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Results in Mathematics

Existence of Basis in Some Whitney Function Spaces

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Abstract

It has been shown that some spaces of Whitney functions on Cantor-type sets have bases.
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I Introduction

In this paper we will be interested in the existence of basis in certain nuclear Fréchet spaces. By a basis we mean a Schauder basis, and whether every nuclear Fréchet space has a basis had been a long standing open problem. After Mityagin and Zobin [?] answered this question in the negative, the related open problem now is the following: are there nuclear Fréchet function spaces without basis? It is believed that if there is such a space one of the candidates is the space of Whitney functions $\mathcal{E}(K)$. In this article we consider two cases of $\mathcal{E}(K)$ and show that they have basis. However our results are existence results. We do not have any results about the form of the basis. But in [?] Goncharov has given the actual construction of a basis in the space, where the Cantor-type set is such that $N_n = N$ for all n.

II Preliminaries

$\mathcal{E}(K)$ and Extension Property

Let $\mathcal{E}(K)$ denote the space of Whitney functions on a perfect compact set $K \subset \mathbb{R}$ with the topology defined by the norms

$$\|f\|_q = |f|_q + \sup\{|(R_y^q f)^{(j)}(x)| \cdot |x-y|^{j-q} : x, y \in K, x \neq y, j \leq q\},\ q = 0, 1, \cdots,$$

where

$$|f|_q = \sup\{|f^{(j)}(x)| : x \in K, j \le q\} \text{ and } R_y^q f(x) = f(x) - T_y^q f(x)$$

is the Taylor remainder. Each function $f \in \mathcal{E}(K)$ is extendable to a C^∞-function on the line. These spaces are Fréchet and quotients of s, the space of rapidly decreasing sequences, hence they are also nuclear. If there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R})$, then we say that the compact set K has the *extension property.* In [?] Tidten applied Vogt's characterization for the splitting of exact sequences of Fréchet spaces and showed that the extension property of K and the dominated norm property (property DN) of the space $\mathcal{E}(K)$ are equivalent.

A Fréchet space X with a fundamental system of seminorms $(\|\cdot\|_q)$ is said to have the property (DN) ([?]) if

$$\exists p\, \forall q\, \exists r, C > 0 : \ \|\cdot\|_q \le t\,\|\cdot\|_p + \frac{C}{t}\|\cdot\|_r, \quad t > 0.$$

Here $p, q, r \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

Cantor Type Sets. We consider the following generalization of the classical Cantor set. Let $I_0 = [0,1]$, $(N_n)_1^\infty$ be a sequence of positive integers with $N_n \ge 2$, and $(l_n)_0^\infty$ be a sequence of positive real numbers such that

$$l_0 = 1, \ N_n \cdot l_n < l_{n-1} \text{ for all } n \ge 1.$$

At the n^{th} step we delete $N_1 \cdot N_2 \cdots N_{n-1} \cdot N_n$ open intervals of length $h_n = \frac{l_{n-1} - N_n \cdot l_n}{N_n - 1}$ from each interval in such a way that all the remaining closed intervals have length l_n. We call the compact set obtained at n^{th} step I_n. Then we set

$$K = K((l_n), (N_n)) = \bigcap_{n=1}^{\infty} I_n.$$

The resulting compact set is called a Cantor type set. Let α_n be defined by $l_1 = 1/e^{\alpha_1}$ and $l_n = l_{n-1}^{\alpha_n}$ for $n \ge 2$. Then $\alpha_n > 1$ for $n \ge 2$ and $l_n = e^{-\alpha_1 \dots \alpha_n}$ for all $n \ge 1$.

We will consider mainly the following special cases for K; $N_n = N$ for all n for some fixed $N \ge 2$ in $\mathbb{N}$ and $N_n \nearrow \infty$ as $n \to \infty$. Then corresponding compact sets will be denoted by K_N and K_∞ respectively.

The following theorem has been proved in [?].
Theorem 1.
a) If $\limsup \alpha_n < N$ then the space $\mathcal{E}(K_N)$ has the property (DN).
b) The space $\mathcal{E}(K_\infty)$ has property (DN) if and only if there is a constant M such that $l_n \ge h_n^M$ for all n.

Diametral Dimension of $\mathcal{E}(K)$. Approximative and diametral dimensions, introduced by Kolmogorov [?], Pełczyński [?] and Bessaga, Pełczyński and Rolewicz [?], were the first linear topological invariants applicable to isomorphic classification of nonnormed Fréchet spaces. We follow the notation of [?].

Let X be a Fréchet space with a fundamental system of neighborhoods (U_q), let

$$d_n(U_q, U_p) = \inf\{\inf\{d > 0 : U_q \subset d\, U_p + L\} : \dim L \le n\}$$

denote the n-th Kolmogorov diameter of U_q with respect to U_p. Then

$$\Gamma(X) = \{(\gamma_n)_{n=0}^{\infty} : \forall p\, \exists q :\ \gamma_n \cdot d_n(U_q, U_p) \to 0 \text{ as } n \to \infty\}.$$

We will consider the counting function corresponding to the diametral dimension

$$\beta(t) = \beta(U_p, U_q, t) = \min\{\dim L : t \cdot U_q \subset U_p + L\}, \quad t > 0.$$

Clearly, the diametral dimension can be characterized in terms of β in the following way.

Proposition. $(\gamma_n) \in \Gamma(X)$ if and only if

$$\forall p\, \exists q\, \forall C\, \exists n_0 :\ \beta(U_p, U_q, C|\gamma_n|) \leq n \text{ for } n \geq n_0.$$

The following two theorems are of crucial importance in this paper. The first one is due to Arslan, Goncharov and Kocatepe [?] and it allows us to calculate the counting function β for the neighborhoods in the space $X = \mathcal{E}(K)$ with $K = K((l_n), (N_n))$.

Theorem 2. Let $X = \mathcal{E}(K)$ with $K = K((l_n), (N_n))$, let p and q, $p < q$ be fixed natural numbers. If $t \leq \frac{1}{5} l_n^{p-q}$, then $\beta(U_p, U_q, t) \leq (q+1)N_1 \cdots N_n$. If $t \geq 5(q-p)!\, l_n^{p-q}$, then $\beta(U_p, U_q, t) \geq N_1 \cdots N_n$.

The second theorem is due to Aytuna, Krone and Terzioğlu and it gives a sufficient condition for the existence of a basis. First we recall the definition of infinite type power series space $\Lambda_\infty(\beta)$. Given a sequence $\beta = (\beta_n)$ of real numbers such that $0 < \beta_n \nearrow \infty$, $\Lambda_\infty(\beta)$ is defined as

$$\Lambda_\infty(\beta) = \{x = (x_n) : \|x\|_k = \sum_{n=1}^{\infty} |x_n| e^{k\beta_n} < \infty,\ k = 0, 1, 2, \dots\}.$$

We note that it is well-known that

$$\Gamma(\Lambda_\infty(\beta)) = \left\{(\gamma_n) : \exists k\, \exists C > 0 :\ |\gamma_n| \leq C e^{k\beta_n},\ \text{all } n\right\}.$$

Theorem 3. Let X be a Fréchet space with property (DN) and assume that X is isomorphic to a quotient space of s. Also assume $\Gamma(X) = \Gamma(\Lambda_\infty(\beta))$ for some $\beta = (\beta_n)$ where $\Lambda_\infty(\beta)$ is stable, i.e. $\sup_n \frac{\beta_{2n}}{\beta_n} < \infty$. Then the space X has a basis.
The proof follows immediately from Lemma 2.3 and Corollary 1.5 in [?].

III Main Result

We note that the second part of the following theorem gives a partial answer to the question in [?].

Theorem 4. **(a)** If $\limsup_{n\to\infty} \alpha_n < N$, then the space $\mathcal{E}(K_N)$ has a basis.
(b) If $\mathcal{E}(K_\infty)$ has property (DN) and $\limsup_{n\to\infty} \alpha_n < \infty$, then the space $\mathcal{E}(K_\infty)$ has a basis.

Proof. The proofs of the two parts are similar. So we shall prove the first part in detail and indicate the changes in the second part.

Define $A(n) = \alpha_1 \dots \alpha_n$ and extend A to the interval $(n, n+1)$ in such a way such that $A(x)$ is strictly increasing and continuous on the interval $[1, \infty)$. Let $S(x) = x$ (in the second part $S(x)$ will be different).

Let $p < q$ be fixed. Then

$$0 \le \frac{1}{l_1^{q-p}} \le \frac{1}{l_2^{q-p}} \le \dots \le \frac{1}{l_{n-1}^{q-p}} \le \frac{1}{l_n^{q-p}} \le \dots \text{ and } \frac{1}{l_n^{q-p}} \to \infty \text{ as } n \to \infty$$

Since $l_{n+1}/l_n < 1/N \le 1/2$, we have $l_{n+m}/l_n < 1/2^m$. So

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{l_{n+m}}{l_n} = 0.$$

We fix m such that $\limsup_{n\to\infty}(l_{n+m}/l_n) < 1/(10(q-p)!)$ and find n_0 such that for all $n \ge n_0$ we have $l_{n-1}/l_{n-1-m} \le 1/(5(q-p)!)$. Then

$$5(q-p)! \, \frac{1}{l_{n-1-m}^{q-p}} \le \frac{1}{l_{n-1}^{q-p}} \text{ and } \frac{1}{l_n^{p-q}} \le \frac{1}{5} \frac{1}{l_{n+m}^{q-p}}, n \ge n_0. \tag{1}$$

Let $t > 0$ be large enough (say $t \ge \frac{1}{l_{n_0-1}^{q-p}} =: t_0$). Find a unique $n \ge n_0$ such that

$$\frac{1}{l_{n-1}^{q-p}} \le t < \frac{1}{l_n^{p-q}} \tag{2}$$

Then by (??)

$$5(q-p)! \, \frac{1}{l_{n-1-m}^{q-p}} \le t \le \frac{1}{5} \frac{1}{l_{n+m}^{q-p}} \tag{3}$$

On the other hand, (??) holds if and only if

$$e^{(q-p)\alpha_1 \dots \alpha_{n-1}} \le t < e^{(q-p)\alpha_1 \dots \alpha_n}$$

which is equivalent to

$$A^{-1}\left(\frac{\ln t}{q-p}\right) < n \le A^{-1}\left(\frac{\ln t}{q-p}\right) + 1 \tag{4}$$

Now by Theorem 2, (??) implies

$$N^{S(n-1-m)} \le \beta\left(U_p, U_q, t\right) \le (q+1)\, N^{S(n+m)}. \tag{5}$$

This last inequality combined with (??) gives

$$N^{S\left(A^{-1}\left(\frac{\ln t}{q-p}\right)-1-m\right)} \le \beta\left(U_p, U_q, t\right) \le (q+1) \cdot N^{S\left(A^{-1}\left(\frac{\ln t}{q-p}\right)+1+m\right)} \text{ for } t \ge t_0. \tag{6}$$

Define $\beta = (\beta_n)$ by $\beta_n = A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right)$. Now we show that $\Gamma(\mathcal{E}(K_N)) = \Gamma(\Lambda_\infty(\beta))$. For every n, choose an integer $k = k(n)$ such that $k \le S^{-1}\left(\frac{\ln n}{\ln N}\right) < k+1$. Then

$$\alpha_1 \dots \alpha_k \le \beta_n < \alpha_1 \dots \alpha_k \cdot \alpha_{k+1}.$$

Let now $(\gamma_n) \in \Gamma\left(\mathcal{E}(K_N)\right)$. Then

$$\forall p\, \exists q\, \forall C\, \exists n_1\, \forall n \geq n_1 : \quad \beta\left(U_p, U_q, C|\gamma_n|\right) \leq n.$$

Assume $n \geq n_1$ and $C|\gamma_n| \geq t_0$. Then the left hand inequality in (??) implies

$$N^{S\left(A^{-1}\left(\frac{\ln(C|\gamma_n|)}{q-p}\right)-1-m\right)} \leq n$$

which is equivalent to

$$C|\gamma_n| \leq e^{(q-p)\cdot A\left(S^{-1}\left(\frac{\ln n}{\ln N}+1+m\right)\right)}.$$

¿From $k \leq S^{-1}\left(\frac{\ln n}{\ln N}\right) < k+1$, it follows that

$$\alpha_1 \ldots \alpha_{k+1+m} \leq A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)+1+m\right) < \alpha_1 \ldots \alpha_{k+2+m}.$$

Thus;

$$1 < \frac{A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)+1+m\right)}{\beta_n} < \alpha_{k+1} \ldots \alpha_{k+2+m}.$$

Since (α_n) is bounded, the sequences $\left(A\left(S^{-1}\left(\frac{\ln n}{\ln N}+1+m\right)\right)\right)$ and (β_n) are equivalent. Thus there exist M and $D > 0$ such that $|\gamma_n| \leq De^{M\beta_n}$ for all n, that is $(\gamma_n) \in \Gamma(\Lambda_\infty(\beta))$.

Conversely assume $(\gamma_n) \in \Gamma(\Lambda_\infty(\beta))$. Then there exists $M > 0$ such that $|\gamma_n| \leq e^{M\beta_n}$ for large n. We show that $(\gamma_n) \in \Gamma(\mathcal{E}(K_N))$, i.e.,

$$\forall p\, \exists q\, \forall C : \; \beta(U_p, U_q, C|\gamma_n|) \leq n, \text{ large } n.$$

Given n, as before let $k = k(n)$ be such that $k \leq S^{-1}\left(\frac{\ln n}{\ln N}\right) < k+1$. We find α such that

$$\limsup_n \alpha_n < \alpha < N. \tag{7}$$

Then $\alpha_n \leq \alpha$ for large n. Since $\lim_{q\to\infty} \frac{q}{(q+1)^{\frac{\ln\alpha}{\ln N}}} = +\infty$, we have an integer q_0 such that

$$2(M+1)\alpha^{m+3} \leq \frac{q}{(q+1)^{\frac{\ln\alpha}{\ln N}}},\; q \geq q_0. \tag{8}$$

Given p, choose q such that $q \geq 2p$ and $q \geq q_0$. Then $q/2 \leq q-p$ and (??) holds. Let $C > 0$ be given. Since $(q+1)^{\frac{\ln\alpha}{\ln N}} = \alpha^{\frac{\ln(q+1)}{\ln N}}$, (??) is equivalent to

$$(M+1)\alpha^{m+3+\frac{\ln(q+1)}{\ln N}} \leq \frac{q}{2}. \tag{9}$$

Choose an integer μ such that $N^\mu \leq q+1 < N^{\mu+1}$. Then $\mu \leq \frac{\ln(q+1)}{\ln N}$. So (??) implies

$$(M+1)\alpha^{m+3+\mu} \leq q-p$$

and therefore

$$(M+1)\alpha_{k-\mu-m-1} \cdots \alpha_{k+1} \leq q-p$$

which implies

$$(M+1)\alpha_1 \dots \alpha_{k+1} \leq (q-p)\alpha_1 \dots \alpha_{k-\mu-m-2}$$

that is

$$(M+1)A(k+1) \leq (q-p)A(k-\mu-m-2).$$

Since $\frac{\ln(q+1)}{\ln N} < \mu + 1$, we have

$$k-\mu-m-2 = k-(\mu+1)-1-m \leq S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N} - 1 - m.$$

Thus

$$\begin{aligned}(M+1)A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right) &\leq (M+1)A(k+1) \\ &\leq (q-p)A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1}{\ln N} - 1 - m\right)\end{aligned}$$

Also $C \leq e^{\beta_n}$ for large n, so $C|\gamma_n| \leq e^{(M+1)\beta_n}$ from which it follows that

$$C|\gamma_n| \leq e^{(q-p)\cdot A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N} - 1 - m\right)}$$

which is equivalent to

$$A^{-1}\left(\frac{\ln(C|\gamma_n|)}{q-p}\right) + 1 + m \leq S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N}. \tag{10}$$

Since $S(x-a) \leq S(x) - a$ for all large x, we have

$$S\left(S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N}\right) \leq \frac{\ln n}{\ln N} - \frac{\ln(q+1)}{\ln N}$$

for all large n. So

$$S\left(A^{-1}\left(\frac{\ln(C|\gamma_n|)}{q-p}\right) + 1 + m\right) \leq \frac{\ln n}{\ln N} - \frac{\ln(q+1)}{\ln N},$$

which implies

$$N^{S\left(A^{-1}\left(\frac{\ln(C|\gamma_n|)}{q-p}\right)+1+m\right)} \leq N^{\frac{\ln n}{\ln N} - \frac{\ln(q+1)}{\ln N}} = \frac{n}{q+1}.$$

So by (??) we get $\beta\left(U_p, U_q; C|\gamma_n|\right) \leq n$ for large n, which means by the Proposition that $(\gamma_n) \in \Gamma(\mathcal{E}(K_N))$.

Next we show that (β_{2n}/β_n) is bounded. Since $S^{-1}(1+x) \leq 1 + S^{-1}(x)$ for all large x, with $k = k(n)$,

$$S^{-1}\left(\frac{\ln(2n)}{\ln N}\right) \leq S^{-1}\left(1 + \frac{\ln n}{\ln N}\right) \leq 1 + S^{-1}\left(\frac{\ln n}{\ln N}\right) < 1 + (k+1).$$

Thus $\beta_{2n} < \alpha_1 \cdots \alpha_k \cdot \alpha_{k+1} \cdot \alpha_{k+2}$, so

$$1 \leq \frac{\beta_{2n}}{\beta_n} \leq \alpha_{k+1} \cdot \alpha_{k+2}.$$

Since (α_n) is bounded, (β_{2n}/β_n) is also bounded. By Theorem 1, $\limsup \alpha_n < N$ implies $\mathcal{E}(K_N)$ has (DN). So by Theorem 3 the space $\mathcal{E}(K_N)$ has a basis.

Next we consider the case of $\mathcal{E}(K_\infty)$ and indicate the places where the proof differs from the previous one. In this case clearly $\lim_{n\to\infty} l_{n+1}/l_n = 0$, so we have $m = 1$.

Let $N := \min\{N_i : N_i > \limsup_{n\to\infty} \alpha_n\}$ and let T_n be defined by $N_n = N^{T_n}$. Then $T_n \nearrow \infty$, and $N_1 \cdot N_2 \cdots N_n = N^{T_1+T_2+\cdots+T_n}$. Then we define $S(n) = T_1 + T_2 + \cdots + T_n$, $n \in \mathbb{N}$. We have

$$S(n) \nearrow \infty, \text{ and } S(k+1) - S(k) = T_{k+1} \nearrow \infty.$$

We extend S to $(n, n+1)$ in such a way that $S(x)$ is strictly increasing and continuous on the interval $[1, \infty)$ and

$$S(x+1) - S(x) \nearrow \infty \text{ as } x \nearrow \infty.$$

Then for a constant c, $S(x-c) \leq S(x) - c$ and $S^{-1}(1+x) \leq 1 + S^{-1}(x)$ both hold for all large x. N, as chosen above, satisfies $\limsup_{n\to\infty} \alpha_n < N$, thus (??) can also be satisfied. Then the previous proof can be repeated.

□

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