# THE NUMBER OF BLOCKS WITH A GIVEN DEFECT GROUP

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*Abstract.* Given a $p$-subgroup $P$ of a finite group $G$, we express the number of $p$-blocks of $G$ with defect group $P$ as the $p$-rank of a symmetric integer matrix indexed by the $N(P)/P$-conjugacy classes in $PC(P)/P$. We obtain a combinatorial criterion for $P$ to be a defect group in $G$.

QUESTION A. *Given a $p$-subgroup $P$ of a finite group $G$, how many $p$-blocks of $G$ have defect group $P$? In particular, when is $P$ a defect group in $G$?*

An answer to this venerable question was given in Robinson [**5**], reformulated in Broué [**2**], and generalised and further illuminated in Broué–Robinson [**3**]. The number of $p$-blocks with defect group $P$ is presented, in those three works, as the $p$-rank of a symmetric integer matrix indexed by certain conjugacy classes of $G$.

QUESTION B. *Supposing that $G$ is a normal subgroup of a finite group $F$, what is the number $f_0(G, F)$ of $F/G$-orbits of defect-zero $p$-blocks of $G$ whose stabilisers in $F/G$ are $p'$-groups?*

Question B is more general than Question A because Brauer's extended first main theorem describes a bijective correspondence between the $p$-blocks of $G$ with defect group $P$, and the $N_G(P)/PC_G(P)$-orbits of defect-zero $p$-blocks $b$ of $PC_G(P)/P$ such that the stabiliser of $b$ in $N_G(P)/PC_G(P)$ is a $p'$-group. In answer to Question B, Theorem 5 below expresses $f_0(G, F)$ as the $p$-rank of a symmetric integer matrix $\Psi(G, F)$ indexed by the $F$-conjugacy classes of $G$. Corollary 6 spells out the new answer thus provided to Question A.

In view of the local reduction indicated above, and also in view of another local reduction conjectured by Alperin [**1**], the number $f_0(G) = f_0(G, G)$ of defect-zero $p$-blocks of $G$ is of especial interest. A description of $f_0(G)$ as the $p$-rank of a symmetric integer matrix indexed by the conjugacy classes of $G$ has already been given by Robinson [**5**], but it is to be noted that the matrix $\Psi(G) = \Psi(G, G)$ has the feature of being independent of $p$.

This work is a synthesis of character-theoretic constructions in Strunkov [**6**] and $G$-algebra-theoretic techniques implicit in Broué [**2**], Broué–Robinson [**3**], Robinson [**5**]. We also make use of a $G$-algebra-theoretic approach to Clifford theory. I am grateful to Robinson for communicating to me an illuminating formulation of material in [**6**].

Let $O$ be a complete local noetherian commutative ring whose residue field $O/J(O)$ has prime characteristic $p$, and whose field of fractions $\kappa$ has characteristic zero. We shall assume that $\kappa$ splits for all the given groups under consideration. When speaking of an integer, we shall always be refering to a rational integer, and we shall identify the integers with the elements of the minimal unital subring of $O$. Also, we shall identify the $p$-blocks of $G$ with the block idempotents of $OG$. Let $\mathrm{Irr}\,(\kappa G)$ denote the set of irreducible $\kappa G$-characters, and $\mathrm{Irr}_0\,(\kappa G)$ the subset consisting of those $\chi \in \mathrm{Irr}\,(\kappa G)$ such that $|G|/\chi(1)$ is coprime to $p$.

The conjugation action of $F$ on its normal subgroup $G$ induces algebra automorphisms of $\kappa G$ and $OG$, and induces permutations of $\mathrm{Irr}\,(\kappa G)$. Recall that, for $H \leqslant K \leqslant F$, the image $(OG)^K_H$ of the relative trace map $\mathrm{Tr}^K_H: (OG)^H \to (OG)^K$ is an ideal of the $K$-fixed subalgebra $(OG)^K$ of $OG$. For each irreducible $\kappa G$-character $\chi$, we write $N_F(\chi)$ for the stabiliser of $\chi$ in $F$, and write $b^F_\chi$ for the primitive idempotent of the commutative algebra $(OG)^F$ such that $\chi(b^F_\chi) = \chi(1)$. Note that $b^F_\chi$ is the sum of the $F$-conjugates of the $p$-block $b^G_\chi$ of $G$ containing $\chi$. We say that a primitive idempotent $b$ of $(OG)^F$ is projective provided $b \in (OG)^F_1$ (when $G = F$, these idempotents are precisely the defect-zero $p$-blocks of $G$).

Proposition 1. *There is a bijective correspondence between the projective primitive idempotents $b$ of $(OG)^F$, and the $F$-orbits of irreducible $\kappa G$-characters $\chi$ such that $|N_F(\chi)|/\chi(1)$ is coprime to $p$, whereby $b$ corresponds to the $F$-orbit of $\chi$ provided $b = b^F_\chi$.*

*Proof.* Let $\chi$ be an irreducible $\kappa G$-character, and put $N = N_F(\chi)$. It is well-known that $b^G_\chi$ is a defect-zero $p$-block of $G$ if, and only if, $|G|/\chi(1)$ is coprime to $p$. When these equivalent conditions hold, the $F$-conjugates of $\chi$ are precisely the irreducible $\kappa G$-characters $\chi'$ such that $b^F_\chi = b^F_{\chi'}$. So it suffices to prove that $b^F_\chi$ is projective if, and only if, $|N|/\chi(1)$ is coprime to $p$.

Suppose that $|N|/\chi(1)$ is coprime to $p$. Then $b^G_\chi$ is a defect-zero block of $OG$, that is, $b^G_\chi = \mathrm{Tr}^G_1\,(\eta)$ for some $\eta \in OG$. Also, $|G: N|$ is coprime to $p$, and $N$ is the stabiliser of $b^F_\chi$ in $F$, hence the primitive idempotent

$$b^F_\chi = \mathrm{Tr}^F_N\,(b^G_\chi) = \mathrm{Tr}^F_1\,(\eta/|G: N|)$$

is projective. Conversely, suppose that $b^F_\chi$ is projective. By Mackey decomposition,

$$(OG)^F_1 \subseteq (OG)^N_1 \subseteq (OG)^G_1.$$

So the idempotent $b^G_\chi = b^F_\chi b^G_\chi$ belongs to the ideal $(OG)^N_1$ of $(OG)^N$, and in particular, $b^G_\chi$ is a defect-zero block of $OG$. Therefore, $|G|/\chi(1)$ is coprime to $p$, and writing $b^G_\chi = \mathrm{Tr}^N_1\,(\mu)$ for some $\mu \in OG$, we have

$$b^G_\chi = \mathrm{Tr}^N_1\,(\mu b^G_\chi) = |N: G|\ \mathrm{Tr}^G_1\,(\mu b^G_\chi)$$

whereupon $|N: G|$ must be coprime to $p$.

In particular, $f_0(G, F)$ is the number of projective primitive idempotents of $(OG)^F$. Let $b^F_G$ be the sum of the projective primitive idempotents of $(OG)^F$.

Thus $b_G^G$ is the sum of the defect-zero $p$-blocks of $G$. We have a direct sum of free cyclic $O$-modules

$$(OG)^F b_G^F = \bigoplus_b (OG)^F b$$

where $b$ runs over the projective primitive idempotents of $(OG)^F$. Iizuka-Watanabe [**4**, Lemma 2] proved the following result in the special case $G=F$.

LEMMA 2: $(OG)^F b_G^F \subseteq ((OG)_1^F)^2 \subseteq (OG)^F b_G^F \oplus J(O)(OG)^F(1-b_G^F)$.

*Proof*: Let $V=(OG)^F b_G^F$ and $U=((OG)_1^F)^2$ as ideals of $(OG)^F$. Clearly, $b_G^F \in V$, so $V \subseteq U$. Now

$$U = \bigoplus_{\chi \in_F \mathrm{Irr}(\kappa G)} U b_\chi^F$$

where the notation indicates that $\chi$ runs over the $F$-orbits of $\mathrm{Irr}(\kappa G)$. Let us fix an irreducible $\kappa G$-character $\chi$. If $b_\chi^F$ is projective, then $b_\chi^F \in V$, so $Ub_\chi^F \subseteq V$. Assuming now that $b_\chi^F$ is not projective, we have $b_G^F b_\chi^F = 0$, and it suffices to show that $Ub_\chi^F \subseteq J(O)(OG)^F$.

First suppose that $\chi \in \mathrm{Irr}_0(\kappa G)$. Then $b_\chi^F \notin U$, so $Ub_\chi^F$ is strictly contained in the free cyclic $O$-module $(OG)^F b_\chi^F$. Therefore $Ub_\chi^F \subseteq J(O)(OG)^F$ in this case. Now suppose that $\chi \notin \mathrm{Irr}_0(\kappa G)$. We observed above that $(OG)_1^F \subseteq (OG)_1^G$, and that the assertion holds when $G=F$. So

$$Ub_\chi^F \subseteq U \subseteq ((OG)_1^G)^2 \subseteq ZOG \cdot b_G^G \oplus J(O) \cdot ZOG(1-b_G^G).$$

But $b_\chi^G b_\chi^F = 0$, so $Ub_\chi^F \subseteq J(O) \cdot ZOG \cap (OG)^F = J(O)(OG)^F$.

For each irreducible $\kappa G$-character $\chi$, let $e_\chi^F$ be the primitive idempotent of $(\kappa G)^F$ such that $\chi(e_\chi^F) = \chi(1)$, and let $\omega_\chi^F$ be the algebra map $(\kappa G)^F \to \kappa$ such that $\omega(e_\chi^F) = 1$. Then $\omega_\chi^F$ is a restriction of the central character $\omega_\chi^G$ associated with $\chi$. That is to say, $\omega_\chi^F = \chi/\chi(1)$ on $(\kappa G)^F$. Let $\omega_G^F : (\kappa G)^F \to \kappa$, and $\nu_G^F : (\kappa G)^F \otimes_\kappa (\kappa G)^F \to \kappa$ be the characters afforded by the translation actions of $(\kappa G)^F$ and $(\kappa G)^F \otimes_\kappa (\kappa G)^F$ on $(\kappa G)^F$. Thus, given $\zeta, \zeta' \in (\kappa G)^F$, the trace of the action of $\zeta$ on $(\kappa G)^F$ is

$$\omega_G^F(\zeta) = \sum_{\chi \in_F \mathrm{Irr}(\kappa G)} \omega_\chi^F(\zeta)$$

while the trace of the action of $\zeta \otimes \zeta'$ on $(\kappa G)^F$ is

$$\nu_G^F(\zeta \otimes \zeta') = \omega_G^F(\zeta\zeta') = \sum_{\chi \in \mathrm{Irr}(\kappa G)} \omega_\chi^F(\zeta)\omega_\chi^F(\zeta').$$

We define linear maps $\varphi_G^F : \kappa G \to \kappa$ and $\psi_G^F : \kappa(G \times G) \to \kappa$ given by

$$\varphi_G^F(g) = |\{(x, w) \in G \times F : g = [x, w]\}|,$$

$$\psi_G^F(g, h) = |\{(x, w, v) \in G \times F \times F : gvh^{-1}v^{-1} = [x, w]\}|$$

for $g, h \in G$. Here, $[x, w] = xwx^{-1}w^{-1}$. The next result shows that $\varphi_G^F$ is a $\kappa G$-character, and that $\psi_G^F$ is a $\kappa(G \times G)$-character. Recall that the irreducible $\kappa(G \times G)$-characters are the $\kappa(G \times G)$-characters of the form $\chi * \chi'$ with

$\chi, \chi' \in \operatorname{Irr}(\kappa G)$, where

$$(\chi * \chi')(g, h) = \chi(g)\chi'(h^{-1}).$$

For any element $x \in G$, we write $[x]_F$ for the set of $F$-conjugates of $x$. For any subset $A \subseteq G$, we write $A^+$ for the sum in $\kappa G$ of the elements of $[x]_F$. Thus $\{[x]_F^+ : x \in G\}$ is a $\kappa$-basis for $(\kappa G)^F$.

LEMMA 3.

(a) $\phi_G^F = \sum_{\chi \in \operatorname{Irr}(\kappa G)} |N_F(\chi)| \chi/\chi(1)$.
(b) $\psi_G^F = \sum_{\chi \in \operatorname{Irr}(\kappa G), vN_F(\chi) \subseteq F} (|N_F(\chi)|/\chi(1))^2 \chi * {}^v\chi$.

*Proof.* Given an element $\zeta \in (\kappa G)^F$, and writing

$$\zeta[x]_F^+ = \sum_{y \in_F G} \zeta_{x,y}[y]_F^+$$

then $\omega_G^F(\zeta) = \sum_{y \in_F G} \zeta_{y,y}$, where the notation indicates that the two sums are indexed by representatives $y$ of the $F$-conjugacy classes of $G$. So for $g, h \in G$, we have

$$\begin{aligned}
\psi_G^F(g, h) &= |\{x, w, v, u) \in G \times F \times F \times F : ugu^{-1}vh^{-1}v^{-1}wxw^{-1} = x\}|/|F| \\
&= \sum_{x \in_F G} |\{(w, v, u) \in F \times F \times F : ugu^{-1}vh^{-1}v^{-1}wxw^{-1} = x\}|/|C_F(x)| \\
&= \sum_{x \in_F G} |\{y, v, u) \in [x]_F \times F \times F : ugu^{-1}vh^{-1}v^{-1}y = x\}| \\
&= v_G^F(\operatorname{Tr}_1^F(g) \otimes \operatorname{Tr}_1^F(h^{-1})) \\
&= \sum_{\chi \in_F \operatorname{Irr}(\kappa G)} \chi(\operatorname{Tr}_1^F(g))\chi(\operatorname{Tr}_1^F(h^{-1}))/\chi(1)^2 \\
&= |G|^2 \sum_{\chi \in_F \operatorname{Irr}(\kappa G), uG \subseteq F, vG \subseteq F} \chi(ugu^{-1})\chi(vh^{-1}v^{-1})/\chi(1)^2 \\
&= \sum_{\chi \in_F \operatorname{Irr}(\kappa G), vN_F(\chi) \subseteq F} (|N_F(\chi)|/\chi(1))^2 \chi(ugu^{-1})\chi(\mathrm{v}h^{-1}\mathrm{v}^{-1}).
\end{aligned}$$

Part (b) is thus established. Part (a) may be proved either by showing, similarly, that

$$\varphi_G^F(g) = \omega_G^F(\operatorname{Tr}_1^F(g)) = \sum_{\chi \in \operatorname{Irr}(\kappa G)} |N_F(\chi)|\chi(g)/\chi(1)$$

or else by observing that $\varphi_G^F(g) = \psi_G^F(g, 1)/|F|$.

LEMMA 4. *Given $g, h \in G$, then:*

(a) $|C_F(g)|$ *divides* $\varphi_G^F(g)$;
(b) $|C_F)(g)|\,|C_F(h)|$ *divides* $\psi_G^F(g, h)$.

*Proof.* The proof of Lemma 3 shows that $\varphi_G^F(g)/|C_F(g)| = \omega_G^F([g]_F^+)$, and $\psi_G^F(g, h)/|C_F(g)|\,|C_F(h)| = v_G^F([g]_F^+ \otimes [h^{-1}]_F^+)$. Any rational number belonging to $O$ must be an integer.

We define a symmetric integer matrix $\Psi(G, F) = (\psi_G^F(g, h))_{g,h \in_F G}$ (indexed by representatives of the $F$-conjugacy classes in $G$). For any integer matrix $\Psi$, let $\mathrm{rk}_p\, \Psi$ denote the $p$-rank of $\Psi$ (the rank of the reduction of $\Psi$ modulo $p$).

THEOREM 5. (a) $f_0(G, F) = \mathrm{rk}_p\, \Psi(G, F)$.
(b) $(OG)^F$ *has a projective primitive idempotent if, and only if,* $\varphi_G^F(x)$ *is coprime to p for some* $x \in G$.

*Proof.* Any projective primitive idempotent $b$ of $(OG)^F$ may be written in the form $b = \mathrm{Tr}_1^F(\sum_{g \in G} b_g g)$ with each $b_g \in O$. Given another projective primitive idempotent $b'$ of $(OG)^F$, the proof of Lemma 3 gives

$$\psi_G^F\left(\sum_{g \in G} b_g g \otimes \sum_{h \in G} b'_h h^{-1}\right) = v_G^F(b \otimes b') = \omega_G^F(bb')$$

which is zero when $b \neq b'$. But $\omega_G^F(b) = 1$ because $b = e_\chi^F$ for some $\chi \in \mathrm{Irr}_0(\kappa G)$. So $f_0(G, F) \leqslant \mathrm{rk}_p\, \Psi_G^F$. The proof of Lemma 3 also shows that

$$\psi_G^F(g, h) = \omega_G^F(\mathrm{Tr}_1^F(g)\, \mathrm{Tr}_1^F(h^{-1}))$$

for $g, h \in G$. But $f_0(G, F)$ is the $O$-rank of $(OG)^F b_G^F$, so Lemma 2 forces $f_0(G, F) \geqslant \mathrm{rk}_p\, \Psi_G^F$, establishing part (a).

Part (b) is the assertion that $\psi_G^F(g, h)$ is coprime to $p$ for some $g, h \in G$ if and only if $\varphi_G^F(x)$ is coprime to $p$ for some $x \in G$. The forwards implication holds because $\psi_G^F(g, h) = \varphi_G^F(g\, \mathrm{Tr}_1^F(h^{-1}))$ for all $g, h \in G$. Proposition 1 and Lemma 3(a) give the reverse implication.

Thanks to Lemma 4(b), $f_0(G, F)$ is, in fact, the $p$-rank of the submatrix of $\Psi(G, F)$ indexed by those representatives $g$ such that $C_F(g)$ is a $p'$-group.

COROLLARY 6. *Let P be a p-subgroup of G. Define* $\bar{C} = PC_G(P)/P$ *and* $\bar{N} = N_G(P)/P$. *Then*:

(a) *the number of p-blocks of G with defect group P is* $\mathrm{rk}_p\, \Psi(\bar{C}, \bar{N})$;
(b) *P is a defect group of a p-block of G if, and only if, there exists some* $g \in \bar{C}$ *such that p does not divide the number of solutions in* $x \in \bar{C}$ *and* $w \in \bar{N}$ *to the equation* $g = [x, w]$.

*Proof.* This is immediate from Proposition 1, Theorem 5, and Brauer's extended first main theorem.

Putting $G = F$ in Theorem 5, or $P = 1$ in Corollary 6, we deduce that $f_0(G)$ is the $p$-rank of the symmetric integer matrix $\Psi(G) := \Psi(G, G)$ indexed by the representatives of the conjugacy classes of $G$. We note, as above, that to calculate $f_0(G)$, we need only consider the submatrix indexed by the representatives of the defect-zero conjugacy classes. Furthermore, we recover the special case of Strunkov [**6**, Theorem 1] asserting that $G$ has a defect-zero $p$-block if, and only if, there exists an element $g \in G$ such that $p$ does not divide the number of ways of expressing $g$ as a commutator $[x, w]$ with $x, w \in G$.

We end by observing another relationship between the character $\varphi_G = \varphi_G^G$ and the number of defect-zero $p$-blocks of $G$. Let us fix a Sylow $p$-subgroup $S$ of $G$. We write $(\mathrm{Res}_S^G(\varphi_G), 1_S)$ to denote the multiplicity of the trivial $\kappa S$-module in the restriction of $\varphi_G$ to $S$. Note that

$$(\mathrm{Res}_S^G(\varphi_G), 1_S) = |\{(x, w) \in G \times G : [x, w] \in S\}| / |S|.$$

PROPOSITION 7. *Modulo p, we have a congruence*

$$|G : S|\, f_0(G) \equiv_p (\mathrm{Res}_S^G(\varphi_G), 1_S).$$

*Proof.* For any $\zeta \in ZOG$, Lemma 3(a) gives

$$\varphi_G(\zeta) = |G| \sum_{\chi \in \mathrm{Irr}(\kappa G)} \omega_\chi(\zeta).$$

In particular, $|G|$ divides $\psi_G(\zeta)$. Let $G_p$ denote the set $p$-elements of $G$. By Iizuka Watanabe [4, Lemmas 3 and 4], if $\chi \in \mathrm{Irr}_0(\kappa G)$ then $\omega_\chi(G_p^+) = 1$, otherwise $p$ divides $\omega_\chi(G_p^+)$. So

$$f_0(G) \equiv_p \varphi_G(G_p^+)/|G|.$$

Let $N = N_G(S)$. A well-known variant of Sylow's theorem asserts that the number of Sylow $p$-subgroups of $G$ containing any given $p$-subgroup is congruent to unity modulo $p$. Applying this to the cyclic $p$-subgroups, we deduce that $|G : N|$ is congruent to unity modulo $p$, and so too is the coefficient in $\mathrm{Tr}_N^G(S^+)$ of each $p$-element. Hence $(\mathrm{Tr}_N^G(S^+) - G_p^{\mp p})/p \in ZOG$, and $p|G|$ divides $\varphi_G(\mathrm{Tr}_N^G(S^+) - G_p^+)$. Therefore

$$\varphi_G(G_p^+)/|S| \equiv_p \varphi_G(\mathrm{Tr}_N^G(S^+))/|S| \equiv_p \varphi_G(S^+)/|S| = (\mathrm{Res}_S^G(\varphi_G), 1_S).$$

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