On the quantization of the chiral solitonic bag model

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A consistent quantization scheme for the two-flavor chiral solitonic bag model with unequal quark masses is developed employing a propagator formulation.

Recently we developed a quantization scheme for the two-flavor chiral solitonic bag model, employing a propagator formalism.[1] Since the original motive behind that work was to compute mass differences among the members of isospin multiplets, the quarks were taken with unequal masses, yielding a perturbative term proportional to $\Delta M_q I^3$ in the collective Hamiltonian. This work was further extended to include strong *CP* violation into the scheme.[2] In the preceding paper it is pointed out[3] that the quantization proposed in Ref. 1 is incomplete. They propose an alternative quantization scheme, employing the so-called cranking formalism.

What we would like to present in this Comment is that the flaw which marred Ref. 1 can easily be cured in the framework of the original formulation without any need to resort to alternative formulations.

The two-flavor chiral bag model is defined by

$$\mathcal{L}=\mathcal{L}_q\theta(R-r)+\mathcal{L}_m\theta(r-R)+\mathcal{L}_B\delta_B \ , \tag{1}$$

where[4]

$$\begin{aligned}
&\mathcal{L}_q=\psi(i\gamma^\mu\partial_\mu-M)\psi \ ,\\
&\mathcal{L}_m=\frac{F_\pi^2}{16}\mathrm{tr}(\partial_\mu U^\dagger\partial_\mu U)+\frac{1}{32a^2}\mathrm{tr}[U^\dagger\partial_\mu U,U^\dagger\partial_\nu U]^2\\
&\qquad+\frac{m_\pi^2F_\pi^2}{8(m_z+m_d)}\mathrm{tr}[M(U+U^\dagger-2I)] \ ,\\
&\mathcal{L}_B=-\tfrac{1}{2}(\bar\psi_L U\psi_R+\bar\psi_R U^\dagger\psi_L) \ ,\\
&M=\mathrm{diag}(m_u,m_d) \ .
\end{aligned}\tag{2}$$

The meson phase is described by the static classical field configuration $U=e^{i\tau\cdot\hat{\mathbf{x}}F(r)}$ with $F(r)$ determined by minimizing the static energy and by imposing the continuity of the axial-vector current at the bag boundary. The quark phase is described by the quantum field operator $\psi(\mathbf{x},t)$.

The standard method[5] to excite the solitonic baryon degrees of freedom, that is, to construct the low-lying quantum states above the semiclassical ground state, is to make the substitution

$$\begin{aligned}
&U(\mathbf{x},t)=A(t)U_s(\mathbf{x})A^\dagger(t) \ ,\\
&\psi(\mathbf{x},t)=A(t)\psi_0(\mathbf{x},t) \ ,
\end{aligned}\tag{3}$$

that is, to quantize the rotational zero modes associated with the collective variables $A(t)$. Here $U_s(\mathbf{x})$ and $\psi_0(\mathbf{x},t)$ are the fields in the rotating (body-fixed) frame.[3] Upon substituting (3) into (2), we get

$$\begin{aligned}
L=L_0+\lambda_m\mathrm{tr}(\dot A^\dagger\dot A)+\frac{i}{2}X^a\int d^sx\,\bar\psi_0\gamma^0\tau^a\psi_0\\
-\tfrac{1}{2}\Delta mR^{3a}\int d^sx\,\bar\psi_0\tau^a\psi_0 \ ,
\end{aligned}\tag{4}$$

where

$$X^a=\mathrm{tr}(\tau^aA^\dagger\dot A),\quad R^{ab}=-\tfrac{1}{2}\mathrm{tr}(A^\dagger\tau^aA\tau^b) \ . \tag{5}$$

In Eq. (4), λ_m is the moment of inertia of the meson phase, associated with the collective rotations, and is given by

$$\lambda_m=\frac{2\pi F_\pi^2}{3}\int_R^\infty dr\, r^2\sin^2F(r)\times\left\{1+\frac{4}{(aF_\pi)^2}\left[\left(\frac{dF}{dr}\right)^2+\frac{\sin^2F(\tau)}{r^2}\right]\right\} . \tag{6}$$

Notice that, since the mesonic Lagrangian is at least quadratic in time derivatives, the approximation of the rotating-frame meson field with the Skyrme solution $U_s(\mathbf{x})$ is consistent.

In order to determine the Lagrangian (4) completely, we need to resort to the known solutions for the chiral hedgehog quark states in the equal-mass case.[6] To make sensible use of these solutions in the framework of perturbation theory, we need to know the equation of motion for the rotating-frame field ψ_0. This differs, however, from the laboratory-frame equations by A dependence buried in the ψ_0's.

Subjecting the laboratory-frame field equation $(i\gamma^\mu\partial_\mu-M)\psi=0$ to the transformation (3), we get

$$(i\gamma^\mu\partial_\mu-M_0+i\gamma^0A^\dagger\dot A+\tfrac{1}{2}\Delta m A^\dagger\tau_3A)\psi_0(\mathbf{x},t)=0 \tag{7}$$

subject to the boundary condition on the bag surface

$$-i\hat{\mathbf{x}}\cdot\gamma\psi_0(\mathbf{x},t)|_{\mathrm{bag}}=e^{i\gamma_5\hat{\mathbf{x}}\cdot\tau F(r)}\psi_0(\mathbf{x},t)|_{\mathrm{bag}} \ . \tag{8}$$

Once this equation is at our disposal, its stationary-state solution $\psi_0(\mathbf{x},t)=\psi_0(\mathbf{x})e^{-i\omega t}$ can be related to the symmetric-case chiral hedgehog quark state solutions $\chi_0(\mathbf{x})$, which satisfy the equation

$$(\omega\gamma_0+i\gamma\cdot\nabla-m_0)\chi_0(\mathbf{x})=0 \tag{9}$$

together with the boundary condition (8). $\chi_0(\mathbf{x})$ is given as[5]

$$\chi_0(\mathbf{x})=\frac{N}{\sqrt{4\pi}}\begin{bmatrix} i\left[\dfrac{E+m_0}{E}\right]^{1/2} j_0(kr)|0\rangle \\ -\left[\dfrac{E-m_0}{E}\right]^{1/2} j_1(kr)(\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})|0\rangle \end{bmatrix} \tag{10}$$

and has some useful properties:

$$\bar{\chi}_0\tau^a\chi_0=\chi_0^\dagger\tau^a\chi_0=0 \ . \tag{11}$$

The relation between ψ_0 and χ_0 is given by

$$\psi_0(\mathbf{x})=\chi_0(\mathbf{x}) - \int d^3y\, S_B(\mathbf{x},\mathbf{y};\omega)(i\gamma^0 A^\dagger \dot{A} + \tfrac{1}{2}\Delta m_q A^\dagger\tau_3 A)\psi_0(\mathbf{y}) \ . \tag{12}$$

Here $S_B(\mathbf{x},\mathbf{y};\omega)$ is the bag propagator defined by

$$(\omega\gamma_0+i\boldsymbol{\gamma}\cdot\nabla-m_0)S_B(\mathbf{x},\mathbf{y};\omega)=\delta^3(\mathbf{x}-\mathbf{y}) \ ,$$
$$[\exp(i\gamma_5\boldsymbol{\tau}\cdot\hat{\mathbf{x}}F)+i\boldsymbol{\gamma}\cdot\hat{\mathbf{x}}]S_B|_{\text{bag}}=0 \ . \tag{13}$$

Equation (12) can be solved perturbatively to any order desired. Since Δm_q is small, it is consistent to solve it to first order in Δm. Furthermore, the collective rotations are adiabatic; thus the rotational velocity $\tau^a A^\dagger \dot{A}$ is also small. Therefore, we will solve (12) to first order in the perturbation sense. To this order the rotating-frame field ψ_0 is given in terms of the symmetric hedgehog quark solutions χ_0 as

$$\psi_0(\mathbf{x})=\chi_0(\mathbf{x})-\int d^3y\, S_B(\mathbf{x},\mathbf{y};\omega) \times(i\gamma^0 A^\dagger \dot{A}+\tfrac{1}{2}\Delta m_q A^\dagger\tau_3 A)\chi_0(\mathbf{y}) \ . \tag{14}$$

Substituting (14) in (4), and retaining up to quadratic terms in rotational velocity X^a (since the mesonic part is already quadratic in X) and making use of (11), we get the complete A-field dependence of the Lagrangian, to first order in Δm:

$$L=L_0-\tfrac{1}{2}\Lambda^{ab}X^aX^b-\frac{i}{4}\Delta m R^{3b}C^{ba}X^a \ , \tag{15}$$

where

$$\Lambda^{ab}=\lambda_m\delta^{ab}+\tfrac{1}{2}\int d^3x\, d^3y[\bar{\chi}_0(\mathbf{x})\tau^a\gamma_0 S_B(\mathbf{x},\mathbf{y};\omega)\gamma_0\tau^b\chi_0(\mathbf{y})+\text{H.c.}] \ ,$$
$$C^{ba}=\int d^3x\, d^3y\{\chi_0^\dagger(\mathbf{x})[\tau^a S_B(\mathbf{x},\mathbf{y};\omega)\tau^b+\tau^b\gamma_0 S_B(\mathbf{x},\mathbf{y};\omega)\gamma_0\tau^a]\chi_0(\mathbf{y})+\text{H.c.}\} \ . \tag{16}$$

The Hamiltonian can now be easily constructed, by taking into account the constraint $A^\dagger A=I$:

$$H=-L_0-\tfrac{1}{2}\Lambda^{ab}X^aX^b \ . \tag{17}$$

This is consistent with the fact that for Lagrangians containing terms linear in velocity, the Hamiltonian is quadratic (to be compared with Ref. 3).

The spin and isospin operators can be computed in the usual manner, applying the Noether term to the transformation $\delta_\tau A=-iAr$ and $\delta_l A=ilA$ (with $r,l=\epsilon^a\tau^a/2$), respectively:

$$-S^a=i\Lambda^{ab}X^b-\tfrac{1}{4}R^{3b}C^{ba}\Delta m, \quad I^a=R^{ab}S^b \ . \tag{18}$$

By using (18), the Hamiltonian can be expressed in terms of spin and isospin operators with further neglect of the terms quadratic in Δm_q:

$$H=-L_0-\tfrac{1}{2}(\Lambda^{-1})^{ab}S^aS^b-\tfrac{1}{4}\Delta m(RC\Lambda^{-1}R^{-1})^{3a}I^a \ . \tag{19}$$

The computation of the last term, which accounts for the mass splitting among the members of isospin multiplets (in addition to the usually negligibly small electromagnetic contributions to the splitting[7]), requires the knowledge of S_B. To compute S_B we employ, as before, the multiple reflection expansion method.[8] Supported by claims in the literature,[9] we will retain only the first reflection term in the expansion

$$S_B(\mathbf{x},\mathbf{y},\omega)=S^0(\mathbf{x},\mathbf{y},\omega) +R^2\int d\Omega_\alpha S^0(\mathbf{x},\boldsymbol{\alpha},\omega)K_\alpha S^0(\boldsymbol{\alpha},\mathbf{y},\omega)+\cdots \ , \tag{20}$$

where

$$K_\alpha=e^{i\gamma_5\hat{\mathbf{n}}_\alpha\cdot\boldsymbol{\tau}F(r)}+i\hat{\mathbf{n}}_\alpha\cdot\boldsymbol{\gamma} \ . \tag{21}$$

Here S^0 is the usual Dirac propagator. It is expanded in partial waves employing the two-component spherical harmonics ϕ_{jlm}:

$$S^0(\mathbf{x},\mathbf{y},\omega)=\sum_{jll'm} S^0_{jll'}(r,r';\omega)\phi_{jlm}(\Omega)\phi^\dagger_{jl'm}(\Omega') \ , \tag{22}$$

where

$$S^0_{jll'}(r,r';\omega)=-ik[\delta_{ll'}(\rho_3\omega+m_0) +k(l'-l)\rho_2]f_l(kr)f_{l'}(kr) \ ,$$
$$f_l(kr)=j_l(kr)\theta(r'-r)+h_l^{(1)}(kr)\theta(r-r') \ . \tag{23}$$

Although S^0 is diagonal in flavor space, the same is not true for the first and higher reflection term. A lengthy analysis, however, shows that both Λ and C matrices are diagonal in flavor space (although the following numerical analysis is carried out to first order only, this diagonality property holds to all orders in multiple reflection expansion). That is,

$$\Lambda^{ab}=\delta^{ab}(\lambda_m+\lambda_q), \quad C^{ba}=\delta^{ab}C \ , \tag{24}$$

where

$$\begin{aligned}\lambda_q&=\tfrac{1}{2}\int d^3x\, d^3y[\bar{\chi}_0(\mathbf{x})\gamma_0 S_B(\mathbf{x},\mathbf{y};\omega)\gamma_0\chi_0(\mathbf{y})+\text{H.c.}]\,,\\ C&=\int d^3x\, d^3y\{\chi_0^\dagger(\mathbf{x})[S_B(\mathbf{x},\mathbf{y};\omega)+\gamma_0 S_B(\mathbf{x},\mathbf{y};\omega)\gamma_0]\chi_0(\mathbf{y})+\text{H.c.}\}\,.\end{aligned}\tag{25}$$

Thus the Hamiltonian can be rewritten as

$$H=-L_0-\frac{\mathbf{S}^2}{2(\lambda_m+\lambda_q)}-\Delta m_q\frac{C}{4(\lambda_m+\lambda_q)}I^3\,.\tag{26}$$

After a lengthy calculation λ_q and C are found as

$$\begin{aligned}C=&-\frac{4\tilde{R}}{aF_\pi\nu^4 j_0^2(\nu)\{\xi[\nu(1+\omega_1^2)-2\omega_1]+\mu\omega_1\}}\\ &\times\int_0^\nu dy\, y^2\Bigg\{\int_0^y dx\, x^2[(\xi+\mu)^2 j_0^2(x)j_0(y)n_0(y)-(\xi-\mu)^2 j_1^2(x)j_1(y)n_1(y)]\\ &\quad+\int_y^\nu dx\, x^2[(\xi+\mu)^2 j_0(x)n_0(x)j_0^2(y)-(\xi-\mu)^2 j_1(x)n_1(x)j_1^2(y)]\\ &\quad+\nu\int_0^\nu dx\, x^2[\cos F\{(\xi+\mu)j_0^2(x)j_0^2(y)[(\xi+\mu)^2\mathcal{R}(h_0^2(\nu))-\nu^2\mathcal{R}(h_1^2(\nu))]\\ &\qquad+(\xi-\mu)j_1^2(x)j_1^2(y)[(\xi-\mu)^2\mathcal{R}(h_1^2(\nu))-\nu^2\mathcal{R}(h_0^2(\nu))]\}\\ &\qquad-\tfrac{2}{3}\nu\sin F\mathcal{R}(h_0(\nu)h_1(\nu))[(\xi+\mu)^2 j_0^2(x)j_0^2(y)+(\xi-\mu)^2 j_1^2(x)j_1^2(y)]]\Bigg\}\,,\\ \lambda_q=&-\frac{\tilde{R}}{aF_\pi\nu^4 j_0^2(\nu)\{\xi[\nu(1+\omega_1^2)-2\omega_1]+\mu\omega_1\}}\\ &\times\int_0^\nu dy\, y^2\Bigg\{\int_0^y dx\, x^2\{(\xi+\mu)^2 j_0^2(x)j_0(y)n_0(y)+(\xi-\mu)^2 j_1^2(x)j_1(y)n_1(y)\\ &\qquad+\nu^2[j_1^2(x)j_0(y)n_0(y)+j_0^2(x)j_1(y)n_1(y)]\}\\ &\quad+\int_y^\nu dx\, x^2\{(\xi+\mu)^2 j_0(x)n_0(x)j_0^2(y)+(\xi-\mu)^2 j_1(x)n_1(x)j_1^2(y)\\ &\qquad+\nu^2[j_1(x)n_1(x)j_0^2(y)+j_0(x)n_0(x)j_1^2(y)]\}\\ &\quad+\nu\int_0^\nu dx\, x^2[\cos F\{(\xi+\mu)j_0^2(x)j_0^2(y)[(\xi+\mu)^2\mathcal{R}(h_0^2(\nu))-\nu^2\mathcal{R}(h_1^2(\nu))]\\ &\qquad-(\xi-\mu)j_1^2(x)j_1^2(y)[(\xi-\mu)^2\mathcal{R}(h_1^2(\nu))-\nu^2\mathcal{R}(h_0^2(\nu))]\\ &\qquad-2\nu^2 j_0^2(x)j_1^2(y)[(\xi-\mu)\mathcal{R}(h_1^2(\nu))-(\xi+\mu)\mathcal{R}(h_0^2(\nu))]\}\\ &\qquad-\tfrac{2}{3}\nu\sin F\mathcal{R}(h_0(\nu)h_1(\nu))[(\xi+\mu)^2 j_0^2(x)j_0^2(y)-(\xi-\mu)^2 j_1^2(x)j_1^2(y)]]\Bigg\}\,,\end{aligned}\tag{27}$$

where

$$\mu=m_0R=\frac{m_0}{aF_\pi}\tilde{R},\quad \nu=kR,\quad \xi=ER,\quad \omega_1=\frac{j_1(\nu)}{j_0(\nu)}\,.\tag{28}$$

We have evaluated the radial integrals in (27) by using the numerical solutions of the equation satisfied by the Skyrme profile $F(r)$:

$$(\tfrac{1}{4}\tilde{r}^2+2\sin^2F)F''+\tfrac{1}{2}\tilde{r}F'+(\sin 2F)F'^2-\tfrac{1}{4}\sin 2F-\frac{\sin^2F\sin 2F}{\tilde{r}^2}=0\tag{29}$$

with the boundary conditions $F(0)=\pi$, $F(\infty)=0$, and $\tilde{r}=aF_\pi r$. Taking $\Delta m_q=3.8$ MeV, $\mu=0.5$, and $a=5.45$, we have plotted $\Delta m_q C/4(\lambda_m+\lambda_q)$ as a function of the bag radius R in Fig. 1. Notice that apart from some negligibly small fluctuations around $R\sim 0.2$ fm (which is probably due to the fact that we truncate our expansion at the first reflection order), the graph for $c/4\lambda_{\text{tot}}$ goes to zero smoothly for $R\to 0$, a gratifying result which lends support on the consistency of our quantization scheme.

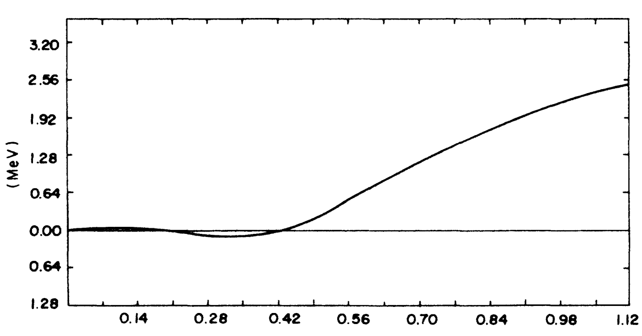

FIG. 1. The $\Delta m_q[C/4(\lambda_m+\lambda_q)]$ as a function of the bag radius.

We would like also to give the simplified expression for the spin operator for completeness (to be compared against Ref. 3):

$$-S^a = i(\lambda_m + \lambda_q)X^a - \tfrac{1}{4}\Delta m C R^{3a} \ . \tag{30}$$

Notice that the spin is partitioned between the meson and quark sectors, as expected. Thus the inconsistency encountered in the previous attempt[1] in this respect is cured as well. We differ from Ref. 3 in the quark content of the spin; their spin is insensitive to quark mass difference. Whereas in our case

$$-S^a_{(q)} = i\lambda_q \operatorname{tr}(\tau^a A^\dagger \dot{A}) - \tfrac{1}{4}\Delta m C R^{3a} \ . \tag{31}$$

That there exists a Δm-dependent term in $S^a_{(q)}$ is quite natural; it follows from the fact that the solutions for the well-defined spin-isospin states (baryons) are no longer solutions to the Laplace equation[4] on the three-sphere. But they are solutions to the equations

$$\left[\nabla^2_{(4)} + iC\Delta m \left[a_0 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_2}\right]\right]\Psi(a) = 0 \tag{32}$$

expressed in terms of the quaternionic variables defined by $A = a_0 + i\mathbf{a}\cdot\boldsymbol{\tau}$.

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