Toward a Generalized Shapiro and Shapiro Conjecture

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To my teacher Oleg Viro on his 60th birthday

Abstract We obtain a new, asymptotically better, bound $g \leqslant \frac{1}{4}d^2 + O(d)$ on the genus of a curve that may violate the generalized total reality conjecture. The bound covers all known cases except $g = 0$ (the original conjecture).

Keywords Shapiro and Shapiro conjecture • Real variety • Discriminant form • Alexander module

1 Introduction

The original (rational) total reality conjecture suggested by B. and M. Shapiro in 1993 states that if all flattening points of a regular curve $\mathbb{P}^1 \to \mathbb{P}^n$ belong to the real line $\mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1$, then the curve can be made real by an appropriate projective transformation of $\mathbb{P}^n$. (The *flattening points* are the points in the source $\mathbb{P}^1$ where the first n derivatives of the map are linearly dependent. In the case $n = 1$, a curve is a meromorphic function, and the flattening points are its critical points.) There are quite a few interesting and not always straightforward restatements of this conjecture, in terms of the Wronsky map, Schubert calculus, dynamical systems, etc.

Although supported by extensive numerical evidence, the conjecture proved extremely difficult to settle. It was not before 2002 that the first result appeared, due to Eremenko and Gabrielov [4], settling the case $n = 1$, i.e., meromorphic functions on $\mathbb{P}^1$. Later, a number of sporadic results were announced, and the conjecture was

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proved in full generality in 2005 by Mukhin et al.; see [6]. The proof, revealing a deep connection between Schubert calculus and the theory of integrable systems, is based on the Bethe ansatz method in the Gaudin model.

In the meanwhile, a number of generalizations of the conjecture were suggested. In this paper, we deal with one of them, see [3] and Problem 1.1 below, replacing the source $\mathbb{P}^1$ with an arbitrary compact complex curve (however, restricting n to 1, i.e., to the case of meromorphic functions). Due to the lack of evidence, the authors chose to state the assertion as a problem rather than a conjecture.

Recall that a *real variety* is a complex algebraic (analytic) variety X supplied with a *real structure*, i.e., an antiholomorphic involution $c\colon X \to X$. Given two real varieties (X,c) and (Y,c'), a regular map $f\colon X \to Y$ is called *real* if it commutes with the real structures: $f \circ c = c' \circ f$.

Problem 1.1 (see [3]). Let (C,c) be a real curve and let $f\colon C \to \mathbb{P}^1$ be a regular map such that

1. All critical points and critical values of f are distinct;
2. All critical points of f are real.

Is it true that f is real with respect to an appropriate real structure in $\mathbb{P}^1$?

The condition that the critical points of f be distinct includes, in particular, the requirement that each critical point be simple, i.e., have ramification index 2.

A pair of integers $g \geqslant 0$, $d \geqslant 1$ is said to have the *total reality property* if the answer to Problem 1.1 is affirmative for any curve C of genus g and map f of degree d. At present, the total reality property is known for the following pairs (g,d):

- $(0,d)$ for any $d \geqslant 1$ (the original conjecture; see [4]);
- (g,d) for any $d \geqslant 1$ and $g > G_1(d) := \frac{1}{3}(d^2 - 4d + 3)$; see [3];
- (g,d) for any $g \geqslant 0$ and $d \leqslant 4$; see [3] and [1].

The principal result of the present paper is the following theorem.

Theorem 1.2. *Any pair (g,d) with $d \geqslant 1$ and g satisfying the inequality*

$$g > G_0(d) := \begin{cases} k^2 - 2k, & \text{if } d = 2k \text{ is even,} \\ k^2 - \dfrac{10}{3}k + \dfrac{7}{3}, & \text{if } d = 2k-1 \text{ is odd} \end{cases}$$

has the total reality property.

Remark 1.3. Note that one has $G_0(d) - G_1(d) \leqslant -\frac{1}{3}(k-1)^2 \leqslant 0$, where $k = [\frac{1}{2}(d+1)]$. Theorem 1.2 covers the values $d = 2,3$ and leaves only $g = 0$ for $d = 4$, reducing the generalized conjecture to the classical one. The new bound is also asymptotically better: $G_0(d) = \frac{1}{4}d^2 + O(d) < G_1(d) = \frac{1}{3}d^2 + O(d)$.

1.1 Content of the Paper

In Sect. 2, we outline the reduction of Problem 1.1 to the question of existence of certain real curves on the ellipsoid and restate Theorem 1.2 in the new terms; see Theorem 2.4. In Sect. 3, we briefly recall V. V. Nikulin's theory of discriminant forms and lattice extensions. In Sect. 4, we introduce a version of the Alexander module of a plane curve suited to the study of the resolution lattice in the homology of the double covering of the plane ramified at the curve. Finally, in Sect. 5, we prove Theorem 2.4 and hence Theorem 1.2.

2 The Reduction

We briefly recall the reduction of Problem 1.1 to the problem of existence of a certain real curve on the ellipsoid. Details can be found in [3].

2.1 The Map Φ

Denote by $\operatorname{conj}\colon z \mapsto \bar{z}$ the standard real structure on $\mathbb{P}^1 = \mathbb{C} \cup \infty$. The *ellipsoid* $\mathbf{E}$ is the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with the real structure $(z, w) \mapsto (\operatorname{conj} w, \operatorname{conj} z)$. (It is indeed the real structure whose real part is homeomorphic to the 2-sphere.)

Let (C, c) be a real curve and let $f\colon C \to \mathbb{P}^1$ be a holomorphic map. Consider the *conjugate map* $\bar{f} = \operatorname{conj} \circ f \circ c\colon C \to \mathbb{P}^1$ and let

$$\Phi = (f, \bar{f})\colon C \to \mathbf{E}.$$

It is straightforward that Φ is holomorphic and real (with respect to the above real structure on $\mathbf{E}$). Hence, the image $\Phi(C)$ is a real algebraic curve in $\mathbf{E}$. (We exclude the possibility that $\Phi(C)$ is a point, for we assume $f \neq \text{const}$; cf. Condition 1.1(1).) In particular, the image $\Phi(C)$ has bidegree (d', d') for some $d' \geqslant 1$.

Lemma 2.1 (see [3]). *A holomorphic map $f\colon C \to \mathbb{P}^1$ is real with respect to some real structure on $\mathbb{P}^1$ if and only if there is a Möbius transformation $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\bar{f} = \varphi \circ f$.* □

Corollary 2.2 (see [3]). *A holomorphic map $f\colon C \to \mathbb{P}^1$ is real with respect to some real structure on $\mathbb{P}^1$ if and only if the image $\Phi(C) \subset \mathbf{E}$ (see above) is a curve of bidegree $(1, 1)$.* □

2.2 The Principal Reduction

Let $p\colon \mathbf{E} \to \mathbb{P}^1$ be the projection to the first factor. In general, the map Φ as above splits into a ramified covering α and a generically one-to-one map β,

$$\Phi\colon C \xrightarrow{\alpha} C' \xrightarrow{\beta} \mathbf{E},$$

so that $d = \deg f = d' \deg\alpha$, where $d' = \deg(p \circ \beta)$, or alternatively, (d', d') is the bidegree of the image $\Phi(C) = \beta(C')$. Then f itself splits into α and $p \circ \beta$. Hence the critical values of f are those of $p \circ \beta$ and the images under $p \circ \beta$ of the ramification points of α. Thus, if f satisfies Condition 1.1(1), the splitting cannot be proper, i.e., either $d = \deg\alpha$ and $d' = 1$ or $\deg\alpha = 1$ and $d = d'$. In the former case, f is real with respect to some real structure on $\mathbb{P}^1$; see Corollary 2.2. In the latter case, assuming that the critical points of f are real, Condition 1.1(2), the image $B = \Phi(C)$ is a curve of genus g with $2g + 2d - 2$ real ordinary cusps (type $\mathbf{A}_2$ singular points, the images of the critical points of f) and all other singularities with smooth branches.

Conversely, let $B \subset \mathbf{E}$ be a real curve of bidegree (d, d), $d > 1$, and genus g with $2g + 2d - 2$ real ordinary cusps and all other singularities with smooth branches, and let $\rho\colon \tilde{B} \to B$ be the normalization of B. Then $f = p \circ \rho\colon \tilde{B} \to \mathbb{P}^1$ is a map that satisfies Conditions 1.1(1) and (2) but is not real with respect to any real structure on $\mathbb{P}^1$; hence, the pair (g, d) does not have the total reality property.

As a consequence, we obtain the following statement.

Theorem 2.3 (see [3]). *A pair (g, d) has the total reality property if and only if there does* not *exist a real curve $B \subset \mathbf{E}$ of degree d and genus g with $2g + 2d - 2$ real ordinary cusps and all other singularities with smooth branches.* □

Thus, Theorem 1.2 is equivalent to the following statement, which is actually proved in the paper.

Theorem 2.4. *Let $\mathbf{E}$ be the ellipsoid, and let $B \subset \mathbf{E}$ be a real curve of bidegree (d, d) and genus g with $c = 2d + 2g - 2$ real ordinary cusps and other singularities with smooth branches. Then $g \leqslant G_0(d)$; see Theorem 1.2.*

Remark 2.5. It is worth mentioning that the bound $g > G_1(d)$ mentioned in the introduction is purely complex: it is derived from the adjunction formula for the virtual genus of a curve $B \subset \mathbf{E}$ as in Theorem 2.3. In contrast, the proof of the conjecture for the case $(g, d) = (1, 4)$ found in [1] makes essential use of the real structure, since an elliptic curve with eight ordinary cusps in $\mathbb{P}^1 \times \mathbb{P}^1$ does in fact exist! Our proof of Theorem 2.4 also uses the assumption that all cusps are real.

2.3 Reduction to Nodes and Cusps Only

In general, a curve B as in Theorem 2.4 may have rather complicated singularities. However, since the proof below is essentially topological, we follow Yu. Orevkov [9] and perturb B to a real *pseudoholomorphic* curve with ordinary nodes (type $\mathbf{A}_1$) and ordinary cusps (type $\mathbf{A}_2$) only. By the genus formula, the number of nodes of such a curve is

$$n = (d-1)^2 - g - c = d^2 - 4d - 1 - 3g. \tag{1}$$

3 Discriminant Forms

In this section, we cite the techniques and a few results of Nikulin [8]. Most proofs can be found in [8]; they are omitted.

3.1 Lattices

A *lattice* is a finitely generated free abelian group L equipped with a symmetric bilinear form $b\colon L\otimes L \to \mathbb{Z}$. We abbreviate $b(x,y) = x\cdot y$ and $b(x,x) = x^2$. Since the transition matrix between two integral bases has determinant ± 1, the determinant $\det L \in \mathbb{Z}$ (i.e., the determinant of the Gram matrix of b in any basis of L) is well defined. A lattice L is called *nondegenerate* if $\det L \neq 0$; it is called *unimodular* if $\det L = \pm 1$ and *p-unimodular* if $\det L$ is prime to p (where p is a prime).

To fix the notation, we use $\sigma_+(L)$, $\sigma_-(L)$, and $\sigma(L) = \sigma_+(L) - \sigma_-(L)$ for, respectively, the positive and negative inertia indices and the signature of a lattice L.

3.2 The Discriminant Group

Given a lattice L, the bilinear form extends to $L\otimes\mathbb{Q}$. If L is nondegenerate, the dual group $L^* = \operatorname{Hom}(L,\mathbb{Z})$ can be regarded as the subgroup

$$\left\{x \in L\otimes\mathbb{Q} \,\middle|\, x\cdot y \in \mathbb{Z} \text{ for all } x \in L\right\}.$$

In particular, $L \subset L^*$, and the quotient L^*/L is a finite group; it is called the *discriminant group* of L and is denoted by $\operatorname{discr} L$ or $\mathcal{L}$. The group $\mathcal{L}$ inherits from $L\otimes\mathbb{Q}$ a symmetric bilinear form $\mathcal{L}\otimes\mathcal{L} \to \mathbb{Q}/\mathbb{Z}$, called the *discriminant form*; when speaking about the discriminant groups, their (anti-)isomorphisms, etc., we always assume that the discriminant form is taken into account. The following properties are straightforward:

1. The discriminant form is nondegenerate, i.e., the associated homomorphism $\mathcal{L} \to \mathrm{Hom}(\mathcal{L}, \mathbb{Q}/Z)$ is an isomorphism;
2. One has $\#\mathcal{L} = |\det L|$;
3. In particular, $\mathcal{L} = 0$ if and only if L is unimodular.

Following Nikulin, we denote by $\ell(\mathcal{L})$ the minimal number of generators of a finite abelian group $\mathcal{L}$. For a prime p, we denote by $\mathcal{L}_p$ the p-primary part of $\mathcal{L}$ and let $\ell_p(\mathcal{L}) = \ell(\mathcal{L}_p)$. Clearly, for a lattice L one has

4. $\mathrm{rk}\, L \geqslant \ell(\mathcal{L}) \geqslant \ell_p(\mathcal{L})$ (for any prime p);
5. L is p-unimodular if and only if $\mathcal{L}_p = 0$.

3.3 Extensions

An *extension* of a lattice S is another lattice M containing L. All lattices below are assumed nondegenerate.

Let $M \supset S$ be a finite index extension of a lattice S. Since M is also a lattice, one has monomorphisms $S \hookrightarrow M \hookrightarrow M^* \hookrightarrow S^*$. Hence, the quotient $\mathcal{K} = M/S$ can be regarded as a subgroup of the discriminant $\mathcal{S} = \mathrm{discr}\, S$; it is called the *kernel* of the extension $M \supset S$. The kernel is an isotropic subgroup, i.e., $\mathcal{K} \subset \mathcal{K}^\perp$, and one has $\mathcal{M} = \mathcal{K}^\perp/\mathcal{K}$. In particular, in view of Sect. 3.2(1), for any prime p one has

$$\ell_p(\mathcal{M}) \geqslant \ell_p(\mathcal{L}) - 2\ell_p(\mathcal{K}).$$

Now assume that $M \supset S$ is a *primitive* extension, i.e., the quotient M/S is torsion free. Then the construction above applies to the finite index extension $M \supset S \oplus N$, where $N = S^\perp$, giving rise to the kernel $\mathcal{K} \subset \mathcal{S} \oplus \mathcal{N}$. Since both S and N are primitive in M, one has $\mathcal{K} \cap \mathcal{S} = \mathcal{K} \cap \mathcal{N} = 0$; hence, $\mathcal{K}$ is the graph of an anti-isometry κ between certain subgroups $\mathcal{S}' \subset \mathcal{S}$ and $\mathcal{N}' \subset \mathcal{N}$. If M is unimodular, then $\mathcal{S}' = \mathcal{S}$ and $\mathcal{N}' = \mathcal{N}$, i.e., κ is an anti-isometry $\mathcal{S} \to \mathcal{N}$. Similarly, if M is p-unimodular for a certain prime p, then $\mathcal{S}'_p = \mathcal{S}_p$ and $\mathcal{N}'_p = \mathcal{N}_p$, i.e., κ is an anti-isometry $\mathcal{S}_p \to \mathcal{N}_p$. In particular, $\ell(\mathcal{S}) = \ell(\mathcal{N})$ (respectively, $\ell_p(\mathcal{S}) = \ell_p(\mathcal{N})$). Combining these observations with Sect. 3.2(4), we arrive at the following statement.

Lemma 3.1. *Let p be a prime, and let $L \supset S$ be a p-unimodular extension of a nondegenerate lattice S. Denote by $\tilde{S}$ the primitive hull of S in L, and let $\mathcal{K}$ be the kernel of the finite index extension $\tilde{S} \supset S$. Then* $\mathrm{rk}\, S^\perp \geqslant \ell_p(\mathcal{S}) - 2\ell_p(\mathcal{K})$. □

4 The Alexander Module

Here we discuss (a version of) the Alexander module of a plane curve and its relation to the resolution lattice in the homology of the double covering of the plane ramified at the curve.

4.1 The Reduced Alexander Module

Let π be a group, and let $\kappa\colon \pi \twoheadrightarrow \mathbb{Z}_2$ be an epimorphism. Set $K = \operatorname{Ker}\kappa$ and define the *Alexander module* of π (more precisely, of κ) as the $\mathbb{Z}[\mathbb{Z}_2]$-module $A_\pi = K/[K,K]$, the generator t of $\mathbb{Z}_2$ acting via $x \mapsto [\bar{t}^{-1}\bar{x}\bar{t}] \in A_\pi$, where $\bar{t} \in \pi$ and $\bar{x} \in K$ are some representatives of t and x, respectively. (We simplify the usual definition and consider only the case needed in the sequel. A more general version and further details can be found in A. Libgober [7].)

Let $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve of even bidegree $(d,d) = (2k,2k)$, and let $\pi = \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$. Recall that $\pi/[\pi,\pi] = \mathbb{Z}_{2k}$; hence, there is a unique epimorphism $\kappa\colon \pi \twoheadrightarrow \mathbb{Z}_2$. The resulting Alexander module $A_B = A_\pi$ will be called the *Alexander module* of B. The *reduced Alexander module* $\tilde{A}_B$ is the kernel of the canonical homomorphism $A_B \to \mathbb{Z}_k \subset \pi/[\pi,\pi]$. There is a natural exact sequence

$$0 \longrightarrow \tilde{A}_B \longrightarrow A_B \longrightarrow \mathbb{Z}_k \longrightarrow 0 \tag{2}$$

of $\mathbb{Z}[\mathbb{Z}_2]$-modules (where the $\mathbb{Z}_2$-action on $\mathbb{Z}_k$ is trivial). The following statement is essentially contained in Zariski [10].

Lemma 4.1. *The exact sequence* (2) *splits*: *one has $A_B = \tilde{A}_B \oplus \operatorname{Ker}(1-t)$, where t is the generator of $\mathbb{Z}_2$. Furthermore, $\tilde{A}_B$ is a finite group free of 2-torsion, and the action of t on $\tilde{A}_B$ is via the multiplication by (-1).*

Proof. Since A_B is a finitely generated abelian group, to prove that it is finite and free of 2-torsion, it suffices to show that $\operatorname{Hom}_{\mathbb{Z}}(\tilde{A}_B,\mathbb{Z}_2) = 0$. Assume the contrary. Then the $\mathbb{Z}_2$-action in the 2-group $\operatorname{Hom}_{\mathbb{Z}}(\tilde{A}_B,\mathbb{Z}_2)$ has a fixed nonzero element, i.e., there is an equivariant epimorphism $\tilde{A}_B \twoheadrightarrow \mathbb{Z}_2$. Hence, π factors to a group G that is an extension $0 \to \mathbb{Z}_2 \to G \to \mathbb{Z}_{2k} \to 0$. The group G is necessarily abelian, and it is strictly larger than $\mathbb{Z}_{2k} = \pi/[\pi,\pi]$. This is a contradiction.

Since $\tilde{A}_B$ is finite and free of 2-torsion, one can divide by 2, and there is a splitting $\tilde{A}_B = \tilde{A}^+ \oplus \tilde{A}^-$, where $\tilde{A}^\pm = \operatorname{Ker}[(1\pm t)\colon \tilde{A}_B \to \tilde{A}_B]$. Then π factors to a group G that is a central extension $0 \to \tilde{A}^+ \to G \to \mathbb{Z}_{2k} \to 0$, and as above, one concludes that $\tilde{A}^+ = 0$, i.e., t acts on $\tilde{A}_B$ via (-1).

Pick a representative $a' \in A_B$ of a generator of $\mathbb{Z}_k = A_B/\tilde{A}_B$. Then obviously, $(1-t)a' \in \tilde{A}_B$, and replacing a' with $a' + \frac{1}{2}(1-t)a'$, one obtains a t-invariant representative $a \in \operatorname{Ker}(1-t)$. The multiple $ka \in \tilde{A}_B$ is both invariant and skew-invariant; since $\tilde{A}_B$ is free of 2-torsion, $ka = 0$, and the sequence splits. $\square$

4.2 The Double Covering of $\mathbb{P}^1 \times \mathbb{P}^1$

Let $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve of even bidegree $(d,d) = (2k,2k)$ and with simple singularities only. Consider the double covering $X \to \mathbb{P}^1 \times \mathbb{P}^1$ ramified at B and denote by $\tilde{X}$ the minimal resolution of singularities of X. Let $\tilde{B} \subset \tilde{X}$ be the proper transform of B, and let $E \subset \tilde{X}$ be the exceptional divisor contracted by the blowdown $\tilde{X} \to X$.

Recall that the minimal resolution of a simple surface singularity is diffeomorphic to its perturbation; see, e.g., [2]. Hence, $\tilde{X}$ is diffeomorphic to the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified at a nonsingular curve. In particular, $\pi_1(\tilde{X}) = 0$, and one has

$$b_2(X) = \chi(X) - 2 = 8k^2 - 8k + 6, \quad \sigma(X) = -4k^2. \tag{3}$$

4.3 An Estimate on the Discriminant Group

Set $L = H_2(\tilde{X})$. We regard L as a lattice via the intersection index pairing on $\tilde{X}$. (Since $\tilde{X}$ is simply connected, L is a free abelian group. It is a unimodular lattice by Poincaré duality.) Let $\Sigma \subset L$ be the sublattice spanned by the components of E, and let $\tilde{\Sigma} \subset L$ be the primitive hull of Σ. Recall that Σ is a negative definite lattice. Further, let $h_1, h_2 \subset L$ be the classes of the pullbacks of a pair of generic generatrices of $\mathbb{P}^1 \times \mathbb{P}^1$, so that $h_1^2 = h_2^2 = 0$, $h_1 \cdot h_2 = 2$.

Lemma 4.2. *If a curve B as above is irreducible, then there are natural isomorphisms* $\tilde{A}_B = \operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Q}/\mathbb{Z}) = \operatorname{Ext}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Z})$*, where $\mathcal{K}$ is the kernel of the extension* $\tilde{\Sigma} \supset \Sigma$.

Proof. One has $A_B = H_1(\tilde{X} \smallsetminus (\tilde{B} + E))$ as a group, the $\mathbb{Z}_2$-action being induced by the deck translation of the covering. Hence, by Poincaré–Lefschetz duality, A_B is the cokernel of the inclusion homomorphism $i^* \colon H^2(\tilde{X}) \to H^2(\tilde{B} + E)$.

On the other hand, there is an orthogonal (with respect to the intersection index form in $\tilde{X}$) decomposition $H_2(\tilde{B} + E) = \Sigma \oplus \langle b \rangle$, where $b = k(h_1 + h_2)$ is the class realized by the divisorial pullback of B in $\tilde{X}$. The cokernel of the restriction $i^* \colon H^2(X) \to \langle b \rangle^*$ is a cyclic group $\mathbb{Z}_k$ fixed by the deck translation. Hence, in view of Lemma 4.1,

$$\tilde{A}_B = \operatorname{Coker}[i^* \colon H^2(\tilde{X}) \to H^2(E)] = \operatorname{Coker}[L^* \to \Sigma^*] = \operatorname{discr} \Sigma / \mathcal{K}^\perp.$$

(We use the splitting $L^* \twoheadrightarrow \tilde{\Sigma}^* \to \Sigma^*$, the first map being an epimorphism, since $L/\tilde{\Sigma}$ is torsion free.) Since the discriminant form is nondegenerate (see Sect. 3.2(1)), one has $\operatorname{discr} \Sigma / \mathcal{K}^\perp = \operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Q}/\mathbb{Z})$.

Since $\mathcal{K}$ is a finite group, applying the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}, \cdot)$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, one obtains an isomorphism $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Q}/\mathbb{Z}) = \operatorname{Ext}_{\mathbb{Z}}(\mathcal{K}, \mathbb{Z})$. $\square$

Corollary 4.3. *In the notation of Lemma 4.2, if B is irreducible and the group $\pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$ is abelian, then $\mathcal{K} = 0$.* $\square$

Corollary 4.4. *In the notation of Lemma 4.2, if B is an irreducible curve of bidegree (d,d), $d = 2k \geqslant 2$, then $\mathcal{K}$ is free of 2-torsion and $\ell(\mathcal{K}) \leqslant d-2$.*

Proof. Due to Lemma 4.2, one can replace $\mathcal{K}$ with $\tilde{A}_B$. Then the statement on the 2-torsion is given by Lemma 4.1, and it suffices to estimate the numbers $\ell_p(\tilde{A}_B) = \ell(\tilde{A}_B \otimes \mathbb{Z}_p)$ for odd primes p.

Due to the Zariski–van Kampen theorem [5] applied to one of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, there is an epimorphism $\pi_1(L \smallsetminus B) = F_{d-1} \twoheadrightarrow \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$, where L is a generic generatrix of $\mathbb{P}^1 \times \mathbb{P}^1$ and F_{d-1} is the free group on $d-1$ generators. Hence, A_B is a quotient of the Alexander module

$$A_{F_{d-1}} = \mathbb{Z}[\mathbb{Z}_2]/(t-1) \oplus \bigoplus\nolimits_{d-2} \mathbb{Z}[\mathbb{Z}_2].$$

For an odd prime p, there is a splitting $A_{F_{d-1}} \otimes \mathbb{Z}_p = A_p^+ \oplus A_p^-$ (over the field $\mathbb{Z}_p$) into the eigenspaces of the action of $\mathbb{Z}_2$, and due to Lemma 4.1, the group $\tilde{A}_B \otimes \mathbb{Z}_p$ is a quotient of $A_p^- = \bigoplus_{d-2} \mathbb{Z}_p$. $\square$

Remark 4.5. All statements in this section hold for pseudoholomorphic curves as well; cf. Sect. 2.3. For Corollary 4.4, it suffices to assume that B is a small perturbation of an algebraic curve of bidegree (d,d). Then one still has an epimorphism $F_{d-1} \twoheadrightarrow \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B)$, and the proof applies literally.

5 Proof of Theorem 1.2

As explained in Sect. 2, it suffices to prove Theorem 2.4. We consider the cases of d even and d odd separately.

5.1 Preliminary Observations

Let $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve of even bidegree (d,d), $d = 2k$. Assume that all singularities of B are simple and let $\tilde{X}$ be the minimal resolution of singularities of the double covering $X \to \mathbb{P}^1 \times \mathbb{P}^1$ ramified at B; cf. Sect. 4.2. As in Sect. 4.3, consider the unimodular lattice $L = H_2(\tilde{X})$.

Let $c\colon \tilde{X} \to \tilde{X}$ be a real structure on $\tilde{X}$, and denote by $L^{\pm}$ the (± 1)-eigenlattices of the induced involution c_* of L. The following statements are well known:

1. $L^{\pm}$ are the orthogonal complements of each other;
2. $L^{\pm}$ are p-unimodular for any odd prime p;
3. One has $\sigma_+(L^+) = \sigma_+(L^-) - 1$.

Since also $\sigma_+(L^+) + \sigma_+(L^-) = \sigma_+(L) = 2k^2 - 4k + 3$, see (3), one arrives at $\sigma_+(L^+) = \sigma_+(L^-) - 1 = (k-1)^2$ and, further, at

$$\operatorname{rk} L^- = (7k^2 - 6k + 5) - \sigma_-(L^+). \tag{4}$$

Remark 5.1. The common proof of Property 5.1(3) uses the Hodge structure. However, there is another (also very well known) proof that also applies to almost complex manifolds. Let $\tilde{X}_{\mathbb{R}} = \operatorname{Fix} c$ be the real part of $\tilde{X}$. Then the normal bundle of $\tilde{X}_{\mathbb{R}}$ in $\tilde{X}$ is i times its tangent bundle; hence, the normal Euler number $\tilde{X}_{\mathbb{R}} \circ \tilde{X}_{\mathbb{R}}$ equals (-1) times the index of any tangent vector field on $\tilde{X}_{\mathbb{R}}$, i.e., $-\chi(\tilde{X}_{\mathbb{R}})$. Now one has $\sigma(L^+) - \sigma(L^-) = \tilde{X}_{\mathbb{R}} \circ \tilde{X}_{\mathbb{R}} = -\chi(\tilde{X}_{\mathbb{R}})$ (by the Hirzebruch G-signature theorem) and $\operatorname{rk} L^+ - \operatorname{rk} L^- = \chi(\tilde{X}_{\mathbb{R}}) - 2$ (by the Lefschetz fixed point theorem). Adding the two equations, one obtains 5.1(3).

5.2 The Case of $d = 2k$ Even

Perturbing, if necessary, B in the class of real pseudoholomorphic curves, see Sect. 2.3, one can assume that all singularities of B are c real ordinary cusps and n ordinary nodes, where

$$c = 2d + 2g - 2 \quad \text{and} \quad n = d^2 - 4d - 1 - 3g; \tag{5}$$

see Theorem 2.3 and (1). Let $n = r + 2s$, where r and s are respectively the numbers of real nodes and pairs of conjugate nodes.

5.3 The Contribution of the Singular Points

Consider the double covering $\tilde{X}$, see Sect. 4.2, lift the real structure on $\mathbf{E}$ to a real structure c on $\tilde{X}$, and let $L^\pm \subset L$ be the corresponding eigenlattices; see Sect. 5.1. In the notation of Sect. 4.3, let $\Sigma^\pm = \Sigma \cap L^\pm$. Then

- Each real cusp of B contributes a sublattice $\mathbf{A}_2$ to Σ^-;
- Each real node of B contributes a sublattice $\mathbf{A}_1 = [-2]$ to Σ^-;
- Each pair of conjugate nodes contributes $[-4]$ to Σ^- and $[-4]$ to Σ^+.

In addition, the classes h_1, h_2 of two generic generatrices of $\mathbf{E}$ span a hyperbolic plane orthogonal to Σ; see Sect. 4.3. It contributes

- A sublattice $[4] \subset L^-$ spanned by $h_1 + h_2$, and
- A sublattice $[-4] \subset L^+$ spanned by $h_1 - h_2$.

(Recall that any real structure reverses the canonical complex orientation of pseudoholomorphic curves.)

5.4 End of the Proof

All sublattices of L^+ described above are negative definite; hence, their total rank $s+1$ contributes to $\sigma_-(L^+)$. The total rank $2c+r+s+1$ of the sublattices of L^- contributes to the rank of $S^- = \Sigma^- \oplus [4] \subset L^-$. Due to (4), one has

$$2c+n+2+\operatorname{rk} S^\perp \leqslant 7k^2-6k+5, \tag{6}$$

where $S^\perp$ is the orthogonal complement of S^- in L^-. All summands of S^- other than $\mathbf{A}_2$ are 3-unimodular, whereas $\operatorname{discr}\mathbf{A}_2$ is the group $\mathbb{Z}_3$ spanned by an element of square $\frac{1}{3}$ mod $\mathbb{Z}$. Let $\tilde{S}^- \supset S^-$ and $\tilde{\Sigma} \supset \Sigma$ be the primitive hulls, and denote by $\mathcal{K}^-$ and $\mathcal{K}$ the kernels of the corresponding finite index extensions; see Sect. 3.3. Clearly, $\ell_3(\mathcal{K}^-) \leqslant \ell_3(\mathcal{K})$, and due to Corollary 4.4 (see also Remark 4.5), one has $\ell_3(\mathcal{K}) \leqslant d-2$. Then, using Lemma 3.1, one obtains $\operatorname{rk} S^\perp \geqslant c-2(d-2)$, and combining the last inequality with (6), one arrives at

$$3c+n-2(d-2) \leqslant 7k^2-6k+3.$$

It remains to substitute the expressions for c and n given by (5) and solve for g to get

$$g \leqslant k^2-2k+\frac{2}{3}.$$

Since g is an integer, the last inequality implies $g \leqslant G_0(2k)$ as in Theorem 2.4.

5.5 The Case of $d = 2k-1$ Odd

As above, one can assume that B has c real ordinary cusps and $n = r+2s$ ordinary nodes; see (5). Furthermore, one can assume that $c > 0$, since otherwise, $g = 0$ and $d = 1$. Then B has a real cusp, and hence a real smooth point P.

Let L_1, L_2 be the two generatrices of $\mathbf{E}$ passing through P. Choose P generic, so that each L_i, $i = 1,2$, intersects B transversally at d points, and consider the real curve $B' = B + L_1 + L_2$ of even bidegree $(2k,2k)$, applying to it the same double covering arguments as above. In addition to the nodes and cusps of B, the new curve B' has $(d-1)$ pairs of conjugate nodes and a real triple (type $\mathbf{D}_4$) point at P (with one real and two complex conjugate branches). Hence, in addition to the classes listed in Sect. 5.3, there are

- $(d-1)$ copies of $[-4]$ in each Σ^+, Σ^- (from the new conjugate nodes),
- A sublattice $[-4] \subset \Sigma^+$ (from the type $\mathbf{D}_4$ point), and
- A sublattice $\mathbf{A}_3 \subset \Sigma^-$ (from the type $\mathbf{D}_4$ point).

Thus, inequality (6) turns into

$$2c+n+2(d-1)+4+2+\operatorname{rk} S^\perp \leqslant 7k^2-6k+5.$$

We will show that $\operatorname{rk} S^{\perp} \geqslant c$. Then, substituting the expressions for c and n, see (5), and solving the resulting inequality in g, one will obtain $g \leqslant G_0(2k-1)$, as required.

5.6 *An Estimate on* $\operatorname{rk} S^{\perp}$

In view of Lemma 3.1, in order to prove that $\operatorname{rk} S^{\perp} \geqslant c$, it suffices to show that $\ell_3(\mathcal{K}) = 0$ (cf. similar arguments in Sect. 5.4).

Perturb B' to a pseudoholomorphic curve B'', keeping the cusps of B' and resolving the other singularities. (It would suffice to resolve the singular points resulting from the intersection $B \cap L_1$.) Then, applying the Zariski–van Kampen theorem [5] to the ruling containing L_1, it is easy to show that the fundamental group $\pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B'')$ is cyclic.

Indeed, let U be a small tubular neighborhood of L_1 in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $L'' \subset U$ be a generatrix transversal to B''. The epimorphism $\pi_1(L_1'' \smallsetminus B'') \twoheadrightarrow \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus B'')$ given by the Zariski–van Kampen theorem factors through $\pi_1(U \smallsetminus B'')$, and the latter group is cyclic.

On the other hand, the new double covering $\tilde{X}'' \to \mathbb{P}^1 \times \mathbb{P}^1$ ramified at B'' is diffeomorphic to $\tilde{X}$, and the diffeomorphism can be chosen identical over the union of a collection of Milnor balls about the cusps of B'. Thus, since $\operatorname{discr} \mathbf{A}_1$ and $\operatorname{discr} \mathbf{D}_4$ are 2-torsion groups, the perturbation does not change $\mathcal{K} \otimes \mathbb{Z}_3$, and Corollary 4.3 (see also Remark 4.5) implies that $\mathcal{K} \otimes \mathbb{Z}_3 = 0$. $\square$

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