11}! \ldots \tau'_{1n}! \ldots \tau'_{m1}! \ldots \tau'_{mn}! \det A_{i,j,s-1}$$

with a unique binary $(s-1) \times (s-1)$ matrix $A_{i,j,s-1}$, and according to the well-known Leibniz formula we find that

$$\begin{aligned}
&\frac{\partial^{w(\tau)}\Delta_s}{\partial x_{11}^{\tau_{11}} \ldots \partial x_{1n}^{\tau_{1n}} \ldots \partial x_{m1}^{\tau_{m1}} \ldots \partial x_{mn}^{\tau_{mn}}} \\
&= \sum_{(i,j)\in \mathcal{N}_s,\ (i,j)=(i_l,j_l)} (-1)^{s+l} \sum_{\nu_{11}=0}^{\tau_{11}} \cdots \sum_{\nu_{mn}=0}^{\tau_{mn}} \binom{\tau_{11}}{\nu_{11}} \cdots \binom{\tau_{mn}}{\nu_{mn}} \frac{\partial^{w(\nu)+1} u_{r(s)}}{\partial x_{ij} \partial x_{11}^{\nu_{11}} \ldots \partial x_{mn}^{\nu_{mn}}} \frac{\partial^{w(\tau)-w(\nu)-1}\Delta_{i,j,s-1}}{\partial x_{11}^{\tau_{11}-\nu_{11}} \ldots \partial x_{mn}^{\tau_{mn}-\nu_{mn}}} \\
&= \sum_{(i,j)\in \mathcal{N}_s,\ (i,j)=(i_l,j_l)} (-1)^{s+l} \binom{\tau_{11}}{\omega_{11}} \cdots \binom{\tau_{mn}}{\omega_{mn}} \quad \omega_{11}! \ldots \omega_{mn}! \quad \tau'_{11}! \ldots \tau'_{mn}! \det A_{i,j,s-1} \\
&= \tau_{11}! \ldots \tau_{1n}! \ldots \tau_{m1}! \ldots \tau_{mn}! \det A_s.
\end{aligned}$$

Further we show that for any integral pair $(i_s, j_s) \notin \mathcal{N}'_{s-1}$, $1 \le i_s \le m$, $1 \le j_s \le n$, it is possible to choose an integral sequence $\tau = (\tau_{11}, \ldots, \tau_{1n}; \ldots; \tau_{m1}, \ldots, \tau_{mn})$ of form (9) satisfying the condition that the last column of the corresponding matrix A_s in equality (5) differs from all its other columns. For any given l, $1 \le l \le s$, we denote by n_l the number of positions (i_l, j), $1 \le j \le n$, where the last column of the matrix

$$H_s = \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{(i,j)\in \mathcal{N}_s,\ 1\le k\le s}$$

has exactly n_l non-zero entries, and set $p_s = n_1 + \ldots + n_s$. If $p_s = 0$, then the last column of the matrix A_s consists only of zero elements, hence it differs from all other columns of A_s. Now we assume that $p_s \ge 1$. It follows from the previously mentioned arguments that for any $l = 1, \ldots, s$ such that $r_i(s) \ge 1$ with $i = i_l$ there is exactly $N_s = \binom{n}{\nu-1}$ distinct binary sequences $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, containing $\nu - 1$ zeros and $n - \nu + 1$ units. If we remove all entries of the vector α_i situated at any $n - n_l$ positions, then we obtain exactly

$$q_l = \sum_{\sigma=\nu_0-1-(n-n_l)}^{\nu_0-1} \binom{n_l}{\sigma}$$

distinct binary sequences of length n_l, where $\nu_0 = \min(\nu, n - \nu + 2)$ and $\binom{n_l}{\sigma} = 0$ for $\sigma > n_l$ or $\sigma < 0$.

Assume that $n_l \geq 1$ for some $l = 1, 2, \dots, s$ and consider the n_l rows of the matrix A_s indexed by integral pairs $(i_l, j) \in \mathcal{N}_s$ with given i_l. Let $k_1, \dots, k_\mu$, $1 \leq k_1 < \dots < k_\mu < s$, be the numbers of all those columns of A_s which intersect the considered rows by non-zero binary vectors of length n_l, say $v_{k_1}, \dots, v_{k_\mu}$. It follows from the structure of the matrix A_s that all remaining entries of the corresponding columns, situated outside of the given n_l positions, are zeros. Since the matrix A'_{s-1} is non-singular and the system U_s is non-degenerate, we find that $\mu \leq \lambda_l = \min(n_l, n-1)$ and, moreover, that the vectors $v_{k_1}, \dots, v_{k_\mu}$ are distinct. Thus we have q_l different possibilities to choose the last column of the matrix A_s with the condition that all their entries situated outside of the given n_l positions are zeros, and only $\mu \leq \lambda_l$ different possibilities for all other columns with the same property. Since $q_l > \lambda_l$ for any $n_l \geq 1$, we can choose the last column in such a way that it differs from all other columns of the matrix A_s. Moreover, if $n_l > 1$, we can choose the last column with the above property as a non-zero vector. Indeed, if $v_0 = 2$, then the set $\{v_{k_1}, \dots, v_{k_\mu}\}$ contains at most one vector of weight 1 or $n_l - 1$, and we have $n_l > 1$ possibilities to choose the last column of A_s as a vector of the same weight. Now, if $v_0 > 2$, we have $q_l > \lambda_l + 1$, and again we can choose the last column of A_s as a non-zero vector.

Now we fix an integral sequence $\tau = (\tau_{11}, \dots, \tau_{1n}; \dots; \tau_{m1}, \dots, \tau_{mn})$ satisfying the just described property and find an integral pair $(i_s, j_s) \notin \mathcal{N}'_{s-1}$, $1 \leq i_s \leq m$, $1 \leq j_s \leq n$, which provides the condition that the matrix A_s in (5) can be reduced to the unit matrix I_s. Set

$$\mathcal{N}''_{s-1} = \mathcal{N}_s \setminus \{(i_1, j_1)\}$$

and consider the matrix

$$H''_{s-1} = \left\| \frac{\partial u_{r(k)}}{\partial x_{ij}} \right\|_{(i,j) \in \mathcal{N}''_{s-1},\ 2 \leq k \leq s}.$$

By the induction assumption we can find an integral pair (i_s, j_s) and an integral sequence $\tau'' = (\tau''_{11}, \dots, \tau''_{1n}; \dots; \tau''_{m1}, \dots, \tau''_{mn})$, $1 \leq \tau''_{ij} \leq \tau_{ij}$, such that the determinant $\Delta''_{s-1} = \det H''_{s-1}$ satisfies the relation

$$\frac{\partial^{w(\tau'')} \Delta''_{s-1}}{\partial x_{11}^{\tau''_{11}} \dots \partial x_{1n}^{\tau''_{1n}} \dots \partial x_{m1}^{\tau''_{m1}} \dots \partial x_{mn}^{\tau''_{mn}}} = \tau''_{11}! \dots \tau''_{1n}! \dots \tau''_{m1}! \dots \tau''_{mn}! \det A''_{s-1}, \tag{10}$$

where A''_{s-1} is a non-singular binary $(s-1) \times (s-1)$ matrix, which can be reduced by elementary row operations to the diagonal matrix I_{s-1}.

Now we fix the integral sequence $\tau'' = (\tau''_{11}, \dots, \tau''_{1n}; \dots; \tau''_{m1}, \dots, \tau''_{mn})$ and the integral pair (i_s, j_s) which satisfy (10), and show that under the above choice of (i_s, j_s) and $\tau = (\tau_{11}, \dots, \tau_{1n}; \dots; \tau_{m1}, \dots, \tau_{mn})$ the matrix A_s in equality (5) possesses all the properties stated in the lemma. Indeed, it follows from Lemma 2 and relations (8), (9) that

$$\frac{\partial^{w(\tau)} \Delta_s}{\partial x_{11}^{\tau_{11}} \dots \partial x_{1n}^{\tau_{1n}} \dots \partial x_{m1}^{\tau_{m1}} \dots \partial x_{mn}^{\tau_{mn}}} = \tau_{11}! \dots \tau_{1n}! \dots \tau_{m1}! \dots \tau_{mn}! \det A_s,$$

where

$$A_s = \|a_{ij}\|_{1 \leq i,j \leq s-1}$$

is a binary $s \times s$ matrix containing $(s-1)\times(s-1)$ submatrices

$$A'_{s-1} = \|a_{ij}\|_{1\le i,j\le s-1}, \quad A''_{s-1} = \|a_{ij}\|_{2\le i,j\le s}.$$

Since the matrices A'_{s-1} and A''_{s-1} may be transformed by elementary row operations to the unit matrix I_{s-1}, as a result the matrix A_s may be reduced by the similar operations to the form

$$\bar{A}_s = \left\|\begin{matrix} 1 & 0 & & 0 & c_{1s} \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ c_{s1} & 0 & & 0 & 1 \end{matrix}\right\|.$$

Because all the columns of the matrix A_s are distinct, all the columns of the matrix $\overline{A}_s$ are distinct as well. In that case $c_{1s}c_{s1} = 0$, and hence the matrix A_s is equivalent to the unit matrix I_s. This proves the lemma.

3. PROOF OF THEOREM 1

We assume that there is a non-degenerate system consisting of $s \le mn$ distinct vector invariants $u_r = u_{r_1,\dots,r_m}$ algebraically dependent over k, and show that this assumption leads to a contradiction.

Let the above system be of the form $U_s = \{u_r \mid r \in \mathcal{M}_s\}$, where $\mathcal{M}_s$ is a subset of the set R_{mn} of cardinality s, and let $T_s = \{t_r \mid r \in \mathcal{M}_s\}$ be the corresponding system of independent variables $t_r = t_{r_1,\dots,r_m}$. From our assumption of algebraic dependence of the vector invariants u_r, $r \in \mathcal{M}_s$ over the field k, it follows that there exists a polynomial $F \in k[T_s]$ such that

$$F(U_s) = 0 \tag{11}$$

identically with respect to $x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn}$. We may assume that F is a polynomial of minimal degree satisfying equality (11). If we differentiate identity (11) with respect to x_{ij}, $1 \le i \le m$, $1 \le j \le n$, we obtain the system

$$\sum_{r\in \mathcal{M}_s} \frac{\partial F(U_s)}{\partial u_r}\frac{\partial u_r}{\partial x_{ij}} = 0, \quad 1 \le i \le m, \quad 1 \le j \le n, \tag{12}$$

consisting of mn linear equations, with respect to the s polynomials $\partial F(U_s)/\partial u_r$, $r \in \mathcal{M}_s$. Since F is a polynomial of minimal degree satisfying relation (11), we have $\partial F(U_s)/\partial u_r \ne 0$ at least for one sequence $r = (r_1, \dots, r_m) \in \mathcal{M}_s$. Therefore to get the desired contradiction, it is sufficient to establish the existence of a set $\mathcal{N}_s = \{(i_1,j_1), \dots, (i_s,j_s)\}$ consisting of s integral pairs (i_l,j_l), $1 \le l \le n$, where $1 \le i_l \le m$, $1 \le j_l \le n$, for which the determinant

$$\Delta_s = \det \left\| \frac{\partial u_r}{\partial x_{ij}} \right\|_{(i,j)\in \mathcal{N}_s,\ r\in \mathcal{M}_s}$$

is not zero.

The existence of such a determinant Δ_s for any s, $1 \le s \le mn$, is provided by Lemma 3, and this concludes the proof.

4. PROOF OF THEOREM 2

In (2) we set $x_j = x_{1j}z_1 + \ldots + x_{mj}z_m$, $1 \le j \le n$. We have

$$(x_{1j}z_1 + \ldots + x_{mj}z_m)^\sigma = \sum_{\sigma_1+\ldots+\sigma_m=\sigma} \frac{\sigma!}{\sigma_1!\ldots\sigma_m!} \left(\sum_{j=1}^{n} x_{1j}^{\sigma_1}\ldots x_{mj}^{\sigma_m}\right) z_1^{\sigma_1}\ldots z_m^{\sigma_m},$$

and then

$$\sum_{j=1}^{n}(x_{1j}z_1 + \ldots + x_{mj}z_m)^\sigma = \sum_{\sigma_1+\ldots+\sigma_m=\sigma} \frac{\sigma!}{\sigma_1!\ldots\sigma_m!} \left(\sum_{j=1}^{n} x_{1j}^{\sigma_1}\ldots x_{mj}^{\sigma_m}\right) z_1^{\sigma_1}\ldots z_m^{\sigma_m}.$$

On the other hand

$$\sum_{1\le j_1<\ldots<j_v\le n} \prod_{s=1}^{v}(x_{1j_s}z_1 + \ldots + x_{mj_s}z_m) = \sum_{r_1+\ldots+r_m=v} u_{r_1,\ldots,r_m} z_1^{r_1}\ldots z_m^{r_m}$$

and then, in view of (2),

$$\sum_{\sigma_1+\ldots+\sigma_m=\sigma} \left(\sum_{j=1}^{n} x_{1j}^{\sigma_1}\ldots z_{mj}^{\sigma_m}\right) z_1^{\sigma_1}\ldots z_m^{\sigma_m}$$

$$= \sum_{l_1+2l_2+\ldots+nl_n=\sigma} c(l_1,\ldots,l_n) \prod_{v=1}^{n} \left(\sum_{r_1+\ldots+r_m=v} u_{r_1,\ldots,r_m} z_1^{r_1}\ldots z_m^{r_m}\right)^{l_v}$$

Hence

$$\sum_{\sigma_1+\ldots+\sigma_m} \frac{\sigma!}{\sigma_1!\ldots\sigma_m!} \left(\sum_{j=1}^{n} x_{1j}^{\sigma_1}\ldots x_{mj}^{\sigma_m}\right) z_1^{\sigma_1}\ldots z_m^{\sigma_m}$$

$$= \sum_{l_1+2l_2+\ldots+nl_n=\sigma} c(l_1,\ldots,l_n) \prod_{v=1}^{n} \left(\sum_{s_{1v}+\ldots+s_{mv}=vl_v} v_{s_{1v},\ldots,s_{mv}} z_1^{s_{1v}}\ldots z_m^{s_{mv}}\right)$$

$$= \sum_{l_1+2l_2+\ldots+nl_n=\sigma} c(l_1,\ldots,l_n) \sum_{\sigma_1+\ldots+\sigma_m=\sigma} \left(\sum_{s_{i1}+\ldots+s_{in}=\sigma_i,\ 1\le i\le m} \prod_{v=1}^{n} v_{s_{1v},\ldots,s_{mv}}\right) z_1^{\sigma_1}\ldots z_m^{\sigma_m}$$

$$= \sum_{\sigma_1+\ldots+\sigma_m=\sigma} \left(\sum_{l_1+2l_2+\ldots+nl_n=\sigma} c(l_1,\ldots,c_n) \sum_{s_{i1}+\ldots+s_{in}=\sigma_i,\ 1\le i\le m} \prod_{v=1}^{n} v_{s_{1v},\ldots,s_{mv}}\right) z_1^{\sigma_1}\ldots z_m^{\sigma_m}$$

and therefore

$$\sum_{j=1}^{n} x_{1j}^{\sigma_1}\ldots x_{mj}^{\sigma_m} = \frac{\sigma_1!\ldots\sigma_m!}{\sigma!} \sum_{l_1+2l_2+\ldots+nl_n=\sigma} c(l_1,\ldots,l_n) \sum_{s_{i1}+\ldots+s_{in}=\sigma_i,\ 1\le i\le m} \prod_{v=1}^{n} v_{s_{1v},\ldots,s_{mv}},$$

which proves the theorem.

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