Fundamental Limits on RSS Based Range Estimation in Visible Light Positioning Systems

Erdal Gonendik and Sinan Gezici

***Abstract*— In this study, theoretical limits are obtained for the accuracy of range (distance) estimation in visible light positioning (VLP) systems. In particular, the Ziv-Zakai bound (ZZB) and the weighted Cramér-Rao bound (WCRB) are derived for range estimation based on received signal strength (RSS) measurements. Also, the maximum *a posteriori* probability (MAP) and the minimum mean-squared error (MMSE) estimators are obtained for RSS based range estimation, and compared against the theoretical limits.**

Keywords: **Visible light, range estimation, Ziv-Zakai bound, Cramér-Rao bound.**

I. Introduction

Indoor positioning via light emitting diodes (LEDs) has recently gathered significant attention [1]. Since LEDs already have widespread use for illumination purposes, their utilization for visible light communication (VLC) and visible light positioning (VLP) can facilitate new applications and services for indoor environments [2], [3]. VLP systems can provide accurate position information since line-of-sight (LOS) is commonly present and much stronger than multipath components in visible light channels. In addition, VLP systems do not suffer from interference as in RF systems, and they can also be employed in environments in which RF emission may be forbidden, e.g ., planes, hospitals, and mines [4].

One of the common approaches in VLP systems is to estimate the position of a VLC receiver based on received signal strength (RSS) measurements between the VLC receiver and a number of LED transmitters [4]–[6]. In [5], a complete VLP system, which achieves sub-meter accuracy via RSS based range estimation and trilateration, is implemented, and comparisons with other positioning systems are presented. In [4], Kalman and particle filtering are used for RSS based position tracking in VLP systems. The study in [6] employs a single LED transmitter and multiple optical receivers for position estimation, where the position of the receiver unit is determined based on RSS measurements at multiple receivers.

Although various studies have been performed on VLP systems, theoretical limits on estimation accuracy have been investigated only in few studies. In [7] and [8], the Cramér-Rao bound (CRB) is derived for RSS based and time-of-arrival based range estimation, respectively, and effects of LED parameters and system configuration are studied. The CRB provides a lower limit on mean-squared errors (MSEs) of unbiased estimators; however, it is not a tight bound in general for low signal-to-noise ratios (SNRs) [9]. In addition, in the presence of prior information about the unknown parameter, which is commonly available in indoor environments, the theoretical limits should also consider the prior knowledge in order to provide a valid bound. Based on these motivations, the Ziv-Zakai bound (ZZB) and the weighted CRB (WCRB) are derived in this study for RSS based range estimation in VLP systems. The ZZB is known for providing tight limits even in low SNR scenarios and the WCRB is an extended version of the CRB that takes prior information into account [9]. Based on a generic formulation with multiple power measurements, the ZZB and the WCRB on range estimation are derived for VLP systems for the first time in the literature. In addition, the maximum *a posteriori* probability (MAP) and the minimum MSE (MMSE) estimators are obtained for range estimation, and their comparisons with the theoretical bounds are provided.

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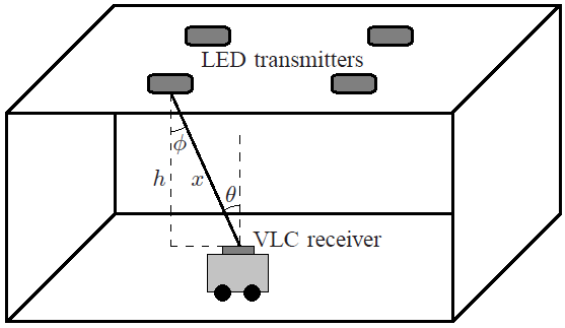

Fig. 1. Visible light positioning (VLP) system.

II. System Model

Consider a VLC system in which LED transmitters are located on the ceiling of a room, and a VLC receiver is located on an object on the floor, as shown in Fig. 1. By utilizing the signals received from the LED transmitters (which have known positions), the VLC receiver estimates its distance (range) to each LED transmitter via RSS measurements and determines its position based on range estimates [5]. The aim in this study is to derive the fundamental limits, namely, the ZZB and the WCRB, on RSS based range estimation in VLP systems.

An LED transmitter at location $\boldsymbol{l}_t \in \mathbb{R}^3$ and a VLC receiver at location $\boldsymbol{l}_r \in \mathbb{R}^3$ are considered. The distance between the LED transmitter and the VLC receiver is denoted by x, which is expressed as $x = \|\boldsymbol{l}_r - \boldsymbol{l}_t\|_2$. Considering M measurements at the VLC receiver, the received power for the ith measurement, $P_{r,i}$, can be modeled as

$$P_{r,i} = \frac{n+1}{2\pi} P_t \cos^n(\phi) \cos(\theta) \frac{A_R}{x^2} \mathbb{I}_{\{\theta \leq \theta_{\mathrm{FOV}}\}} + \eta_i \qquad (1)$$

for $i = 1, \ldots, M$, where P_t denotes the transmit power, n is the Lambertian order, A_R is the area of the photo detector at the VLC receiver, ϕ is the irradiation angle, θ is the incidence angle, θ_{FOV} denotes the field of view of the photo detector, and η_i is the noise in the ith measurement [5], [7]. Also,

$\mathbb{I}_{\{\theta\leq\theta_{\rm FOV}\}}$ represents an indicator function, which is equal to 1 if $\theta \leq \theta_{\rm FOV}$ and zero otherwise. The measurement noise η_i is modeled as a zero-mean Gaussian random variable with variance σ_i^2 [7], and it is assumed that the noise is independent for each measurement. Multiple measurements can be realized in a VLC receiver by performing measurements at different times (under stationarity conditions) or by employing multiple closely located photo detectors at the receiver.

For brevity of analytical expressions, it is assumed, similar to [6]–[8], that the LED transmitter is pointing downwards (which is commonly the case) and the photo detector at the VLC receiver is pointing upwards such that $\phi = \theta$ and $\cos(\phi) = \cos(\theta) = h/x$, where h denotes the height of the LED transmitter relative to the VLC receiver, as shown in Fig. 1. (Possible extensions to different transmitter and receiver orientations are discussed in Section VII.) Also, as in [4], [6]–[8], it is assumed that the height of the VLC receiver is known. This assumption holds in various practical scenarios; e.g., when the VLC receiver is attached to a robot or a cart that is tracked via a VLC system (e.g., Fig. 3 in [1]). Under these assumptions and considering scenarios in which the LED transmitter is in the field of view of the VLC receiver, the measurements in (1) can be expressed as

$$P_{r,i} = \frac{n+1}{2\pi} P_t \left(\frac{h}{x}\right)^{n+1} \frac{A_R}{x^2} + \eta_i \triangleq \frac{k}{x^{n+3}} + \eta_i \triangleq g(x) + \eta_i \tag{2}$$

for $i = 1, \ldots, M$, where k is a known constant that depends on n, P_t, h, and A_R. Let $\boldsymbol{y} = [P_{r,1}\ P_{r,2} \cdots P_{r,M}]$ denote the measurement vector. Then, the conditional probability density function (PDF) of the measurement vector is represented by $p(\boldsymbol{y}|x)$.

For the considered system model in Fig. 1, the range parameter, x, can be modeled to lie between h and $\bar{h}$, where $\bar{h} \triangleq h/\cos(\theta_{\rm FOV})$.[1] Therefore, the prior PDF of x is represented by a generic density $w(x)$, which is zero for $x \notin [h, \bar{h}]$. As a special case, when x is uniformly distributed over $[h, \bar{h}]$, the prior PDF is given by

$$w(x) = \mathbb{I}_{\{h\leq x\leq\bar{h}\}} / \left(\bar{h} - h\right). \tag{3}$$

III. Ziv-Zakai Bound (ZZB)

The ZZB provides a lower limit on the MSE by relating it to the probability of error in a binary hypothesis-testing problem. For a prior PDF that is zero for $x \notin [h, \bar{h}]$, it is given by [9]

$$\xi \geq \frac{1}{2} \int_0^{\bar{h}-h} \int_h^{\bar{h}-\delta} (w(\varphi) + w(\varphi+\delta)) \mathrm{P}_{\min}(\varphi, \varphi+\delta) d\varphi\, \delta\, d\delta \tag{4}$$

where $\xi = \mathrm{E}\{|\hat{x} - x|^2\}$ is the mean-squared error of an arbitrary estimator $\hat{x}$, $w(\cdot)$ denotes the prior PDF of parameter x, and $\mathrm{P}_{\min}(\varphi, \varphi+\delta)$ represents the probability of error corresponding to the MAP decision rule for the following hypothesis-testing problem:

$$\mathcal{H}_0 \ :\ p(\boldsymbol{y}|x = \varphi)\ ,\quad \mathcal{H}_1 \ :\ p(\boldsymbol{y}|x = \varphi + \delta) \tag{5}$$

[1] It is assumed that no communications occur when the LED transmitter is not in the field of view of the VLC receiver. In fact, the theoretical results in this study are valid for any finite value of $\bar{h}$ with $\bar{h} > h$.

For the estimation problem described in Section II, $p(\boldsymbol{y}|x)$ is obtained from the model in (2) as

$$p(\boldsymbol{y}|x) = \exp\left\{-\sum_{i=1}^{M} \frac{(P_{r,i} - g(x))^2}{2\sigma_i^2}\right\} \prod_{i=1}^{M} \frac{1}{\sqrt{2\pi}\,\sigma_i} \tag{6}$$

where $x \in [h, \bar{h}]$ and $P_{r,i}$ corresponds to the ith element of $\boldsymbol{y}$, as defined earlier. Since x has a PDF denoted by $w(x)$, the prior probabilities of hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ in (5) are given by $w(\varphi)/(w(\varphi)+w(\varphi+\delta))$ and $w(\varphi+\delta)/(w(\varphi)+w(\varphi+\delta))$, respectively. Then, from (5) and (6), the MAP decision rule can be expressed, after some manipulation, as

$$\sum_{i=1}^{M} \frac{P_{r,i}}{\sigma_i^2} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\lessgtr}} (g(\varphi+\delta) + g(\varphi)) \sum_{i=1}^{M} \frac{1}{2\sigma_i^2} + \Lambda(\varphi, \varphi+\delta) \tag{7}$$

for $\delta \geq 0$, where $\Lambda(\varphi, \varphi+\delta) \triangleq \log(w(\varphi+\delta)/w(\varphi))/(g(\varphi) - g(\varphi+\delta))$, with $\log$ denoting the natural logarithm. The probability of error for the rule in (7) can be obtained as

$$\mathrm{P}_{\min}(\varphi, \varphi+\delta) = \frac{w(\varphi)\mathrm{P}_{\mathrm{e},0} + w(\varphi+\delta)\mathrm{P}_{\mathrm{e},1}}{w(\varphi) + w(\varphi+\delta)} \quad \text{with} \tag{8}$$

$$\mathrm{P}_{\mathrm{e},i} = Q\left(\frac{g(\varphi) - g(\varphi+\delta)}{2} \sqrt{\sum_{i=1}^{M} \frac{1}{\sigma_i^2}} + (-1)^{i+1} \tilde{\Lambda}(\varphi, \varphi+\delta)\right) \tag{9}$$

for $i \in \{0, 1\}$, where $\tilde{\Lambda}(\varphi, \varphi+\delta) \triangleq \Lambda(\varphi, \varphi+\delta)/\sqrt{\sum_{i=1}^{M} \sigma_i^{-2}}$ and $Q(t) = (1/\sqrt{2\pi}) \int_t^\infty \mathrm{e}^{-\tau^2/2} d\tau$ denotes the Q-function. The ZZB in (4) can be evaluated based on (8) and (9) for any given prior PDF. In particular, for the uniform prior PDF in (3), $\Lambda(\varphi, \varphi+\delta)$ in (7) (hence, $\tilde{\Lambda}(\varphi, \varphi+\delta)$ in (9)) becomes zero $\forall \varphi, \delta$, and the ZZB in (4) reduces, based on (2), (8), and (9), to

$$\xi \geq \frac{1}{\bar{h} - h} \int_0^{\bar{h}-h} \int_h^{\bar{h}-\delta} Q\left(\frac{\tilde{k}}{\varphi^{n+3}} - \frac{\tilde{k}}{(\varphi+\delta)^{n+3}}\right) d\varphi\, \delta\, d\delta \tag{10}$$

where $\tilde{k} \triangleq 0.5k\sqrt{\sum_{i=1}^{M} \sigma_i^{-2}}$. The ZZBs in (4) and (10) can be evaluated accurately via numerical approaches as the integral limits are finite. Also, they provide tight limits on MSEs of optimal estimators for RSS based range estimation in VLP systems, as investigated in Section VI.

IV. Weighted CRB (WCRB)

The CRB belongs to the family of covariance inequality bounds [9]. For a given value of parameter x, the *conditional CRB* can be calculated as in [7] based on (2) and (6) as follows:

$$\begin{aligned} \xi \geq (\mathrm{J_F}(x))^{-1} &= \left(\mathrm{E}\left\{\left(\frac{d \log p(\boldsymbol{y}|x)}{dx}\right)^2\right\}\right)^{-1} \\ &= \frac{(g'(x))^{-2}}{\sum_{i=1}^{M} \frac{1}{\sigma_i^2}} = \frac{x^{2n+8}}{k^2(n+3)^2 \sum_{i=1}^{M} \frac{1}{\sigma_i^2}} \end{aligned} \tag{11}$$

where ξ represents the MSE of an unbiased estimator.

In the presence of prior information about the unknown parameter, the *Bayesian CRB (BCRB)* can be considered to provide a lower limit on the MSE of any estimator [9].

However, for some prior distributions, such as the uniform distribution, the BCRB does not exist (since an assumption in the derivation of the BCRB is violated), in which case the *weighted CRB (WCRB)* is commonly employed. The WCRB is given by [9]

$$\xi \geq \frac{\left(\mathrm{E}\{q(x)\}\right)^2}{\mathrm{E}\{q^2(x)\mathrm{J_F}(x)\} + \mathrm{E}\left\{q^2(x)\left(\frac{d\log(w(x)q(x))}{dx}\right)^2\right\}} \quad (12)$$

where ξ denotes the MSE of any estimator, the expectations are with respect to x, $\mathrm{J_F}(x)$ is the conditional Fisher information defined in (11), $w(x)$ is the prior PDF of x, and $q(x)$ is a weighting function. Similarly to [9], the following weighting function is employed in this study:

$$q(x) = \left(\frac{x-h}{\bar{h}-h}\right)^c \left(1-\frac{x-h}{\bar{h}-h}\right)^c \quad (13)$$

if $x \in [h, \bar{h}]$, and $q(x) = 0$ otherwise, where c is a parameter that can be adjusted to improve the bound in (12). From (13), $\mathrm{E}\{q(x)\}$ is calculated as

$$\mathrm{E}\{q(x)\} = \int_h^{\bar{h}} w(x)q(x)dx = \beta(c+1, c+1) \quad (14)$$

where the final expression is obtained for the uniform prior PDF in (3) with $\beta(a,b) \triangleq \int_0^1 x^{a-1}(1-x)^{b-1}dx$ representing the beta function. Also, the second term in the denominator of (12) can be obtained for the uniform prior as

$$\mathrm{E}\left\{q^2(x)\left(\frac{d\log(w(x)q(x))}{dx}\right)^2\right\} = \frac{c\,\beta(2c+1, 2c-1)}{(\bar{h}-h)^2}\,. \quad (15)$$

Finally, based on (11) and (13), the first term in the denominator of (12) can be stated for the uniform prior as follows:

$$\mathrm{E}\{q^2(x)\mathrm{J_F}(x)\} = k^2(n+3)^2 \sum_{i=1}^{M} \frac{1}{\sigma_i^2} \times \int_0^1 x^{2c}(1-x)^{2c}\left((\bar{h}-h)x + h\right)^{-2n-8} dx\,. \quad (16)$$

From (14)–(16), the WCRB in (12) can be evaluated. In addition, maximization over c is performed to find the tightest lower bound. Since the calculation of the WCRB for a given value of c involves single integrals only, the maximization over c leads to comparable computational complexity with that of the ZZB in (10).

V. MAP and MMSE Estimators

To compare the theoretical bounds against practical estimators, the MAP and MMSE estimators are derived for the range parameter, x, in this section. The MAP estimator [10] for the range can be expressed as

$$\hat{x}_{\mathrm{MAP}}(\boldsymbol{y}) = \underset{x\in[h,\bar{h}]}{\arg\max}\; w(x)p(\boldsymbol{y}|x)\,. \quad (17)$$

For the uniform prior PDF in (3), the MAP estimator in (17) reduces, based on (2) and (6), to

$$\underset{x\in[h,\bar{h}]}{\arg\min} \sum_{i=1}^{M} \frac{(P_{r,i} - kx^{-n-3})^2}{2\sigma_i^2}\,. \quad (18)$$

To obtain a closed-form solution for $\hat{x}_{\mathrm{MAP}}(\boldsymbol{y})$, the first-order derivative of the objective function in (18) is calculated as follows:

$$k(n+3)x^{-n-4}\left(\sum_{i=1}^{M}\frac{P_{r,i}}{\sigma_i^2} - kx^{-n-3}\sum_{i=1}^{M}\frac{1}{\sigma_i^2}\right). \quad (19)$$

The sign of the first-order derivative depends on the expression in the big parentheses, which is a monotone increasing function of x for $x \geq 0$. Then, the MAP estimator in (18) can be obtained as

$$\hat{x}_{\mathrm{MAP}}(\boldsymbol{y}) = \begin{cases} h\,, & \text{if } \sum_{i=1}^{M}\frac{P_{r,i}}{\sigma_i^2} > \frac{k}{h^{n+3}}\sum_{i=1}^{M}\frac{1}{\sigma_i^2} \\ \bar{h}\,, & \text{if } \sum_{i=1}^{M}\frac{P_{r,i}}{\sigma_i^2} < \frac{k}{(\bar{h})^{n+3}}\sum_{i=1}^{M}\frac{1}{\sigma_i^2} \\ f(\boldsymbol{y})\,, & \text{otherwise} \end{cases} \quad (20)$$

where $f(\boldsymbol{y}) \triangleq \left(k\sum_{i=1}^{M}\frac{1}{\sigma_i^2}\Big/\sum_{i=1}^{M}\frac{P_{r,i}}{\sigma_i^2}\right)^{1/(n+3)}$.

On the other hand, the MMSE estimator is given by [10]

$$\hat{x}_{\mathrm{MMSE}}(\boldsymbol{y}) = \mathrm{E}\{x|\boldsymbol{y}\} = \int x p(x|\boldsymbol{y})dx = \left(\int_h^{\bar{h}} x\, w(x)p(\boldsymbol{y}|x)dx\right) \Big/ \left(\int_h^{\bar{h}} w(x)p(\boldsymbol{y}|x)dx\right) \quad (21)$$

where $p(\boldsymbol{y}|x)$ is as in (6).

It is noted that the MAP estimator has lower computational complexity than the MMSE estimator. However, the MMSE estimator can achieve lower MSEs in general since it is the optimal estimator in terms of MSE minimization [10].

VI. Numerical Results

In this section, numerical examples are presented to investigate the ZZB and the WCRB and to compare them against the MAP and MMSE estimators for RSS based range estimation in VLP systems. The following parameters are employed for the model in (1): $n = 1$, $A_R = 1\,\mathrm{cm}^2$, and $\theta_{\mathrm{FOV}} = 60^o$ [4], [5]. Also, the uniform prior is considered, and $h = 4\,\mathrm{m}$, and $\bar{h} = h/\cos(\theta_{\mathrm{FOV}}) = 8\,\mathrm{m}$ are used in (3). The variances of the measurement noise η_i in (1) are taken to be equal; that is, $\sigma_i^2 = \sigma^2$ for $i = 1, \ldots, M$, and σ^2 is set to $10^{-13}\,\mathrm{A}^2$ [7].

In Fig. 2, the root mean-squared errors (RMSEs) for the MAP estimator, the MMSE estimator, the ZZB, and the WCRB[2] are plotted versus the transmit power P_t for $M = 1$ (i.e., single measurement). It is observed that the ZZB provides a very tight lower limit for the performance of the optimal MMSE estimator whereas the WCRB gets loose at high values of P_t. As P_t increases, the measurements in (2) get very accurate and the prior information becomes negligible compared to the information gathered from the measurements. In such situations, the WCRB may not be a tight bound when the conditional CRB is a function of the unknown parameter [9], which is the case in the considered problem (cf. (11)). Another observation from Fig. 2 is that the MAP estimator has higher RMSEs than the MMSE estimator for low power levels. This is due to the fact that as P_t goes to zero, the MAP estimator in (20) results in either h or $\bar{h}$ whereas the MMSE estimator in (21) converges to $\mathrm{E}\{x\} = 0.5(h+\bar{h})$.

[2]For each P_t, the optimal value of c is calculated and the tightest WCRB is employed (see (12)–(16) in Section IV).

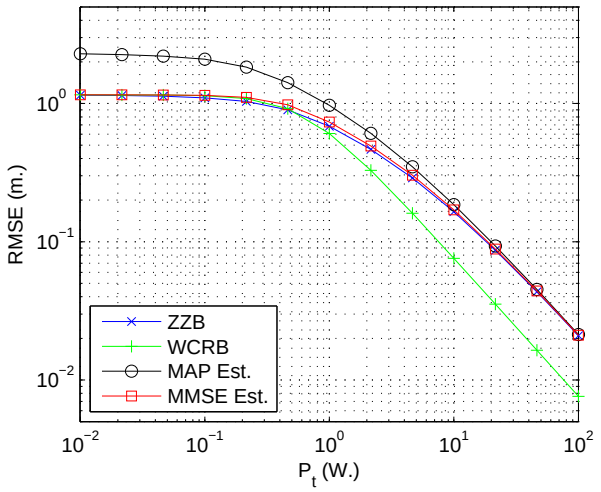

Fig. 2. RMSEs of the MAP and MMSE estimators, together with the ZZB and the WCRB for $M = 1$.

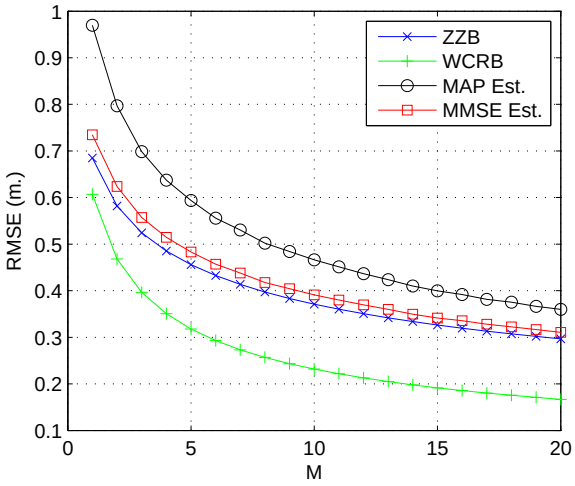

Fig. 3. RMSEs of the MAP and MMSE estimators, together with the ZZB and the WCRB for $P_t = 1$ W.

In Fig. 3, the RMSEs for the MAP estimator, the MMSE estimator, the ZZB, and the WCRB are plotted versus the number of measurements, M, for a transmit power of $P_t = 1$ W. Similar to Fig. 2, the ZZB provides a very tight limit for the performance of the optimal MMSE estimator and the WCRB is not very tight in general. As expected, the RMSE decreases with M; however, a diminishing return is observed as M increases. It is also noted that performing multiple independent measurements can result in significant reductions in the RMSE (e.g., 34.2% reduction in the RMSE of the MMSE estimator with $M = 5$).

The numerical results indicate that the ZZB can be used to provide guidelines for determining the parameters of a VLP system with a desired level of ranging accuracy. For example, from (10) and the definition of k presented after (10), it is deduced, based on the monotone decreasing property of the Q-function, that the ZZB reduces as k and/or $\sqrt{\sum_{i=1}^{M} \sigma_i^{-2}}$ increases. Since each new measurement results in a larger $\sqrt{\sum_{i=1}^{M} \sigma_i^{-2}}$ term, the ZZB can be decreased by employing multiple measurements, as expected. In particular, if $\sigma_i^2 = \sigma^2$ for $i = 1, \ldots, M$, the ZZB reduces as $\sqrt{M}/\sigma$ increases. On the other hand, parameter k, which is defined as in (2), can be increased by, e.g., employing higher transmit powers or photo detectors with larger areas. In fact, a larger area A_R of the photo detector also results in a higher noise variance since the variance of the shot noise (which is commonly the dominant component of the noise at VLC receivers) is proportional to A_R [8]. However, as A_R increases, the k term in (10) still increases with $\sqrt{A_R}$ (since k is proportional to A_R), resulting in lower ZZBs.

VII. Concluding Remarks and Future Work

In this study, the ZZB and the WCRB have been derived for RSS based range estimation in VLP systems. In addition, the MAP and MMSE estimators have been obtained for range estimation, and performance comparisons have been presented. It has been observed that the ZZB provides a very tight lower limit on the performance of the optimal MMSE estimator while the WCRB is not very tight in general. Hence, the ZZB can provide guidelines for the design of VLP systems; e.g., for choosing an LED transmitter with required parameters such as the transmit power.

The assumption that the LED transmitter and the VLC receiver are pointing downwards and upwards, respectively, can be relaxed to some extent based on arguments similar to those in Section II-C of [7]. In that case, $g(x)$ in (2) and $\bar{h}$ can be updated accordingly and the generic ZZB expression specified by (4), (8), and (9) still holds. Also, the WCRB can be obtained based on (12) by modifying (11) and (14)–(16) accordingly. As future work, range/position estimation with multiple LED transmitters can be considered in the presence of unknown height for the VLC receiver.

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