MULTI-LOCATION ASSORTMENT OPTIMIZATION UNDER CAPACITY CONSTRAINTS

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

INDUSTRIAL ENGINEERING

By

Başak Bebitoğlu

August 2016

Multi-Location Assortment Optimization Under Capacity Constraints
By Başak Bebitoğlu
August 2016

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Alper Şen(Advisor)

Özlem Çavuş İyigün

Sinan Gürel

Approved for the Graduate School of Engineering and Science:

Levent Onural
Director of the Graduate School

ABSTRACT

MULTI-LOCATION ASSORTMENT OPTIMIZATION UNDER CAPACITY CONSTRAINTS

Başak Bebitoğlu
M.S. in Industrial Engineering
Advisor: Alper Şen
August 2016

We study the assortment optimization problem in an online setting where a retailer determines the set of products to carry in each of its distribution centers under a capacity constraint so as to maximize its expected profit (revenue minus the shipping costs). It is assumed that each distribution center is primarily responsible for a geographical location whose customers' choice is governed by a separate multinomial logit model. A distribution center can satisfy a demand of a region that it is not primarily responsible for, but this incurs an additional shipping cost for the retail company. We consider two variants of this problem. In the first variant, customers have access to the entire assortment in all locations but in the second variant, the online retail company can select which product to show to each region. Under each variant, we first assume that there is a constant shipping cost for all products between any two location. In the second case, we allow the shipping costs to differ based on the origin and destination. We develop conic quadratic mixed integer programming formulations and suggest a family of valid inequalities to strengthen these formulations. Numerical experiments show that our conic approach, combined with valid inequalities over-perform the mixed integer linear programming formulation and enables us to solve large instances optimally. Finally, we study the effect of various factors such as no-purchase preference, capacity constraint and shipping cost on company's profitability and assortment selection.

Keywords: online retailing, multi-locational assortment optimization, MMNL consumer choice model, conic integer programming.

ÖZET

KAPASİTE KISITLARI ALTINDA ÇOK KONUMLU ÜRÜN ÇEŞİDİ ENİYİLEMESİ

Başak Bebitoğlu
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Alper Şen
Ağustos 2016

Bu çalışmada, bir elektronik perakendecinin kendine ait her bir dağıtım merkezinde, kapasite sınırları içinde beklenen karı en çoklamak amacıyla hangi ürünleri taşıması gerektiğini belirleyen bir ürün çeşidi en iyilemesi problemi ele alınmıştır. Her bir dağıtım merkezi, öncelikli olarak belirlenen bir coğrafik bölgeden sorumludur ve o alandaki tüketicilerin ürün seçimi MNL tüketici seçimi modeli ile gösterilmiştir. Perakendeci, bir bölgeden gelen bir talebi, başka bir bölgeden ilave bir nakliyat ücreti ödeyerek karşılayabilmektedir. Bu çalışma, problemin iki varyantını inceler. İlk varyantta tüketiciler, bütün bölgelerdeki ürünlere internet sitesi aracılığıyla ulaşabilirken, ikinci varyantta perakendeci, her bölgede hangi ürünlerin gösterileceğini seçebilmektedir. Her varyant için nakliyat ücretleri ilk olarak sabit, daha sonra başlangıç ve bitiş noktasına göre değişken olarak alınmıştır. Tanımlanan her problem için konik tamsayılı programlama formülasyonu geliştirilmiş ve bu formülasyon kuvvetlendirecek eşitsizlikler önerilmiştir. Yapılan sayısal çalışmalarda, önerilen konik yaklaşımın, eşitsizliklerle birleştiğinde, doğrusal yaklaşımdan daha üstün bir performans göstererek daha büyük problemler çözebilmemize olanak sağladığı gözlemlenmiştir. Son olarak, değişen parametre değerlerinin şirket karı ve ürün seçiminde ortaklaştırmaya etkileri incelenmiştir.

Anahtar sözcükler: elektronik perakendecilik, çok konumsal ürün çeşidi en iyilemesi, MMNL tüketici seçimi modeli, konik tamsayılı programlama.

Acknowledgement

First, I would like to thank my advisor Alper Şen for being a great mentor at every step of my master studies. Not only he guided me on my thesis, but also he supported me to make decisions for life and turned this challenging experience into a pleasant one with his sense of humor. His excellence has been a tremendous source of inspiration for me. It was a great opportunity for me to work with such a professor in every aspect.

I would like to acknowledge the insightful comments and feedbacks of Özlem Çavuş and Sinan Gürel, which has helped shape this dissertation.

I would like to thank all the wonderful people here in Department of Industrial Engineering who overlapped with me during my study. I would like to thank all professors and graduate students who have contributed towards making my experience a pleasant one.

Special thanks to Sinem Savaşer and Ömer Burak Kınay for providing a welcome break from courses and research. I would especially like to thank Onur Altıtaş without whom, none of my achievements would be possible. His constant support and encouragement was invaluable for me and I feel so fortunate to have such a great person in my life.

Finally, I would like to acknowledge the continuous self-sacrifice of my parents Nuşe Bebitoğlu and İlhan Bebitoğlu. I would also like to thank my sister Burcu Bebitoğlu for her lifetime support.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The purchasing behaviour, in most of the cases, depends highly on what kind of products the retailers choose to offer to customers and has an enormous impact on sales and gross margin. Consumers often do not know what kind of products they will purchase and even if they arrive into the system with the intention of purchasing a fixed product, they do not know which specific variant best fit their needs. As a particular example, we can consider a consumer that wants to buy a new cellphone. Even if he has the intention of buying a new phone, his purchasing behaviour depends on different attributes of the phone such as price, operating system, camera features etc. However, if he does not find a product that exceeds his threshold utility among the alternatives that are offered, he leaves the system to check other stores. Hence, a common question arises for the retailers: which products to make available to customers, i.e. the product assortment, so as to maximize the profitability?

It has been also documented that consumers derive additional utility from broader product lines [1], consequently, retailers have started to compete by expanding their product variety. There has been an appreciable growth in the number of products available in the market [2], [3]. However, the large number of product options together with limited shelf-space created a trade-off between filling the space with the most popular but relatively cheaper products or with the less popular but more profitable products. This situation has contributed to the complexity of this decision. In addition, the assortment is typically chosen before any sales have been observed for some candidate products and it is difficult to know the effect of products that are not available on the shelf beforehand. All these have made assortment decisions more challenging compared to pricing or advertising decisions [4].

Brandes and Brandes also highlight the importance of choosing the assortment appropriately in one of Harvard Business Review articles: "Succeeding in the retail business means being good on a number of dimensions, they include service, pricing, marketing, and location selection, but assortment is number one" [5]. Paralleling the growth of importance of this problem in the industry, there has also been a growing number of studies in operations research/operations management literature. One can see Kök et al. [6] for an extensive review. Retailers face significant challenges to understand the mapping from assortment decisions to consumer behaviour as this mapping should synthesize complex aspects of purchase decisions such as "substitution behaviour, consumers' collection and aggregation of information, consumer heterogeneity, and the effect of competition" [7]. Hence, any analytical model in the academic literature needs to map this behaviour. This mapping is called the consumer choice model.

There are three commonly used consumer choice models in assortment planning literature: multinomial logit model, exogenous demand model and locational choice model. Before proceeding to explain these models in more detail, we will define a brief notation. Let N be the set of products in the subcategory, $N=\{1, 2, .., n\}$ indexed by j. Let S be the subset of products carried by the retailer. Let π_j be the unit price for product j. The most commonly used demand model is the multinomial logit (MNL) model. One can see [8], [9], [10] for the characteristics, limitations and assumptions of MNL. The MNL model is based on a utility that a customer gets from consuming a product. For any product, the utility has two components $U_j = u_j + \varepsilon_j$ where u_j is the deterministic component and ε_j is the random component. The ε_j's for all $j \in S$ are independent and identically distributed random variables having Gumbel distribution with mean zero and variance $\mu^2\pi^2/6$. Given that, the probability that a customer purchases product j from a given assortment set S is given by $p_j(S) = v_j/(v_0 + \sum_{k \in S} v_k)$, where $v_j = e^{((u_j - \pi_j)/\mu)}$ and v_0 corresponds to the no-purchase option.

While the MNL model is easy to incorporate with other marketing variables like price and can be efficiently estimated, it has two significant shortcomings. First, it has the Independence of Irrelevant Alternatives (IIA) property which states that the ratio of choice probabilities of two alternatives, i.e., a product's market share relative to another product is independent of the other alternatives in the assortment. The red bus/blue bus example illustrates this property. Consider an individual going to work and has the same probability of using his/her own car and taking the bus every morning, i.e, P(car)=P(bus)=0.5. Assume now that a new bus service is offered. The only difference of this new bus service is the color. Then, he/she will be indifferent between taking an already existing blue bus or the new red bus, as they are basically the same. One would expect that the probability of choosing a bus or a car will remain the same 0.5. However, the MNL models the probabilities as P(car)=1/3 and P(bus)=2/3 as the choice set is car, blue bus, red bus. The second weakness is that the total penetration of the assortment to the market and the substitution rates within that assortment cannot be independently defined in the MNL model.

Starting with the work of Guadagni and Little [11], MNL has been extensively used in marketing literature to estimate the demand for a group of differentiated products. Since then, MNL and its nested version, nested logit (NL) model have become the most popular discrete choice models in analytical works on pricing and assortment planning. In one of the pioneering work in this area, van Ryzin and Mahajan [10] studied the problem of finding the optimal set of products to offer when inventory has to be kept for each product in the assortment using a newsvendor framework. Following this work, many other researchers have also worked on similar problems using MNL, those will be presented under literature review, in the following chapter.

NL Model has been introduced by Ben-Akiva and Lerman [9]. In NL model, a nested process is used for modeling choice. The choice set has its disjoint subsets called categories and customers first choose a subset according to MNL model with a certain probability $P_{N_s}(N)$ where N_s is the partitioned subset. Then he/she chooses a variant within that subset. Hence, the overall probability of choosing product j according to this model becomes $P_j(N) = P_{N_s}(N) \times P_j(N_s)$. Although finding a suitable nesting structure can be challenging in practice [12],[13], this model has been widely used in analytical models of pricing and assortment planning as it overcomes the IIA property that MNL model holds. In the next chapter, we present an overview of the literature that uses NL model in the assortment planning problem.

In this study, we consider the assortment optimization problem under a generalization of the MNL model, mixed multinomial logit (MMNL) model, which does not have either of those limitations. The MMNL model will be explained further in the following sections.

The second commonly used model in the literature is the exogenous demand model. Unlike MNL, this model does not consider any form of consumer utility and initial purchase probabilities and substitution rates are specified externally. While this model can be considered as flexible and does not suffer from many of the problems associated with MNL, it may be tough to estimate the model parameters. It is also difficult to obtain managerial insights through these models. Major assortment papers that use exogenous demand model include Smith and Agrawal [14] and Kök and Fisher [15].

The third model, locational choice model, is proposed by Lancaster [16] and is based on the original work of Hotelling [17]. In this model, each product is represented by a vector of attributes in characteristics space. Also, each consumer defines his/her ideal point in mind which combines his/her most preferred combinations of those attributes. The utility that a consumer gets from consuming product j is given by $U_j = R - r_j - g(y, z_j)$, where R is a constant and z_j is the location of the product j, y is the consumer's ideal product and $g(y, z_j)$ is some measure of distance between y and z_j. IIA property of MNL does not hold in this model, however, it may again be tough to define and measure the attributes in the characteristics space and make necessary estimations. Some of the authors that use this model in assortment planning are McBride and Zufryden [18], Kohli and Sukumar [19], Gaur and Honhon [20].

Apart from the challenge of modeling the consumer choice accurately, another very important aspect of retailing is making the fulfillment decisions. In this study, we consider fulfillment decisions under an online retailing environment. There is a growing stream of online retailing and this relatively new trend has started disrupting the business models of the traditional retailers. According to Forrester Research Inc., US online retail is expected to reach $373 billion by the end of 2016 and will grow more than $500 billion by 2020 [21]. Moreover, based on 10k statements of various online retailers such as Amazon.com Inc., Bluefly Inc., Overstock.com Inc., outbound shipping and handling costs account for 5-7 % of revenues, equating to $19-$26 billion in the US in 2016 [22].

The situation in fact is similar in Turkey. Turkish informatics industry association (TUSIAD) chairman states that e-commerce volume showed a strong increase in the share of retail spending. The e-commerce volume in Turkey has reached 18.9 billion Turkish Liras ($5.49 billion) last year, after it increased by 35% in 2014 compared to the previous year. E-commerce in Turkey now represents 1.6% of all retail business among European countries [23]. These numbers show that the industry has gone and will continue to go under a challenging transformation in a short period of time. As the value of the online retailing keeps growing exponentially each year, it is highly important that the both global and local companies manage their online operations better.

In the traditional retail supply chain, vendors typically supply distribution centers, which in turn supply retail stores. The end consumer visits these stores and if they decide to buy an item, they choose an item that is available on the shelf or else leave the store without any purchase. All consumers are served immediately and the assortment is only limited by the physical space/capacity of the store itself.

Online retailing, on the other hand, is different than conventional retailing in a number of ways. First, there is a network of warehouses that keeps the inventory. The demand coming from a particular region can be fulfilled from any of these warehouses. In this context, these are called fulfillment centers. Hence, the space is not limited with the physical capacity of a single retail store and in most of the cases, customers are able to see any item that is available in the network of fulfillment centers. This creates an opportunity of holding a deeper product variety. Additionally, the structure of the fulfillment centers is not hierarchical. The fulfillment centers in the distribution network may either be large in order to hold a wide variety of products or small in size to maximize geographical coverage for the most popular products. Any of these fulfillment centers can serve any customer, and they can also replenish each other. Another important difference between the online retail supply chains and the conventional retail supply chains is that the online retailer decides where to fulfill the demand as opposed to customer choosing which store to visit. In this study, we will consider an online retail business and model an assortment problem considering all the aspects presented above.

This study considers the problem of choosing the optimal assortment of different fulfillment centers located in different areas/cities, of an online retailing company. The company has to decide a subset of products for each of those different centers so as to maximize the total profit. Each fulfillment center has its own capacity, therefore cannot carry the entire product line. In addition, the number of different customer classes and the number of fulfillment center are the same. It is assumed that the group of people living nearby a specific fulfillment center in a given geographical area show similar purchasing behaviour and they belong to a specific customer class. Each customer class' demand is modeled as a separate MNL, which leads to the fact that the firm also has to consider the preferences of different customer segments while choosing different assortments for those centers. There exist different customer classes living in different locations and having different preferences for a set of products. If there was a single location from which these customers can be served, this problem would be equivalent to a single-location assortment problem having multiple customer classes, therefore it can be formulated as MMNL, as each customer class in different locations has their own MNL model with different choice parameters. Although the assortment optimization under MNL can be solved efficiently, there does not exist a polynomial time algorithm to solve the problem under MMNL model even for uncapacitated case. One can see Bront et al. [24] and Rusmevichientong et al.[25] for the proof of NP-hardness of the assortment optimization problem under MMNL choice model.

The setting modeled in this thesis, is very similar to that of an e-retailer company such as Amazon. Hence, it considers all the aspects of an online retailing mentioned above. One can think that the assortment problem can be solved separately for each location and warehouse. However, this does not hold true for various reasons. First of all, when customers enter the website, in most of the cases, they have access to the entire assortment available in all warehouses, not only the closest or the centralized one. Similarly, the retailer may choose to satisfy demand from any of the fulfillment centers. Even though the demand comes from a specific location, the warehouse in that location is not necessarily the one to carry that product. However, if the product is not available in the assigned distribution center, the company has to ship from another center; which comes at an additional cost. Another reason that the locations are inseparable is that the company may choose to carry, for example, a less desirable product in one region, simply because that second most desirable product is more popular in other locations.

Given the above setting, we are interested in solving two variants of the problem. In the first one, the entire assortment in all distribution centers is offered as a common assortment to each and every customer. In the second one, the retailer can customize what it wants to show to each customer in a particular region.

In both approaches, we assume that a given customer demand can be satisfied from any of the centers with an extra cost. Considering this, the problems are analyzed under two settings. In the first setting, shipping the product from a different distribution center has an additional cost to the company, but this cost is constant no matter where it is coming from. However, in the second setting, we assume that the costs of shipping from different locations are also different, so the decision of “from where to ship” is also important. We analyze both settings for those two variants of problems above.

The different versions of the assortment problem under MMNL are formulated using the traditional mixed integer linear programming (MILP) approach. Then, with the help of the pioneering work of Şen et al. [26], a new formulation is developed using conic quadratic mixed integer programming. Together with McCormick estimators, the new conic formulations is stronger than the MILP formulations and lead to faster solution times. In addition to this new formulation, the effects of different factors such as “capacities”, “no purchase preferences” and “transportation costs between locations” on assortment depth and assortment commonality are investigated numerically.

The rest of the thesis is presented as follows: In Section 2, the related literature on the assortment optimization is given, in Section 3, further details of the problem are explained and formulations are presented. Section 4 provides the results of the computational study as well as the insights obtained. Finally, Section 5 concludes the study and provides avenue for future research.

Chapter 2

Literature Review

2.1 Assortment Planning Problem under MNL Model

This chapter provides a review of literature on assortment optimization under the MNL random utility model of consumer choice and its variants. Although MNL has been widely used in marketing and economics literature since 1980's, the application of it to assortment and inventory planning can be considered as a relatively new but a quickly growing field. In this field, the intention is to formulate and optimize a set of products to be carried under a set of certain constraints. Although the retailer might have more than one store and want to carry different assortments for each store, the literature focuses mostly on finding an assortment for a single store, which can also be interpreted as carrying a common assortment at all stores. Therefore, we will provide a related literature of those who uses that idea under the MNL and its variants, and then review a small number of works that focus on multi-location, multi-echelon setting.

To begin with, van Ryzin and Mahajan [10] formulate the assortment planning problem where the demand follows a stochastic choice process in which individual purchase decisions are made according to MNL model. The retailer maximizes the utility under the well-known news-vendor model. van Ryzin and Mahajan brings those two fundamental streams of research together and creates a pioneering work which is followed by many other researchers. Each customer considers a set of products S offered by the store and make a purchase accordingly. As explained briefly in the first chapter, the random component of the utilities of those products in set S are assumed to be Gumbel distribution according to MNL model.This assumption leads us to the standard result of MNL: the probability that a customer purchases item $j \in S$ is given as $p_j(S) = v_j/(v_0+\sum_{k\in S} v_k)$, where $v_j = e^{(u_j/\mu)}$ are called the preferences and v_0 corresponds to the no-purchase option. The products are indexed in descending order of their preference, i.e., $v_1 \geq v_2 \geq ... \geq v_n$, costs $c_j = c$ and prices $r_j = r$ for all j. They assume that variants are not substitutes, so the customer does not dynamically substitute his/her first choice with the second. If it is out of stock, the sales is lost. They call this static substitution. In their study, using the above setting, firstly, they show that the profit is maximized either by adding the variant with the next highest utility value v_j or not. Secondly, by using this result, they characterize the so-called popular set $P= \{\{1\}, \{1, 2\}, \{1, 2, 3\}, ..., \{1, 2, ..., N\}\}$ and state that the optimal assortment will always be one of the sets in P. This reduces the number of possible optimal assortment set combinations from $2^{|N|}$ to $|N|$.

They provide insights on how various factors affect the optimal assortment and show that a deeper assortment with a sufficiently high price and sufficiently high no-purchase preference v_0 is more profitable. This result is later generalized for concave cost functions by Cachon et al. [27].

Mahajan and van Ryzin [28] later extend their work to cover determination of initial inventory levels and dynamic substitution effects of stock-outs. They use a sample path analysis to show that under very general assumptions on the demand process, total sales of each product are concave in their own inventory levels and the marginal value of an additional unit of the given product is decreasing in the inventory levels of all other products. They show that the expected profit function does not have any concavity property and hence global optimum is difficult to find. They propose a stochastic gradient algorithm and compare it with two heuristics. They deduce that the substitution may highly effect the gross profit.

Finally, they conclude that under substitution, one should stock more of the more popular variants and less of the less popular variants than a traditional newsboy analysis implies.

Chong et al. [29] present an empirically based modeling framework for managers to assess the revenue and lost sales impact of alternative category assortments.

Talluri and van Ryzin [30] show that the optimal assortment includes a certain number of products with the largest revenues which is referred as nested-by-revenue assortments. They show that more general choice models lead to a more difficult problem.

In 2005, Cachon et al.[27] extend van Ryzin and Mahajan [10] model by analyzing the effects of consumer choice under search, i.e., the customers may still want to search for other stores to lessen the uncertainty. They also study the no-search model and conclude that profit obtained by the assortment found with no-search model is significantly less than the one that takes into account search. They supported their findings with the numerical results, which show that it may be optimal to include an unprofitable product to the assortment. They also show that no-search model performs well with the categories having so many variants like jewelery. On the other hand, it does not perform well on the overlapping assortment search, where the products may be common in different stores.

Li [31] studies the assortment problem under MNL choice model. They attempt to find the optimal assortment first by assuming a continuous store traffic. They use a measure called profit rate to evaluate the profitability of each variant and then show that the optimal assortment should include the first few items that have the highest profit rate. Second, they assume a discrete store traffic. When the store traffic is discrete, the optimal solution is more difficult to obtain. They propose a profit rate heuristic, which is inspired by the result for the case of continuous store traffic and show special cases in which the heuristics yields the optimal solution. Finally, they point out the importance of measuring the profitability of each product when the demand is random and there is cannibalization.

Maddah and Bish [32] model the assortment selection problem of differentiated variants by their secondary attributes, along with the inventory levels and prices under an MNL model within a newsvendor type of setting. They find that the optimal assortment has items with the smallest unit cost under some special cases, and propose a dominance relationship for the general case that significantly reduces the number of assortments to be analyzed. They also observe structural properties of the optimal prices based on their numerical study and propose a heuristic which performs quite effective on many numerical tests.

Smith [4] formulates an optimization problem considering more general preference models for heterogeneous customer classes and shows the importance of including preference heterogeneity. However, this leads to complex nonlinear objective functions that are difficult to optimize and thus requires heuristics.

Hopp and Xu [33] study a previously intractable, game theoretic problem to test their static approximation of a dynamic customer demand substitution behaviour. Unlike the previous models considering dynamic substitution, such as Mahajan and van Ryzin [28], this approximation creates a much simpler inclusion of the dynamic behaviour to product pricing, inventory and assortment decisions. They study a duopoly of price, service and a assortment. This is modeled as an assortment sub-game where the decision is the product assortment for both sides, considering customer perception, unit product cost and demand uncertainty. They show that there exists a pure strategy Nash equilibrium for the product assortment competition and identify a condition for uniqueness. They also find that competition on price and assortment results in a larger total number of products and a higher aggregate inventory level in a duopolist market as opposed to a monopoly.

Miller et al. [34] simultaneously examine the problem for infrequently purchased products and incorporating customer heterogeneity at the same time. They assess the robustness of such assortments with regard to shifts in customer preferences and develop an integer program to reach optimality.

Rusmevichientong et al. [35] consider both static and dynamic assortment optimization problems with MNL choice model and capacity constraint. In the static approach, they basically assume customers have similar preferences on products and the parameters of MNL are already known. Thus, they develop a profit-maximizing algorithm and find the optimal assortment under a capacity constraint. In dynamic approach, they develop a policy that first estimates the choice parameters of MNL from a past data by exploiting the structural properties found in static problem, then optimizes the profit. Finally, they experiment their policy with a numerical study based on an online retailer's sales data.

Rusmevichientong and Topaloglu [36] formulate a robust assortment optimization problem under MNL model with uncertainty in the choice parameters of MNL , i.e., the parameters of MNL are unknown. They maximize the worst-case expected revenue over a set of likely parameter values, called the uncertainty set. They consider static and dynamic cases. In the static setting, the inventory is ignored, whereas in the dynamic setting, there is a limited initial inventory that must be allocated over time. They also provide a family of uncertainty sets for the decision maker to control the trade-off between increasing the average revenue and protecting against the worst-case scenario. They perform a numerical study to show that the robust approach combined with the proposed family of uncertainty sets is beneficial when there is significant uncertainty in the parameter values and provides more than 10% improvement in the worst-case performance.

Wang [37] again study the problem of finding the optimal assortment and prices under the capacity constraints. They provide an efficient algorithm to simplify the combinatorial optimization problem to a problem of finding a unique fixed point of a single dimensional decreasing function.

Davis et al. [38] consider bounds on cardinality, display location of the selected products and precedences while studying assortment optimization problem. They model them separately and are able to reformulate these different types using a set of totally unimodular constraints. Thus, show that it is possible to solve this fractional binary problem as a relaxed linear problem.

Topaloğlu [39] formulates a non-linear program to solve an assortment and its stocking problem where the duration of time that each set of assortment is offered on shelf and the number of each product in the assortment are decision variables. They show that this problem is hard to solve because the objective function lacks concavity and the number of variables grows exponentially with the number of products. By using the structure of MNL, they reformulate the problem to be decomposable and make the number variables grow linearly. They find a well-behaved dynamic programming approach. However, as the approach requires discretizing the state space, they also come up with two alternative approximate algorithms that yield good insights for offer sets.

Another recent work is done by Goyal et al. [40] on a single-period capacitated assortment and inventory planning problem under dynamic substitution and stochastic demand. Although they study this approach under random preferences, they show that one can adapt their algorithm to MNL choice model. They show that this problem is NP-hard "even when there is only one customer and all possible preferences include only two product types" and show that the approximation to this problem under general preferences cannot be better than a factor of 1-1/e. Then, they develop a polynomial time approximation scheme (PTAS) that guarantees a near-optimal solution for assortment optimization problem with dynamic substitution for the first time. Lastly, they perform a numerical study to prove the significance of their approach.

The last article presented in this section is written by Besbes and Saure [7]. This is one of the most recently published articles and considers a game-theoretic approach to assortment planning under MNL. Unlike most of the studies that consider a monopolistic setting, this paper includes a competitive environment. They show that under some display constraints, there exists an equilibrium when the predefined sets of products available to retailers do not overlap. This results in a direct corollary that "competition leads a firm to offer a broader set of products compared to when it is operating as a monopolist, and to broader offerings in the market compared to a centralized planner" [7].

2.2 Assortment Planning Problem under Nested Logit Model

There has been various studies under assortment optimization considering nested logit model due to its less exposure to IIA property.

Kok and Xu [41] explore joint assortment optimization and pricing problems under the nested logit model, where the decision is to find optimal set of products offered and their corresponding prices in order to maximize the expected revenue. They consider two structures of hierarchy. The first one assumes that customers first choose the brand, then decides on which product to purchase within that brand. On the contrary, the second structure chooses the product type first. They come up with properties that can be used by managers to rule out non-optimal assortments while being able to choose the best prices under a hierarchical choice process.

Davis, Gallego and Topaloğlu [42], for the first time, consider the assortment optimization problem when there is a large number of nests. They investigate four cases where the problem can be solved exactly or approximately. They are able to show that it is optimal to offer a nested-by-revenue assortment within each nest for the case where "the dissimilarity parameters of the nests are less than one and customers always make a purchase within the selected nest". Unlike the first case, the other practically important three versions do not conform to the standard version of the nested model. Hence, after stating that those versions are NP-hard, they provide a parsimonious collection of possible assortments that guarantees a worst-case performance.

Gallego and Topaloğlu [43] study the assortment problem under capacity and space constraints when the consumer chooses according to the nested logit model. When there is cardinality constraint on the assortment offered in each nest, they show that the problem is efficiently solvable by using a linear program. When there is a space constraint, i.e., each product fills a certain space, the problem becomes NP-hard and they are able to formulate a tractable linear program with a performance guarantee on the chosen assortment. They also study a joint pricing and assortment problem under the nests and based on their previous findings, show that it can be solved efficiently.

Feldman and Topaloğlu [44] study the assortment problem under cardinality and space constraints in each nests similar to Gallego and Topaloğlu [43]. They show that the problem under a cardinality constraint is tractable by solving a linear program and give, for the first time, and exact solution method. For the model with space constraint, they provide a 4-approximation algorithm which scales polynomially with the number of nests. They compare the results of their approximation with the upper bounds on the optimal solution using another linear program in order to demonstrate the performance of their algorithm.

Li and Rusmevichientong [45] provide an efficient greedy heuristic for the two-level nested assortment optimization problem, which has the fastest known running time. They, for the first time, provide a necessary and sufficient condition for an optimal assortment. They also exhibit a "lumpy" structure and exploit those two to come up with an iterative algorithm to find the optimal assortment.

Assuming fixed prices, Li et al. [46] study the problem of finding the optimal assortment to maximize revenue under the d-level nested logit model, d being an arbitrary number of levels. To solve this problem, they develop an efficient algorithm with a running time of $O(d\ n\ \log n)$ where n is the number of products. They also study price optimization problem under this structure.

The final work that is going to be presented in this section is the recent work done by Rayfield et al. [47] on the problem of finding the optimal assortment together with the prices of the products in that assortment. The prices should be selected in between a pre-defined upper and a lower bound specific to the product. They give an approximation method to solve this problem under the nested logit consumer choice model, and prove that its performance is fast. They also study the pricing problem when the offered products are determined in advance.

2.3 Assortment Planning Problem under MMNL Model

In reality, customers do not have to subscribe to the same choice model as in MNL. This is valid especially in the case of online retailing. Websites never serve to a single market and there is almost always customer heterogeneity in terms of preferences. Thus, MMNL model is important in terms of representing those multiple customer segments with different preferences. Obviously, this model is also the most appropriate one for the context of this thesis and thus, it is utilized for our study. It also does not have either of the limitations of MNL. Although being more realistic, unlike the polynomially solvable MNL problems [35], this assortment optimization problem under MMNL is proven to be NP-hard [24] even when the number of customer classes is two [25]. Note that the number of customer classes being one corresponds to the classical assortment problem with MNL model.

MMNL choice model has mentioned for the first time by Cardell and Dunbar [48] and Boyd and Mellman [49]. In order to model the consumer choice more realistically, a growing stream of the management science/operations research literature has started to focus on assortment optimization under mixed multinomial logit model. Various authors also mention it as mixtures of MNL [44], MNL with random choice parameters [25] and latent-class MNL [50]. McFadden and Train [51] shows that MMNL model can actually be used as an approximation to any discrete choice model based on random utility maximization. Moreover, an new approach called choice-based, deterministic, linear programming model (CDLP) also studies this problem as a subproblem in revenue management [52, 53, 24].

In order to understand the studies further in this term, this section presents a related literature of the articles that deals with the assortment problem under MMNL model.

Bront et al. [24] and Mendez-Diaz et al. [50] formulate the assortment optimization problem under MMNL as a mixed integer linear program. Bront et al. [24] propose a greedy heuristic to provide reasonable solutions. Mendez-Diaz et al. [50] provide a computationally fast branch and cut algorithm that utilizes five families of specific valid inequalities and yields near-optimal solutions.

Rusmevichientong et al. [25] determine two specific cases where revenue-ordered assortments are optimal. Furthermore, they provide an approximation guarantee of $\min\{G, \lceil n/2 \rceil\}$ when the number of customer segments and products are small, where G is possible realizations for the vector of mean utilities and n is the number of products. When the number is large, they provide an approximation guarantee of $e \; log \; (er_1/r_n)$ to the revenue-ordered assortment where $r_i, \; i = 1...n$ are the revenues associated with the products. They also extend their model to the multi-period capacity allocation problem and finally, in an extensive numerical study, they demonstrate that revenue-ordered assortments perform well.

Similar to the setting in this thesis, Feldman and Topaloğlu [44] study the problem of choosing the optimal assortment under the MMNL model. They develop an approach that establishes tight upper bounds on the optimal expected revenue in an efficient period of time. By doing this, they aim to assess the optimality gap of any heuristic by checking the gap between the upper bound and the revenue provided by that heuristic. They conduct a numerical study and show that the upper bound deviates only 0.83% in the worst case from the optimal solution.

One of the most recent work on assortment optimization under MMNL model is studied by Şen et al [26]. They consider the capacitated, single-location assortment optimization problem under MMNL model. They, for the first time, reformulate the problem as a conic quadratic MIP. By doing this, they aim to solve the problem for large instances optimally in a very short time. They strengthen their formulations further with McCormick inequalities and by using benchmark instances, they show that it is better than any other formulation provided in the literature. We will extend the setting of this paper in this thesis and provide a multi-locational formulation.

2.4 Multi-Location Approach to Assortment Planning

One can see that the literature search presented so far is almost exclusively for single store settings. Assortment planning for multi-location supply chains is obviously an open research area. This is also stated in Kök et al.'s review on assortment planning [6]. Exploiting this need, we are going to consider a multi-location setting and to the best of our knowledge, there has only been two studies in this area, both of which consider two-tier supply chains.

Singh et al. [54] study the assortment and stocking decisions building on van Ryzin and Mahajan model [10] considering two different supply chain structures: traditional channel and drop-shipping channel. In the traditional channel, retailers own inventory, however in the drop-shipping channel, the wholesaler stocks and owns inventory and makes those decisions for multiple retailers while retailers have to pay a per unit fee for drop-shipping. This fee provides the benefit of risk-pooling structure and can be seen in online retailing. Consequently, when the number of retailers is large, the drop-shipping charge per retailer decreases and the product variety increases. They study the conditions on the parameters under which the retailers or the wholesaler or both prefer the drop-shipping channel. For an integrated firm with multiple retailers, the authors also find that a hybrid supply chain structure may be cost-efficient when the more popular products are stocked at the retailer while the less popular products are stocked at the warehouse and drop-shipped to the customers. Moreover, a firm following such a hybrid strategy will offer a larger assortment than an otherwise identical firm that fulfills orders exclusively from its retail locations.

In another work, Aydın and Hausman [55] consider van Ryzin and Mahajan [10] model in a decentralized supply chain with one supplier and one retailer.

Their study shows that since the profit margins are lower than that of a vertically integrated (centralized) supply chain, the retailer chooses a narrower assortment than the supply chain optimal assortment. The supplier/manufacturer can coordinate this by paying a “per product” fee to the retailer and ensures that both parties remains more profitable. By this way, retailer may also agree to broaden its assortment and maximizes supply-chain revenue. An example of it can be seen in grocery industry.

2.5 Our contribution

This chapter presents a literature review on assortment planning problems integrated with MNL model and its versions. However, the majority of the literature focuses on a single location setting except for the last two studies presented ([54], [55]). Our study, on the other hand, provide a multi-locational formulation to assortment optimization problem under MMNL. We extend the model of Şen et al. [26] and consider the case when there are different and fixed shipping costs between locations and find the assortment that maximizes the expected revenue under these conditions. We also provide a numerical study and generate insights that may be useful in guiding decision makers in multi-location settings.

Chapter 3

Models

3.1 Problem Definition

We consider the capacitated assortment optimization problem under a multi-locational setting when there is a heterogeneity in customer preferences. This is an open research area [6] and has practical impact considering the recent growth of online retailing industry. In this thesis, we consider different customer classes that are willing to purchase a product from an online-retailing company. We denote the set of customer classes by M. Naturally, customers are able to select a product among the alternatives that are available on the company website. The set of all those possible products is denoted by N. The consumer choice in each customer class is assumed to be governed by a separate MNL model. They have different preferences for the offered products as well as different no-purchase preferences. Therefore, one can assume the overall demand for the company follows a mixtures of MNL model. We explain how this model is used in the next section.

The online-retailer has a distribution center in each geographical area for which a customer class is assigned. Hence, the set of distribution centers are also denoted by M. However, the company can satisfy the online demand from any of its distribution centers and in our case, they do not charge the customer for shipping the product from a distant distribution center. Now, assume that there is a demand coming from customer class $i \in M$ for a product. If the product is available in distribution center i, i.e., the same geographical area, it is assumed to have no extra shipping cost. However, if the retailer has to ship from another location, this incurs an extra cost for the retailer. In addition, there is a capacity constraint for each center, thus the retailer should decide which products to offer in which location considering the customer preferences, shipping costs and the capacities.

In this thesis, we consider the above structure and are particularly interested in finding the assortments for each and every distribution center that maximize the company's total profit. We consider two variants of the problem. In the first variant, customers can access the entire assortment in all distribution centers when they enter the website. We call this variant "common assortment". In the second one, the retailer can select which products to show to each customer class in each region. This variant is called "customized assortments". Moreover, for both variants, we assume two different settings. In the first setting, the shipping costs are taken as constant. However, in the second setting, we relax this assumption. Thus, shipping costs vary by shipping location, i.e. shipping from a closer location will cost less compared to a distant one.

It is shown that even the uncapacitated version of the single location problem under MMNL is NP-hard [24, 25]. Hence the majority of the articles in the literature resort to heuristics and approximations. In contrast, we take a mathematical programming approach to obtain the exact solution. We first provide a mixed integer linear programming approach to each variant. We then present a conic quadratic mixed integer programming approach, and show that by adding the respective valid McCormick inequalities, we are able to get a stronger formulation and shorter solution times. We also examine how commonalities of the assortments in different locations change with respect to different parameters such as no purchase preferences, shipping costs and capacities.

3.2 Problem Formulation

The following table presents the sets and the parameters that are necessary to explain the given models in this chapter.

Table 3.1: **Sets and Parameters**

Sets		
M	:	Set of customer classes and distribution centers
N	:	Set of products
Parameters		
λ_i	:	Probability that the demand originates from customer class i
v_{ij}	:	Preference of product j in customer class i
v_{i0}	:	No purchase preference in customer class i
$v_{i[k]}$	:	k^{th} largest of preferences v_{it}, $t \in N$
π_{ij}	:	Unit revenue of product j in customer class i
$\overline{\pi_i}$	:	The product with the highest revenue that is available to customer class i
τ	:	Fixed cost of shipping any product between any two locations
κ_i	:	Capacity of fulfillment center i
κ	:	Total capacity of the system

3.2.1 Modeling the First Variant: Common Assortment

We first assume in this variant that the customers are able to see the entire assortment available in all distribution centers and make a purchase accordingly once they enter the website. Thus, as long as the company chooses to include a product in one of the distribution centers, it becomes available online to all customer classes. Each demand originated from a customer class $i \in M$ comes from its respective geographical area where the company also has a distribution center. Thus, when a product demand comes from customer class i, the company checks the availability of that product in the same location, i.e., the distribution center i. If the product is available in the same geographical area, i.e., the distribution center i, the demand can be satisfied with a zero cost. However, if it is not available in that center, it incurs an extra shipping cost. We initially assume that this cost of shipping is assumed to be fixed regardless of where the product is shipped from.

In the rest of the thesis, we refer this problem with those two basic assumptions as the baseline model. We later relax these assumptions and obtain more complicated models that may also apply to online business models. Before we explain the mathematical program for this baseline model, we provide below the consumer choice model that governs the consumer behavior at each location.

The consumer choice in each customer class is represented by a separate MNL discrete choice model. This model is based on the utility that a customer gets by consuming a product in the assortment. Each product $j \in N$ has a deterministic utility component u_j and a random component ε_j. Thus, the overall utility of product j is represented by $U_j = u_j + \varepsilon_j$. The random component is modeled as a Gumbel random variable characterized by $Pr\{X \leq \varepsilon\}= e^{-e^{-\frac{\varepsilon}{\mu+\gamma}}}$ where $\gamma = 0.57722$ is the Euler's constant. The mean is zero and variance is $\mu^2\pi^2/6$. In addition to the utilities associated with products, consumers also associate a utility to not purchasing any product. This is called no-purchase option and is represented by $\{0\}$. Given an assortment S with the no-purchase option, the probability of a rational customer who wants to maximize its utility to buy product j is $p_j(S) = Pr\{U_j = \max_{k\in S\cup\{0\}}(U_k)\}$. Moreover, if we let π_j be the unit revenue gained by selling product j, the probability of a customer to purchase product j can be expressed as the following closed form [8]:

$$p_j(S) = \frac{v_j}{v_0 + \sum_{k\in S} v_k}, \tag{3.1}$$

where $v_j = e^{((u_j-\pi_j)/\mu)}$ and v_0 corresponds to the no-purchase preference. One can also call this expression "the market share of product j".

Following the above expression for the probability of purchasing product j, the expected revenue gained over an arriving customer at location i given an assortment S offered to region i is

$$\frac{\sum_{j\in S} \pi_{ij} v_{ij}}{v_{i0} + \sum_{j\in S} v_{ij}}. \tag{3.2}$$

Now, let the binary variable o_{ij} take 1 if product j is available in fulfillment center i and 0 otherwise. Let the binary variable x_j take 1 if product j is available in at least one of the fulfillment centers. As we denote N to be all possible products that can be offered to a region, the expected revenue for an arriving customer at location i can also be expressed as

$$\frac{\sum_{j\in N} \pi_{ij} v_{ij} x_j}{v_{i0} + \sum_{j\in N} v_{ij} x_j}. \tag{3.3}$$

In order to obtain the expected profit in a given location, we need to subtract the shipping costs incurred by the products that are not available in distribution center's assortment S_i but are in the overall assortment S. The expected transportation cost incurred for an arriving customer at location i then becomes

$$\frac{\sum_{j\in S\setminus S_i} \tau v_{ij}}{v_{i0} + \sum_{j\in S} v_{ij}}. \tag{3.4}$$

Now let o_{ij} be 1 if the product j is carried in location i and 0 otherwise. Clearly, $x_j = 1$ if and only if $\sum_i o_{ij} \geq 1$. Then, we can express (3.4) as

$$\frac{\sum_{j\in N} \tau v_{ij}(x_j - o_{ij})}{v_{i0} + \sum_{j\in N} v_{ij} x_j}. \tag{3.5}$$

In order to obtain the expected profit for an arriving customer, we need to sum up the above expression (3.3) over each customer segment and subtract the expected transportation costs incurred (3.5). By summing those expressions over the set M, we can write our objective function as shown below (3.6). Thus, we can formulate the baseline model as a mathematical program as follows:

$$\max \quad \sum_{i\in M} \lambda_i \left[\frac{\sum_{j\in N} \pi_{ij} v_{ij} x_j}{v_{i0} + \sum_{j\in N} v_{ij} x_j}\right] - \sum_{i\in M} \lambda_i \left[\frac{\sum_{j\in N} \tau v_{ij}(x_j - o_{ij})}{v_{i0} + \sum_{j\in N} v_{ij} x_j}\right] \tag{3.6}$$

s.t

$$\sum_{j\in N} o_{ij} \leq \kappa_i, \ \forall i \in N, \tag{3.7}$$

$$x_j \geq o_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.8}$$

$$x_j \leq \sum_{i\in M} o_{ij}, \ \forall j \in N, \tag{3.9}$$

$$o_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, \tag{3.10}$$

$$x_j \in \{0,1\}, \ \forall j \in N. \tag{3.11}$$

The first constraint (3.7) stands for the cardinality constraint. The total number of products chosen for the assortment in location $i \in M$ is not allowed to exceed the capacity κ_i for that location. Constraint (3.8) is to ensure that product j is not available in the system if it is available in at least one of the distribution centers. Finally, (3.9) states that if product j is not carried in any of the centers, then it should not also be in the system.

3.2.1.1 MILP Formulation of the Baseline Model

Bront et al.[24], Mendez-Diaz et al.[50] and Sen et al.[26] formulate the traditional capacitated assortment optimization model as an MILP. Thus, by using the idea behind those papers, the above generic model can be transformed into a linear program. In order to do that, we first need to linearize the terms in the objective function of the above mathematical program. First, let $y_i = 1/(v_{i0} + \sum_{j\in N} v_{ij}x_j)$ and $q_{ij} = x_j - o_{ij}$. By doing that, the objective function becomes:

$$\max \qquad \sum_{i\in M}\sum_{j\in N} \lambda_i \pi_{ij} v_{ij} x_j y_i - \sum_{i\in M}\sum_{j\in N} \lambda_i \tau v_{ij} q_{ij} y_i \tag{3.12}$$

Now, the bilinear terms $x_j y_i$ and $q_{ij} y_i$ should be linearized. Following the approach in Wu [56], to linearize the bilinear terms, we define new continuous variables $z = xy$ and $t = qy$ and add the following inequalities to the formulation: For z, we add $y - z \leq M - Mx, 0 \leq z \leq y$ and $z \leq Mx$ and for t, we add $y - t \leq M - Mq, 0 \leq t \leq y$ and $t \leq Mq$ where M is a sufficiently large number. We can replace M with an upper bound on y: $1/v_{i0}$. Performing these changes, the final MILP formulation with constant shipping costs becomes the following:

$$\max \quad \sum_{i\in M}\sum_{j\in N} \lambda_i \pi_{ij} v_{ij} z_{ij} - \sum_{i\in M}\sum_{j\in N} \lambda_i \tau v_{ij} t_{ij} \tag{3.13}$$

s.t

$$
\begin{aligned}
& & \sum_{j \in N} o_{ij} \leq \kappa_i, \ \forall i \in N, & \quad (3.14) \\
& & x_j \geq o_{ij}, \ \forall i \in M, \forall j \in N, & \quad (3.15) \\
& & x_j \leq \sum_{i \in M} o_{ij}, \ \forall j \in N, & \quad (3.16) \\
& & q_{ij} = x_j - o_{ij}, \ \forall i \in M, \forall j \in N, & \quad (3.17) \\
& & v_{i0} y_i + \sum_{j \in N} v_{ij} z_{ij} = 1, \ \forall i \in M, & \quad (3.18) \\
& (MILP\ for\ z_{ij}) & v_{i0}(y_i - z_{ij}) \leq 1 - x_j, \ \forall i \in M, \forall j \in N, & \quad (3.19) \\
& & 0 \leq z_{ij} \leq y_i, \ \forall i \in M, \forall j \in N, & \quad (3.20) \\
& & v_{i0} z_{ij} \leq x_j, \ \forall i \in M, \forall j \in N, & \quad (3.21) \\
& (MILP\ for\ t_{ij}) & v_{i0}(y_i - t_{ij}) \leq 1 - q_{ij}, \ \forall i \in M, \forall j \in N, & \quad (3.22) \\
& & 0 \leq t_{ij} \leq y_i, \ \forall i \in M, \forall j \in N, & \quad (3.23) \\
& & v_{i0} t_{ij} \leq q_{ij}, \ \forall i \in M, \forall j \in N, & \quad (3.24) \\
& & o_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, & \quad (3.25) \\
& & x_j \in \{0, 1\}, \ \forall j \in N, & \quad (3.26) \\
& & q_{ij} \geq 0, \ \forall i \in M, \forall j \in N, & \quad (3.27) \\
& & z_{ij} \geq 0, \ \forall i \in M, \forall j \in N, & \quad (3.28) \\
& & t_{ij} \geq 0, \ \forall i \in M, \forall j \in N, & \quad (3.29) \\
& & y_i \geq 0, \ \forall i \in M. & \quad (3.30)
\end{aligned}
$$

The first three constraints are same as what is given in the initial mathematical formulation. The following two constraints are for the new variables that have been added to the objective function and the last six inequalities (3.19)-(3.24) are to linearize the bilinear terms.

We will show that this formulation is not scaling especially when the capacity constraints are tight. Our conic quadratic MIP formulation will be a remedy for that problem. In the next section, we will provide how this formulation can be changed into a conic programming and will provide additional tight McCormick inequalities to strengthen it.

3.2.1.2 Conic Formulation of the Baseline Model

Second order cone program on $x \in \mathbb{R}^n$ is of the following form:

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t} \quad & \\ & \|A_i x + b_i\| \leq c_i^T x + d_i, \;\; i = 1, ..., N \end{aligned}$$

where $f \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{(n_i-1)\times n}$, $b_i \in \mathbb{R}^{n_i-1}$, $c_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$ and $\|u\| = (u^T u)^{1/2}$ is the L2 norm. We call $\|A_i x + b_i\| \leq c_i^T x + d_i$ a conic quadratic constraint (second order cone constraint) of dimension n_i. This problem is a convex optimization problem since the objective is a convex function and constraints define a convex set. For a detailed review and applications of second order cone programming, we refer to Lobo et al. [57].

Several convex optimization problems can be reformulated as a second order cone program such as linear programs, quadratically constrained linear programs, quadratic programs, quadratically constrained quadratic programs and many other non-linear convex optimization problems involving hyperbolic constraints. We can cast conic quadratic inequalities as hyperbolic constraints/ rotated cone constraints using the following [57]:

$$\|[2w,\; x - y]\| \leq x + y \qquad \Longleftrightarrow \qquad w^2 \leq xy \;\; : \quad x, y \geq 0.$$

In this section, we will employ these hyperbolic inequalities in the reformulation of our conic model.

In order to formulate the problem as a second order cone program, we first need to transform the objective function into a minimization. To do that, first let $\overline{\pi_i} = \max_{j \in N} \pi_{ij}$. Instead of maximizing the revenue, we minimize the gap between the maximum possible gain by selling only the product with the highest revenue to an arriving customer ($\sum_{i \in M} \lambda_i \overline{\pi}_i$) and the current profit function as a whole given in baseline model's objective. Since $\sum_{i \in M} \lambda_i \overline{\pi}_i$ is constant, these two formulations are equivalent. The accuracy of the models can be checked simply by summing up the maximization (linear) and minimization (conic) objective, which will yield nothing but the parameter $\sum_{i\in M}\lambda_i\overline{\pi}_i$.

Then, the minimization objective function (3.13) can be written as

$$\min \quad \sum_{i\in M}\lambda_i\left[\overline{\pi_i}-\left[\frac{\sum_{j\in N}\pi_{ij}v_{ij}x_j}{v_{i0}+\sum_{j\in N}v_{ij}x_j}-\frac{\sum_{j\in N}\tau v_{ij}(x_j-o_{ij})}{v_{i0}+\sum_{j\in N}v_{ij}x_j}\right]\right]. \tag{3.31}$$

Again, using the definition of $y_i = 1/(v_{i0}+\sum_{j\in N}v_{ij}x_j)$ and $q_{ij}=x_j-o_{ij}$ posed in MILP formulation, (3.31) becomes

$$\min \quad \sum_{i\in M}\lambda_i\left[\overline{\pi_i}v_{i0}y_i+\sum_{j\in N}(\overline{\pi_i}-\pi_{ij})v_{ij}x_jy_i+\sum_{j\in N}\tau v_{ij}q_{ij}y_i\right]. \tag{3.32}$$

Since the coefficients of the objective function are non-negative, we only need to use lower bounds on y, z and t variables. The minimization formulation is therefore:

$$\min \quad \sum_{i\in M}\lambda_i\overline{\pi}_iv_{i0}y_i+\sum_{i\in M}\sum_{j\in N}\lambda_i(\overline{\pi}_i-\pi_{ij})v_{ij}z_{ij}+\sum_{i\in M}\sum_{j\in N}\lambda_i\tau v_{ij}t_{ij} \tag{3.33}$$

s.t

$$\sum_{j\in N}o_{ij}\le\kappa_i,\ \forall i\in N, \tag{3.34}$$

$$x_j\ge o_{ij},\ \forall i\in M,\forall j\in N, \tag{3.35}$$

$$x_j\le\sum_{i\in M}o_{ij},\ \forall j\in N, \tag{3.36}$$

$$q_{ij}=x_j-o_{ij},\ \forall i\in M,\forall j\in N, \tag{3.37}$$

$$y_i\ge\frac{1}{v_{i0}+\sum_{j\in N}v_{ij}x_j},\ \forall i\in M, \tag{3.38}$$

$$z_{ij}\ge x_jy_i,\ \forall i\in M,\forall j\in N, \tag{3.39}$$

$$t_{ij}\ge q_{ij}y_i,\ \forall i\in M,\forall j\in N, \tag{3.40}$$

$$o_{ij}\in\{0,1\},\ \forall i\in M,\forall j\in N, \tag{3.41}$$

$$x_j\in\{0,1\},\ \forall j\in N, \tag{3.42}$$

$$q_{ij}\ge 0,\ \forall i\in M,\forall j\in N, \tag{3.43}$$

$$z_{ij} \geq 0,\ \forall i \in M, \forall j \in N, \tag{3.44}$$

$$t_{ij} \geq 0,\ \forall i \in M, \forall j \in N, \tag{3.45}$$

$$y_i \geq 0,\ \forall i \in M. \tag{3.46}$$

Now, define $w_i = v_{i0} + \sum_{jinN} v_{ij}x_j$. Without loss of generality, we can also state $x_j = x_j^2$ and $q_{ij} = q_{ij}^2$ as they are binary variables. By doing that, we are able to state constraints (3.38), (3.39) and (3.40) in rotated cone form. Then, the constraints become:

$$y_i w_i \geq 1, \tag{3.47}$$

$$z_{ij} w_i \geq x_j^2, \tag{3.48}$$

$$t_{ij} w_i \geq q_{ij}^2, \tag{3.49}$$

respectively. We also add $v_{i0}y_i + \sum_{jinN} v_{ij}z_{ij} \geq 1$ to strengthen the continuous relaxation of the formulation.

The final conic quadratic MIP formulation with fixed shipping cost is

$$\min \quad \sum_{i \in M} \lambda_i \overline{\pi}_i v_{i0} y_i + \sum_{i \in M}\sum_{j \in N} \lambda_i(\overline{\pi}_i - \pi_{ij}) v_{ij} z_{ij} + \sum_{i \in M}\sum_{j \in N} \lambda_i \tau v_{ij} t_{ij} \tag{3.50}$$

s.t

$$\sum_{j \in N} o_{ij} \leq \kappa_i,\ \forall i \in N, \tag{3.51}$$

$$x_j \geq o_{ij},\ \forall i \in M, \forall j \in N, \tag{3.52}$$

$$x_j \leq \sum_{i \in M} o_{ij},\ \forall j \in N, \tag{3.53}$$

$$q_{ij} = x_j - o_{ij},\ \forall i \in M, \forall j \in N, \tag{3.54}$$

$$v_{i0}y_i + \sum_{j \in N} v_{ij}z_{ij} \geq 1,\ \forall i \in M, \tag{3.55}$$

$$w_i = v_{i0} + \sum_{j \in N} v_{ij}x_j,\ \forall i \in M, \tag{3.56}$$

$$(CONIC) \quad y_i w_i \geq 1,\ \forall i \in M, \tag{3.57}$$

$$z_{ij} w_i \geq x_j^2,\ \forall i \in M, \forall j \in N, \tag{3.58}$$

$$t_{ij} w_i \geq q_{ij}^2,\ \forall i \in M, \forall j \in N, \tag{3.59}$$

$$o_{ij} \in \{0,1\},\ \forall i \in M, \forall j \in N, \tag{3.60}$$

$$x_j \in \{0,1\},\ \forall j \in N, \tag{3.61}$$

$$q_{ij} \geq 0,\ \forall i \in M, \forall j \in N, \tag{3.62}$$

$$z_{ij} \geq 0,\ \forall i \in M, \forall j \in N, \tag{3.63}$$

$$t_{ij} \geq 0,\ \forall i \in M, \forall j \in N, \tag{3.64}$$

$$y_i \geq 0,\ \forall i \in M, \tag{3.65}$$

$$w_i \geq 0,\ \forall i \in M. \tag{3.66}$$

Note that even though we assume consumer homogeneity (single segment) at each location, the formulations we provide can be extended for multiple segments in each location as well. In that case, the overall model will still be MMNL, but with more classes. Thus, the number of binary decision variables do not change, but one now needs to define continuous decision variables (q, z, t, y, w) for each segment at each location.

3.2.1.3 Strengthening the Formulation with McCormick Inequalities

We can further strengthen our formulation by using valid McCormick inequalities for the bilinear terms z and t. Before stating those inequalities, we suggest upper and lower bounds y_i which makes use of the capacity constraint. To that end, we observe the following propositions hold:

Proposition 1. *The following global bounds on variables y_i, $i \in M$, are valid:*

$$y_i^l := \frac{1}{v_{i0} + \sum_{k=1}^{\kappa} v_{i[k]}} \leq y_i, \tag{3.67}$$

$$y_i^u := \frac{1}{v_{i0}} \geq y_i. \tag{3.68}$$

Once again, note that $\kappa = \sum_{i \in M} \kappa_i$. While y_i is reaching to its lowest possible value, the denominator of y_i^l should attain its maximum value. v_{i0} should always appear in the denominator as the consumer always has the possibility to leave without purchase. In addition to no-purchase option, by adding the first κ products with the highest preference values, one can reach a tight and valid lower bound on y_i for all $i \in M$. y_i^u is straightforward.

Proposition 2. *The following conditional bounds on variables y_i, $i \in M$ are valid:*

$$x_j = 0 \;\Rightarrow\; y^l_{i|x_j=0} := \frac{1}{v_{i0} + \sum_{k=1}^{\kappa} \overline{v}_{i[k]}} \le y_i, \tag{3.69}$$

$$x_j = 1 \;\Rightarrow\; \begin{cases} y^l_{i|x_j=1} := \dfrac{1}{v_{i0} + v_{ij} + \sum_{k=1}^{\kappa-1} \overline{v}_{i[k]}} \le y_i, \\ y^u_{i|x_j=1} := \dfrac{1}{v_{i0} + v_{ij}} \ge y_i, \end{cases} \tag{3.70}$$

$$q_{ij} = 0 \;\Rightarrow\; y^l_{i|q_{ij}=0} := \frac{1}{v_{i0} + \sum_{k=1}^{\kappa} v_{i[k]}} \le y_i, \tag{3.71}$$

$$q_{ij} = 1 \;\Rightarrow\; \begin{cases} y^l_{i|q_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij} + \sum_{k=1}^{\kappa-1} \overline{v}_{i[k]}} \le y_i, \\ y^u_{i|q_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij}} \ge y_i, \end{cases} \tag{3.72}$$

where $\overline{v}_{i[k]}$ is the k^{th} largest of preferences v_{it}, $t \in N \setminus \{j\}$.

Proposition 2 states bounds on variables y_i conditioned on x_j and q_{ij} respectively. When $x_j = 0$, product j is not included in the assortment. Then, to obtain the lower bound on y_i, we can add the first κ products with the highest preference values which does not include product j. While $x_j = 1$, we follow the same idea to find $y^l_{i|x_j=1}$. It is known that product j is included in the assortment for sure. Hence, we have one less capacity to fill with the products with highest preference values and we exclude product j from that sum of the highest ordered preferences. In order to find $y^u_{i|x_j=1}$ in (3.70), we also need to add v_{ij} to the denominator in (3.68), as we know product j is included for sure. Then, to find the upper bound, the least possible value that may appear in the denominator becomes $v_{i0} + v_{ij}$.

Finding bounds on y_i conditioned on q_{ij} is a bit more challenging. We can attain $q_{ij} = x_j - o_{ij} = 0$ in two possible ways. Either $x_j = o_{ij} = 0$ or $x_j = o_{ij} = 1$.

Thus, while constructing (3.71), we cannot assume that product j is not in the assortment set and simply exclude v_{ij} as in (3.69). So, we sum up the first κ largest preferences. On the other hand, the idea behind (3.72) is the same as (3.70). This is because the only way $q_{ij} = 1$ is to have $x_j = 1$ and $o_{ij} = 0$; which indicates that we have j in the assortment set.

Using the above propositions, we can write the following valid McCormick inequalities [58] for each bilinear term.
For $z_{ij} = x_j y_i$, we add:

$$(MC\ for\ z_{ij}) \quad z_{ij} \leq y^u_{i|x_j=1} x_j,\ \forall i \in M, \forall j \in N, \tag{3.73}$$

$$z_{ij} \geq y^l_{i|x_j=1} x_j,\ \forall i \in M, \forall j \in N, \tag{3.74}$$

$$z_{ij} \leq y_i - y^l_{i|x_j=0}(1 - x_j),\ \forall i \in M, \forall j \in N, \tag{3.75}$$

$$z_{ij} \geq y_i - y^u_i(1 - x_j),\ \forall i \in M, \forall j \in N. \tag{3.76}$$

For $t_{ij} = q_{ij} y_i$, we add:

$$(MC\ for\ t_{ij}) \quad t_{ij} \leq y^u_{i|q_{ij}=1} q_{ij},\ \forall i \in M, \forall j \in N, \tag{3.77}$$

$$t_{ij} \geq y^l_{i|q_{ij}=1} q_{ij},\ \forall i \in M, \forall j \in N, \tag{3.78}$$

$$t_{ij} \leq y_i - y^l_{i|q_{ij}=0}(1 - q_{ij}),\ \forall i \in M, \forall j \in N, \tag{3.79}$$

$$t_{ij} \geq y_i - y^u_i(1 - q_{ij}),\ \forall i \in M, \forall j \in N. \tag{3.80}$$

The formulation and valid inequalities to strength the formulation are now complete for our baseline model. In the next section, we study the first variant when we have no longer fixed shipping costs between locations. We formulate the problem as an MILP and conic MIP and provide the necessary McCormick inequalities.

3.2.2 Common Assortment Under Different Shipping Costs

In this section, we present a new $|M| \times |M|$ matrix Φ, where the entry Φ_{iw} represents the w^{th} preferred location to ship from for location i. For example, assume below matrix is the preference matrix of the system with three distribution centers:

$$\Phi = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

If the first center was to ship a product from somewhere to itself, the first preference of it would again be itself. The second preference would be to ship from center 2, the last preference of it would be to ship from center 3. Similarly, the preferences of the second and third center is {2,1,3} and {3,2,1} respectively.

As with the new model, different shipping costs/distances between locations matter. Thus, we also introduce a new decision variable p_{ikj}, which takes 1 if the product j is shipped from location k to location i and 0 otherwise. Below, first we present the new MILP formulation with different shipping costs:

$$\max \quad \sum_{i \in M} \sum_{j \in N} \lambda_i \pi_{ij} v_{ij} z_{ij} - \sum_{i \in M} \sum_{k \in M} \sum_{j \in N} \lambda_i \tau_{ikj} v_{ij} t_{ikj} \tag{3.81}$$

s.t

$$\sum_{j \in N} o_{ij} \leq \kappa_i, \ \forall i \in N, \tag{3.82}$$

$$x_j \geq o_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.83}$$

$$x_j \leq \sum_{i \in M} o_{ij}, \ \forall j \in N, \tag{3.84}$$

$$p_{ikj} \leq o_{kj}, \ \forall i, k \in M, \forall j \in N, \tag{3.85}$$

$$\sum_{k \in M} p_{ikj} = x_j, \ \forall i \in M, \forall j \in N, \tag{3.86}$$

$$0 \leq p_{ikj} \leq 1, \ \forall i, k \in M, \forall j \in N, \tag{3.87}$$

$$p_{i\Phi_{iwj}} \geq o_{\Phi_{iwj}} - \sum_{s=1}^{w-1} o_{\Phi_{isj}}, \ \forall i, w \in M, j \in N, \tag{3.88}$$

$$v_{i0}y_i + \sum_{j \in N} v_{ij}z_{ij} = 1, \ \forall i \in M, \tag{3.89}$$

$$(MILP\ for\ z_{ij}) \quad v_{i0}(y_i - z_{ij}) \leq 1 - x_j, \ \forall i \in M, \forall j \in N, \tag{3.90}$$

$$0 \leq z_{ij} \leq y_i, \ \forall i \in M, \forall j \in N, \tag{3.91}$$

$$v_{i0}z_{ij} \leq x_j, \ \forall i \in M, \forall j \in N, \tag{3.92}$$

$$(MILP\ for\ t_{ikj}) \quad v_{i0}(y_i - t_{ikj}) \leq 1 - p_{ikj}, \ \forall i, k \in M, \forall j \in N, \tag{3.93}$$

$$0 \leq t_{ikj} \leq y_i, \ \forall i, k \in M, \forall j \in N, \tag{3.94}$$

$$v_{i0}t_{ikj} \leq p_{ikj}, \ \forall i, k \in M, \forall j \in N, \tag{3.95}$$

$$o_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, \tag{3.96}$$

$$x_j \in \{0, 1\}, \ \forall j \in N, \tag{3.97}$$

$$p_{ikj} \geq 0, \ \forall i, k \in M, \forall j \in N, \tag{3.98}$$

$$z_{ij} \geq 0, \ \forall i \in M, \forall j \in N, \tag{3.99}$$

$$t_{ikj} \geq 0, \ \forall i, k \in M, \forall j \in N, \tag{3.100}$$

$$y_i \geq 0, \ \forall i \in M. \tag{3.101}$$

where $z_{ij} = x_j y_i$, $t_{ikj} = p_{ikj} y_i$.

Now, with the introduction of p_{ikj} variable to the model, the cost side of the baseline model (3.13) slightly changes. Instead of $x_j - o_{ij}$, we use p_{ikj} directly. This is because in (3.85)-(3.86), we force product j to be shipped from some location to i if it is available in any center, and 0 if the product is not offered in anywhere. We define it as a non-negative continuous variable which takes values between [0,1] (3.87) and in (3.88) we force it to take its extreme values. By this way, we avoid the complexity of adding one more binary variable, which has a significant effect on solution times. To demonstrate the idea, assume the preference matrix in the example above and say we want to find the best possible location to ship product j to location 1. Assume $o_{1j} = 0, \ o_{2j} = 1, \ o_{3j} = 1$. Looking at the Φ matrix above, we induce that it is best to ship from location 2. Therefore, we want to force p_{12j} to take 1. Now, by adding constraint (3.88), we will have the following constraints in open form:

$$p_{11j} \geq o_{1j} = 0, \tag{3.102}$$

$$p_{12j} \geq o_{2j} - (o_{1j}) = 1, \tag{3.103}$$

$$p_{13j} \geq o_{3j} - (o_{1j} + o_{2j}) = 0. \tag{3.104}$$

These constraints, together with $p_{11j} + p_{12j} + p_{13j} = 1$ forces the solution to be $(p_{11j}, p_{12j}, p_{13j}) = (0, 1, 0)$. The constraints (3.90)-(3.95) labeled with $(MILP\ for\ z_{ij})$ and $(MILP\ for\ t_{ikj})$ are the Big-M constraints obtained by the same way of linearization of the bilinear terms, presented in section (3.2.1.1). Now, we will see the conic formulation of this model.

3.2.2.1 Conic Formulation for Different Shipping Costs

In order to transform MILP model to a conic program, we apply the same steps presented in the baseline model. Thus, we first subtract the MILP objective from $\sum_{i \in M} \lambda_i \overline{\pi}_i$ to obtain a minimization in the objective function. Then, we put lower bounds on y, z and t variables: $y_i \geq (1/(v_{i0} + \sum_{j \in N} v_{ij} x_j))$, $z_{ij} \geq x_j y_i$, $t_{ikj} \geq p_{ikj} y_i$. Finally, we introduce $w_i = v_{i0} + \sum_{j \in N} v_{ij} x_j$ to formulate lower bound inequalities as rotating cone constraints, labeled as (CONIC) in the model below.

The final conic formulation with different shipping costs is as follows:

$$\min \quad \sum_{i \in M} \lambda_i \overline{\pi}_i v_{i0} y_i + \sum_{i \in M} \sum_{j \in N} \lambda_i (\overline{\pi}_i - \pi_{ij}) v_{ij} z_{ij} + \sum_{i \in M} \sum_{k \in M} \sum_{j \in N} \lambda_i \tau_{ikj} v_{ij} t_{ikj} \tag{3.105}$$

s.t

$$\sum_{j \in N} o_{ij} \leq \kappa_i, \ \forall i \in N, \tag{3.106}$$

$$x_j \geq o_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.107}$$

$$x_j \leq \sum_{i \in M} o_{ij}, \ \forall j \in N, \tag{3.108}$$

$$p_{ikj} \leq o_{kj}, \ \forall i, k \in M, \forall j \in N, \tag{3.109}$$

$$\sum_{k\in M} p_{ikj} = x_j,\ \forall i\in M, \forall j\in N, \tag{3.110}$$

$$0 \le p_{ikj} \le 1,\ \forall i,k\in M, \forall j\in N, \tag{3.111}$$

$$p_{i\Phi_{iw}j} \ge o_{\Phi_{iw}j} - \sum_{s=1}^{w-1} o_{\Phi_{is}j},\ \forall i,w\in M, j\in N, \tag{3.112}$$

$$v_{i0}y_i + \sum_{j\in N} v_{ij}z_{ij} = 1,\ \forall i\in M, \tag{3.113}$$

$$w_i = v_{i0} + \sum_{j\in N} v_{ij}x_j,\ \forall i\in M, \tag{3.114}$$

$$(CONIC)\quad y_i w_i \ge 1,\ \forall i\in M, \tag{3.115}$$

$$z_{ij}w_i \ge x_j^2,\ \forall i\in M, \forall j\in N, \tag{3.116}$$

$$t_{ikj}w_i \ge p_{ikj}^2,\ \forall i\in M, \forall j\in N, \tag{3.117}$$

$$o_{ij}\in\{0,1\},\ \forall i\in M, \forall j\in N, \tag{3.118}$$

$$x_j\in\{0,1\},\ \forall j\in N, \tag{3.119}$$

$$p_{ikj}\ge 0,\ \forall i,k\in M, \forall j\in N, \tag{3.120}$$

$$z_{ij}\ge 0,\ \forall i\in M, \forall j\in N, \tag{3.121}$$

$$t_{ikj}\ge 0,\ \forall i,k\in M, \forall j\in N, \tag{3.122}$$

$$y_i\ge 0,\ \forall i\in M, \tag{3.123}$$

$$w_i\ge 0,\ \forall i\in M. \tag{3.124}$$

3.2.2.2 Strengthening the Formulation with McCormick Inequalities

The formulation can be further strengthened by McCormick estimators. The valid inequalities for z_{ij}, i.e. the formulation given in (2.73)-(2.76) will be the same for this formulation, too. On the other hand, for t_{ikj}, we can write below bounds on y variable:

Proposition 3. *The following global bounds on variables y_i, $i\in M$, are valid:*

$$p_{ikj}=0 \ \Rightarrow\ \begin{cases} y^l_{i|p_{ikj}=0} := \dfrac{1}{v_{i0}+\sum_{k=1}^{\kappa} v_{i[k]}} \le y_i, \\ y^u_{i|p_{ikj}=0} := \dfrac{1}{v_{i0}} \ge y_i, \end{cases} \tag{3.125}$$

$$p_{ikj} = 1 \;\Rightarrow\; \begin{cases} y^l_{i|p_{ikj}=1} := \dfrac{1}{v_{i0} + v_{ij} + \sum_{k=1}^{\kappa-1} \overline{v}_{i[k]}} \le y_i, \\ y^u_{i|p_{ikj}=1} := \dfrac{1}{v_{i0} + v_{ij}} \ge y_i. \end{cases} \tag{3.126}$$

Using those conditional bounds in Proposition 3, we come up with the following McCormick inequalities for t_{ikj}:

$$(MC\ for\ t_{ikj}) \quad t_{ikj} \le y^u_{i|p_{ikj}=1} p_{ikj},\ \forall i, k \in M, \forall j \in N, \tag{3.127}$$

$$t_{ikj} \ge y^l_{i|p_{ikj}=1} p_{ikj},\ \forall i, k \in M, \forall j \in N, \tag{3.128}$$

$$t_{ikj} \le y_i - y^l_{i|p_{ikj}=0}(1 - p_{ikj}),\ \forall i, k \in M, \forall j \in N, \tag{3.129}$$

$$t_{ikj} \ge y_i - y^u_{i|p_{ikj}=0}(1 - p_{ikj}),\ \forall i, k \in M, \forall j \in N. \tag{3.130}$$

Combining those with z_{ij} inequalities, we come up with a stronger formulation of which we study the effectiveness in the next chapter.

3.2.3 Modeling the Second Variant: Customized Assortments

Our assumption in the first variant was the following, each customer class were able to see the entire assortment in each location once they enter the website and they were able to choose and purchase any product from the entire assortment. In this variant, we relax this assumption. The company is now able to choose what product to show to which customer class as well as the assortment in each location. In this case, the company may even choose not to show a product j to customer class i, even if it is available in distribution center i. To that end, we introduce a new binary variable m_{ij} to the model instead of x_j variable used in earlier formulations. The binary variable m_{ij} takes 1 if product j is shown to customer class i, and 0 otherwise.Then, the total revenue obtained at location i under MNL model is expressed as:

$$\frac{\sum_{j \in N} \pi_{ij} v_{ij} m_{ij}}{v_{i0} + \sum_{j \in N} v_{ij} m_{ij}}. \tag{3.131}$$

Now, we need to express the shipping costs incurred. For a given customer class i and product j, we want to subtract the shipping cost if and only if that product is shown to class i, but not available in that location. This is only possible whenever $m_{ij} = 1$ but $o_{ij} = 0$. Therefore, we introduce a new continuous variable a_{ij} that takes on the following values based on m_{ij} and o_{ij}:

m_{ij}	o_{ij}	a_{ij}
1	0	1
1	1	0
0	1	0
0	0	0

We can force these values to hold by adding $a_{ij} \geq m_{ij} - o_{ij}$, $a_{ij} \leq 1 - o_{ij}$ and $a_{ij} \leq m_{ij}$ to the model. By using this new variable, we express the total cost of shipment to location i as

$$\frac{\sum_{j \in N} \tau v_{ij} a_{ij}}{v_{i0} + \sum_{j \in N} v_{ij} m_{ij}}. \tag{3.132}$$

3.2.3.1 MILP Formulation Under a Fixed Shipping Cost

Let $y_i = 1/(v_{i0} + \sum_{j \in N} v_{ij} m_{ij})$, $z_{ij} = m_{ij} y_i$ and $t_{ij} = a_{ij} y_i$. Then, following the approach taken for the baseline model, we can reformulate the problem as an MILP:

$$\max \quad \sum_{i \in M} \sum_{j \in N} \lambda_i \pi_{ij} v_{ij} z_{ij} - \sum_{i \in M} \sum_{j \in N} \lambda_i \tau v_{ij} t_{ij} \tag{3.133}$$

s.t

$$\sum_{j \in N} o_{ij} \leq \kappa_i, \ \forall i \in M, \tag{3.134}$$

$$m_{kj} \leq \sum_{i \in M} o_{ij}, \ \forall k \in M, \forall j \in N, \tag{3.135}$$

$$\sum_{i \in M} m_{ij} \geq o_{kj}, \ \forall k \in M, \forall j \in N, \tag{3.136}$$

$$a_{ij} \geq m_{ij} - o_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.137}$$

$$a_{ij} \leq 1 - o_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.138}$$

$$a_{ij} \leq m_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.139}$$

$$v_{i0}y_i + \sum_{j \in N} v_{ij}z_{ij} = 1, \ \forall i \in M, \tag{3.140}$$

$$(MILP\ for\ z_{ij}) \quad v_{i0}(y_i - z_{ij}) \leq 1 - m_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.141}$$

$$0 \leq z_{ij} \leq y_i, \ \forall i \in M, \forall j \in N, \tag{3.142}$$

$$v_{i0}z_{ij} \leq m_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.143}$$

$$(MILP\ for\ t_{ij}) \quad v_{i0}(y_i - t_{ij}) \leq 1 - a_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.144}$$

$$0 \leq t_{ij} \leq y_i, \ \forall i \in M, \forall j \in N, \tag{3.145}$$

$$v_{i0}t_{ij} \leq a_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.146}$$

$$o_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, \tag{3.147}$$

$$m_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, \tag{3.148}$$

$$a_{ij} \geq 0, \ \forall i \in M, \forall j \in N, \tag{3.149}$$

$$z_{ij} \geq 0, \ \forall i \in M, \forall j \in N, \tag{3.150}$$

$$t_{ij} \geq 0, \ \forall i \in M, \forall j \in N, \tag{3.151}$$

$$y_i \geq 0, \ \forall i \in M. \tag{3.152}$$

We replace (3.15)-(3.16) with (3.135)-(3.136) in order to ensure that product j cannot be offered to a customer class if it is not carried in any of the distribution centers (3.135). If it is available, it is offered to at least one customer class (3.136). Constraints (3.137)-(3.139) ensure that the shipping cost variable a_{ij} takes the appropriate value as discussed above. (3.141)-(3.146) deal with the linerization of $m.y$ and $a.y$. Finally, we define non-negative continuous and binary variables. In the next seciton, we will reformulate the problem as a conic MIP.

3.2.3.2 Conic Formulation Under a Fixed Shipping Cost

The conic formulation of the second variant under constant shipping costs is as follows:

$$\min \quad \sum_{i \in M} \lambda_i \overline{\pi}_i v_{i0} y_i + \sum_{i \in M} \sum_{j \in N} \lambda_i (\overline{\pi}_i - \pi_{ij}) v_{ij} z_{ij} + \sum_{i \in M} \sum_{j \in N} \lambda_i \tau v_{ij} t_{ij} \tag{3.153}$$

s.t

$$
\begin{aligned}
& \sum_{j \in N} o_{ij} \leq \kappa_i, \ \forall i \in M, && (3.154)\\
& m_{kj} \leq \sum_{i \in M} o_{ij}, \ \forall k \in M, \forall j \in N, && (3.155)\\
& \sum_{i \in M} m_{ij} \geq o_{kj}, \ \forall k \in M, \forall j \in N, && (3.156)\\
& a_{ij} \geq m_{ij} - o_{ij}, \ \forall i \in M, \forall j \in N, && (3.157)\\
& a_{ij} \leq 1 - o_{ij}, \ \forall i \in M, \forall j \in N, && (3.158)\\
& a_{ij} \leq m_{ij}, \ \forall i \in M, \forall j \in N, && (3.159)\\
& v_{i0} y_i + \sum_{j \in N} v_{ij} z_{ij} \geq 1, \ \forall i \in M, && (3.160)\\
& w_i = v_{i0} + \sum_{j \in N} v_{ij} m_{ij}, \ \forall i \in M, && (3.161)\\
(CONIC) \quad & y_i w_i \geq 1, \ \forall i \in M, && (3.162)\\
& z_{ij} w_i \geq m_{ij}^2, \ \forall i \in M, \forall j \in N, && (3.163)\\
& t_{ij} w_i \geq a_{ij}^2, \ \forall i \in M, \forall j \in N, && (3.164)\\
& o_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, && (3.165)\\
& m_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, && (3.166)\\
& a_{ij} \geq 0, \ \forall i \in M, \forall j \in N, && (3.167)\\
& z_{ij} \geq 0, \ \forall i \in M, \forall j \in N, && (3.168)\\
& t_{ij} \geq 0, \ \forall i \in M, \forall j \in N, && (3.169)\\
& y_i \geq 0, \ \forall i \in M, && (3.170)\\
& w_i \geq 0, \ \forall i \in M. && (3.171)
\end{aligned}
$$

While transforming linear program to a conic program, we first change the objective function to a minimization in the same way presented in the baseline model. The constraints from (3.154) to (3.159) are common for both the linear and conic model of this problem. We already defined $y_i = 1/(v_{i0} + \sum_{j \in N} v_{ij} m_{ij})$, so in (3.160), we put a lower bound on y_i to strengthen the relaxation. Following the same idea in the baseline model, we then let $w_i = v_{i0} + \sum_{j \in N} v_{ij} m_{ij}$ (3.161) and state y, z and t in rotated cone form in (3.162)-(3.164). Finally, we introduce all variables to the model in (3.165)-(3.171).

3.2.3.3 Strengthening the Formulation with McCormick Inequalities

We follow the same structure that we have followed in the first variant in order to get upper and lower bounds on variables m_{ij} and a_{ij}.

Proposition 4. *The following global bounds on variables m_{ij} and a_{ij} are valid:*

$$m_{ij} = 0 \;\Rightarrow\; \begin{cases} y^l_{i|m_{ij}=0} := \dfrac{1}{v_{i0} + \sum_{k=1}^{\kappa} v_{i[k]}} \le y_i, \\ y^u_{i|m_{ij}=0} := \dfrac{1}{v_{i0}} \ge y_i, \end{cases} \tag{3.172}$$

$$m_{ij} = 1 \;\Rightarrow\; \begin{cases} y^l_{i|m_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij} + \sum_{k=1}^{\kappa-1} \overline{v}_{i[k]}} \le y_i, \\ y^u_{i|m_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij}} \ge y_i, \end{cases} \tag{3.173}$$

$$a_{ij} = 0 \;\Rightarrow\; \begin{cases} y^l_{i|a_{ij}=0} := \dfrac{1}{v_{i0} + \sum_{k=1}^{\kappa} v_{i[k]}} \le y_i, \\ y^u_{i|a_{ij}=0} := \dfrac{1}{v_{i0}} \ge y_i, \end{cases} \tag{3.174}$$

$$a_{ij} = 1 \;\Rightarrow\; \begin{cases} y^l_{i|a_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij} + \sum_{k=1}^{\kappa-1} \overline{v}_{i[k]}} \le y_i, \\ y^u_{i|a_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij}} \ge y_i. \end{cases} \tag{3.175}$$

The valid McCormick inequalities for z_{ij} and t_{ij} then are as follows:

$$(MC\ for\ z_{ij}) \quad z_{ij} \le y^u_{i|m_{ij}=1} m_{ij}, \;\; \forall i \in M, \forall j \in N, \tag{3.176}$$

$$z_{ij} \ge y^l_{i|m_{ij}=1} m_{ij}, \;\; \forall i \in M, \forall j \in N, \tag{3.177}$$

$$z_{ij} \le y_i - y^l_{i|m_{ij}=0}(1 - m_{ij}), \;\; \forall i \in M, \forall j \in N, \tag{3.178}$$

$$z_{ij} \ge y_i - y^u_{i|m_{ij}=0}(1 - m_{ij}), \;\; \forall i \in M, \forall j \in N, \tag{3.179}$$

$$(MC\ for\ t_{ij}) \quad t_{ij} \le y^u_{i|a_{ij}=1} a_{ij}, \;\; \forall i, k \in M, \forall j \in N, \tag{3.180}$$

$$t_{ij} \ge y^l_{i|a_{ij}=1} a_{ij}, \;\; \forall i, k \in M, \forall j \in N, \tag{3.181}$$

$$t_{ij} \le y_i - y^l_{i|a_{ij}=0}(1 - a_{ij}), \;\; \forall i, k \in M, \forall j \in N, \tag{3.182}$$

$$t_{ij} \ge y_i - y^u_{i|a_{ij}=0}(1 - a_{ij}), \;\; \forall i, k \in M, \forall j \in N. \tag{3.183}$$

By adding these valid inequalities to conic model, we show that we strengthen our formulation and decrease the gap between the linear programming relaxation and the integer model. In the next section, we again relax the constant shipping cost assumption and let the costs vary by location.

3.2.4 Customized Assortments Under Different Shipping Costs

Similar to the model with different shipping costs in the first variant, we relax the fixed cost assumptions. Thus, we replace a_{ij} variables with p_{ikj} which represents the decision of shipping product j from location k to location i. Then, we first let $z_{ij} = m_{ij} y_i$, $t_{ikj} = p_{ikj} y_i$ and again let $y_i = 1/(v_{i0} + \sum_{j \in N} v_{ij} m_{ij})$. Using these variables, we formulate an MILP model for the problem of choosing the optimal assortment for each location while deciding which product to show to which customer class. The linear programming formulation with different shipping costs is as follows:

$$\max \quad \sum_{i \in M} \sum_{j \in N} \lambda_i \pi_{ij} v_{ij} z_{ij} - \sum_{i \in M} \sum_{j \in N} \lambda_i \tau_{ikj} v_{ij} t_{ikj} \tag{3.184}$$

s.t

$$\sum_{j \in N} o_{ij} \leq \kappa_i, \ \forall i \in M, \tag{3.185}$$

$$m_{kj} \leq \sum_{i \in M} o_{ij}, \ \forall k \in M, \forall j \in N, \tag{3.186}$$

$$\sum_{i \in M} m_{ij} \geq o_{kj}, \ \forall k \in M, \forall j \in N, \tag{3.187}$$

$$p_{ikj} \leq m_{ij}, \ \forall i, k \in M, \forall j \in N, \tag{3.188}$$

$$\sum_{k \in M} p_{ikj} = m_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.189}$$

$$f_{i \Phi_{iw} j} \geq o_{\Phi_{iw} j} - \sum_{s=1}^{w-1} o_{\Phi_{is} j}, \ \forall i, w \in M, j \in N, \tag{3.190}$$

$$p_{ikj} \leq f_{ikj}, \ \forall i, k \in M, \forall j \in N, \tag{3.191}$$

$$f_{ikj} \leq o_{kj}, \ \forall i, k \in M, \forall j \in N, \tag{3.192}$$

$$\sum_{k \in M} f_{ikj} \leq 1, \forall i \in M, \forall j \in N, \tag{3.193}$$

$$v_{i0} y_i + \sum_{j \in N} v_{ij} z_{ij} = 1, \ \forall i \in M, \tag{3.194}$$

$$(MILP \ for \ z_{ij}) \quad v_{i0}(y_i - z_{ij}) \leq 1 - m_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.195}$$

$$0 \leq z_{ij} \leq y_i, \ \forall i \in M, \forall j \in N, \tag{3.196}$$

$$v_{i0} z_{ij} \leq m_{ij}, \ \forall i \in M, \forall j \in N, \tag{3.197}$$

$$(MILP \ for \ t_{ikj}) \quad v_{i0}(y_i - t_{ikj}) \leq 1 - p_{ikj}, \ \forall i, k \in M, \forall j \in N, \tag{3.198}$$

$$0 \leq t_{ikj} \leq y_i, \ \forall i, k \in M, \forall j \in N, \tag{3.199}$$

$$v_{i0} t_{ikj} \leq p_{ikj}, \ \forall i, k \in M, \forall j \in N, \tag{3.200}$$

$$0 \leq p_{ikj} \leq 1, \ \forall i, k \in M, \forall j \in N, \tag{3.201}$$

$$0 \leq f_{ikj} \leq 1, \ \forall i, k \in M, \forall j \in N, \tag{3.202}$$

$$o_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, \tag{3.203}$$

$$m_{ij} \in \{0, 1\}, \ \forall i \in M, \forall j \in N, \tag{3.204}$$

$$z_{ij} \geq 0, \ \forall i \in M, \forall j \in N, \tag{3.205}$$

$$t_{ikj} \geq 0, \ \forall i, k \in M, \forall j \in N, \tag{3.206}$$

$$y_i \geq 0, \ \forall i \in M. \tag{3.207}$$

As before, the objective function maximizes the expected profit which is expected revenue minus the expected transportation costs. Constraints (3.185)-(3.187) are same as the constraints that were used in earlier formulation. In (3.188), we prevent any kind of shipment from any location if product j is not visible to class i as there will not be any purchase for that product in that location. Subsequently, in (3.189), we ensure that the product will be supplied to location i from at least one location, if it is offered for the customers in that location. Following this, we add (3.190) to enforce the same logical relationship that is used and explained in (3.88), in a different cost setting. But this time, we replace the continuous p_{ikj} variable with a continuous f_{ikj}. We use the variable f_{ikj} to identify the closest location to ship product j to location i (if any) and use it to be an upper bound for p_{ikj}, the actual shipment decision variable. This allows the retailer ship a product to a region from the closest region only if the retailer decides to offer that product in that region. (3.192) prevents the shipment of a product from location k if it is not available in the assortment of distribution center k. (3.193) then bounds it with 1 so that when combined with (3.190), it acts as if it is a binary variable. We add (3.194) by definition of y_i. Finally, the inequalities labeled with ($MILP\ for\ z_{ij}$) and ($MILP\ for\ t_{ikj}$) are added as the Big-M constraints for the bilinear terms $m.y$ and $p.y$.

In the next section, we are going to reformulate the problem as a conic program.

3.2.4.1 Conic Formulation for Different Shipping Costs

The conic formulation of the second variant under different shipping costs setting is presented below. To construct this formulation, we first transform the MILP objective to a minimization problem. We keep the MILP constraints (3.185)-(3.193) as they are. Then by definition of y_i, we add (3.218). Although this constraint is redundant, it not trivial to add this, as we realized that it is a useful valid cut that affects solution times. Then, we let $w_i = v_{i0} + \sum_{jinN} v_{ij} m_{ij}$ (3.219) to express the lower bounds on y, z and t as conic constraints. Doing this, those variables can be bounded as the following: $y \geq 1/w$, $z \geq m/w$ and $t \geq p/w$. As m and P takes binary values, it will not be wrong to state $m = m^2$ and $p = p^2$ holds. Using this, the above inequalities become $yw \geq 1$, $zw \geq m^2$ and $tw \geq p^2$. We add those inequalities as constraints which are labeled as (CONIC). Then we finalize the model by introducing all non-negative binary and continuous variables to the model.

$$\min \quad \sum_{i\in M} \lambda_i \overline{\pi}_i v_{i0} y_i + \sum_{i\in M} \sum_{j\in N} \lambda_i (\overline{\pi}_i - \pi_{ij}) v_{ij} z_{ij} + \sum_{i\in M} \sum_{k\in M} \sum_{j\in N} \lambda_i \tau_{ikj} v_{ij} t_{ikj} \tag{3.208}$$

s.t

$$\sum_{j\in N} o_{ij} \leq \kappa_i, \ \forall i \in M, \tag{3.209}$$

$$m_{kj} \leq \sum_{i\in M} o_{ij}, \ \forall k \in M, \forall j \in N, \tag{3.210}$$

$$\sum_{i\in M} m_{ij} \geq o_{kj}, \ \forall k \in M, \forall j \in N, \tag{3.211}$$

$$p_{ikj} \leq m_{ij},\ \forall i, k \in M, \forall j \in N, \tag{3.212}$$

$$\sum_{k \in M} p_{ikj} = m_{ij},\ \forall i \in M, \forall j \in N, \tag{3.213}$$

$$f_{i\Phi_{iw}j} \geq o_{\Phi_{iw}j} - \sum_{s=1}^{w-1} o_{\Phi_{is}j},\ \forall i, w \in M, j \in N, \tag{3.214}$$

$$p_{ikj} \leq f_{ikj},\ \forall i, k \in M, \forall j \in N, \tag{3.215}$$

$$f_{ikj} \leq o_{kj},\ \forall i, k \in M, \forall j \in N, \tag{3.216}$$

$$\sum_{k \in M} f_{ikj} \leq 1, \forall i \in M, \forall j \in N, \tag{3.217}$$

$$v_{i0} y_i + \sum_{j \in N} v_{ij} z_{ij} \geq 1,\ \forall i \in M, \tag{3.218}$$

$$w_i = v_{i0} + \sum_{j \in N} v_{ij} m_{ij},\ \forall i \in M, \tag{3.219}$$

$$(CONIC) \quad y_i w_i \geq 1,\ \forall i \in M, \tag{3.220}$$

$$z_{ij} w_i \geq m_{ij}^2,\ \forall i \in M, \forall j \in N, \tag{3.221}$$

$$t_{ikj} w_i \geq p_{ikj}^2,\ \forall i, k \in M, \forall j \in N, \tag{3.222}$$

$$o_{ij} \in \{0, 1\},\ \forall i \in M, \forall j \in N, \tag{3.223}$$

$$m_{ij} \in \{0, 1\},\ \forall i \in M, \forall j \in N, \tag{3.224}$$

$$0 \leq p_{ikj} \leq 1,\ \forall i, k \in M, \forall j \in N, \tag{3.225}$$

$$0 \leq f_{ikj} \leq 1,\ \forall i, k \in M, \forall j \in N, \tag{3.226}$$

$$z_{ij} \geq 0,\ \forall i \in M, \forall j \in N, \tag{3.227}$$

$$t_{ikj} \geq 0,\ \forall i, k \in M, \forall j \in N, \tag{3.228}$$

$$y_i \geq 0,\ \forall i \in M, \tag{3.229}$$

$$w_i \geq 0,\ \forall i \in M. \tag{3.230}$$

3.2.4.2 Strengthening the Formulation with McCormick Inequalities

Proposition 5. *The following global bounds on variables m_{ij} and p_{ikj} are valid:*

$$m_{ij} = 0 \;\Rightarrow\; \begin{cases} y^l_{i|m_{ij}=0} := \dfrac{1}{v_{i0} + \sum_{k=1}^{\kappa} v_{i[k]}} \leq y_i, \\ y^u_{i|m_{ij}=0} := \dfrac{1}{v_{i0}} \geq y_i, \end{cases} \tag{3.231}$$

$$m_{ij} = 1 \;\Rightarrow\; \begin{cases} y^l_{i|m_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij} + \sum_{k=1}^{\kappa-1} \overline{v}_{i[k]}} \leq y_i, \\ y^u_{i|m_{ij}=1} := \dfrac{1}{v_{i0} + v_{ij}} \geq y_i, \end{cases} \tag{3.232}$$

$$p_{ikj} = 0 \;\Rightarrow\; \begin{cases} y^l_{i|p_{ikj}=0} := \dfrac{1}{v_{i0} + \sum_{k=1}^{\kappa} v_{i[k]}} \leq y_i, \\ y^u_{i|p_{ikj}=0} := \dfrac{1}{v_{i0}} \geq y_i, \end{cases} \tag{3.233}$$

$$p_{ikj} = 1 \;\Rightarrow\; \begin{cases} y^l_{i|p_{ikj}=1} := \dfrac{1}{v_{i0} + v_{ij} + \sum_{k=1}^{\kappa-1} \overline{v}_{i[k]}} \leq y_i, \\ y^u_{i|p_{ikj}=1} := \dfrac{1}{v_{i0} + v_{ij}} \geq y_i. \end{cases} \tag{3.234}$$

Then, the valid McCormick inequalities for z_{ij} and t_{ikj} are as follows:

$$(MC\ for\ z_{ij}) \quad z_{ij} \leq y^u_{i|m_{ij}=1} m_{ij}, \;\; \forall i \in M, \forall j \in N, \tag{3.235}$$

$$z_{ij} \geq y^l_{i|m_{ij}=1} m_{ij}, \;\; \forall i \in M, \forall j \in N, \tag{3.236}$$

$$z_{ij} \leq y_i - y^l_{i|m_{ij}=0}(1 - m_{ij}), \;\; \forall i \in M, \forall j \in N, \tag{3.237}$$

$$z_{ij} \geq y_i - y^u_{i|m_{ij}=0}(1 - m_{ij}), \;\; \forall i \in M, \forall j \in N, \tag{3.238}$$

$$(MC\ for\ t_{ikj}) \quad t_{ikj} \leq y^u_{i|p_{ikj}=1} p_{ikj}, \;\; \forall i, k \in M, \forall j \in N, \tag{3.239}$$

$$t_{ikj} \geq y^l_{i|p_{ikj}=1} p_{ikj}, \;\; \forall i, k \in M, \forall j \in N, \tag{3.240}$$

$$t_{ikj} \leq y_i - y^l_{i|p_{ikj}=0}(1 - p_{ikj}), \;\; \forall i, k \in M, \forall j \in N, \tag{3.241}$$

$$t_{ikj} \geq y_i - y^u_{i|p_{ikj}=0}(1 - p_{ikj}), \;\; \forall i, k \in M, \forall j \in N. \tag{3.242}$$

Chapter 4

Numerical Study and Results

This chapter presents the results of the numerical study that we have conducted to each problem variant under two different sets of parameters. We first compare the performances of linear and conic formulations under the fixed cost setting and discuss the effects of different parameters such as no-purchase preference, capacity, shipping cost on profitability and assortment selection for both problem variants. Then we present the run results of the problems under different costs setting and repeat the same analysis. We first start our discussion by introducing parameter sets.

The numerical study is done via two different set of problems both of which have five different randomly generated samples. The first set includes instances with 50 products ($|N| = 50$) and 5 customer classes ($|M| = 5$). The probabilities of different customer classes arriving into the system are taken to be the same. This means that the demand originates from customer class i with probability $\lambda_i = 1/|M|$. There are different prices for each product which are randomly generated from a uniform distribution between 1 and 3 and the prices are the same across customer classes: $\pi_{ij} = \pi_j$. The customer preferences for products, v_{ij}, are uniformly generated between 0 and 1 and no-purchase preferences are taken as either 5 or 10. The capacities of the warehouses κ_i are assumed to be the same and the total capacities, $\kappa = \sum_{i \in M} \kappa_i$, are taken as 10, 20, 40 and

50 respectively. Thus, at each sample we have 2×4=8 no-purchase and capacity combinations, which yield a total of 40 different instances.

The second set consists of 100 products ($|N| = 100$) and 10 customer classes ($|M| = 10$) and the data is generated the same way as above. The only addition is that the models are also run for the capacity 100 as well as 10, 20, 40 and 50. So for this set, we have 10 combinations at each sample, adding up to 50 instances in total.

In order to study the effectiveness of our formulation and the impact of the change in parameters to commonalities, two sets of instances are solved via the commercial solver Gurobi-64 bit. The instances are solved on a computer with an Intel Core-i5 3.20 GHz processor and 10 GB RAM operating on 64-bit Windows 8. The default settings of the solver are used except the "barrier algorithm" at the root node of the MIP and "outer approximation approach" when solving continuous relaxations of the conic program. Time limit is given as 1800 seconds.

4.1 Fixed Cost Setting

In the fixed cost setting, the cost of shipping products between any two locations are taken as $\tau = 0.5$. Below, we present the run results for the fixed cost setting. Table 4.1- 4.2 present those results for the first variant of our problem where the customers are able to see the entire assortment. Table 4.3- 4.4 on the other hand, present results of the fixed cost setting under the second variant where the e-retailer can choose what product to show to each customer class at each location. For each instance, four different solution approaches are compared in the given tables. "MILP", corresponds to the results of linear programming formulations that are formerly presented in Chapter 3 under "Problem Formulations". "MILP+MC" corresponds to those same formulations where Big-M constraints, labeled as (MILP), are removed and replaced with stronger McCormick constraints that are indicated as (MC) under the corresponding section. "CONIC" corresponds to the results of conic programming formulations with the constraints labeled as (CONIC) and finally, "CONIC+MC" corresponds to the conic formulation plus McCormick inequalities (MC).

The results represent the average values of five samples in each data set. For each combination of no-purchase preference v_0 and total capacity κ, the first column "Gap" under each solution approach represents the average percentage gap between the continuous relaxation objective and the optimal integer objective value of that model, i.e., the integrality gap at the root node. In order to find the optimal integer objective values, we use the objective values of "CONIC+MC" approach as this model usually leads to the optimal one within the time limit, or else leads to the best objective values among all models. Hence, the results of it are taken as the optimal conic objective value. Recall that the optimal conic (minimization) objective and the optimal linear (maximization) objective together sum up to $\sum_{i \in M} \lambda_i \overline{\pi}_i$, we have used this property to obtain the optimal integer maximization objective. The average values of those integer objectives are presented in the last two column.

The second column "Nodes" denotes the number of nodes explored in branch-and-bound algorithm. In the third column "Time", there exists two lines. For the instances that are solved within the time limit, the first line reports the solution time of the model in terms of seconds. The second line reports the average remaining percentage optimality gap for the instances that could not be solved within the pre-determined time limit. Next to those lines, the numbers of such instances are reported within parenthesis. One can also find the results of each five samples of both sets separately for both variants in Appendix A.

The results of the first set of problem instances under first variant is presented in Table 4.1. As observed in previous studies [26], the MILP formulation works poorly even though the capacity constraint is not binding. The root gap reaches up to 54.78% which leads to excessive branching when there is a tight capacity constraint. The minimum root gap is reported as 8.53% when there is no binding capacity. Moreover, we see that the MILP formulation is able to solve only 13 instances out of 40 until optimality. In addition, the remaining optimality gaps at termination are considerably large for the unsolved instances.

Table 4.1: **Common assortment under a fixed cost: Sample averages for problems with 50 products and 5 classes.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	43.39%	573665	450.96 (5) –	7.82%	45360.60	38.33 (5) –	10.20%	56142	102.38 (5) –	1.73%	3950	2.91 (5) –	1.2842	1.6914
	20	19.21%	659083	– 2.41% (5)	5.17%	3148882.00	– 1.90% (5)	15.02%	702512	– 1.64% (5)	2.93%	71983	225.98 (5) –	1.5448	1.4308
	40	10.94%	2093653	991.18 (1) 0.50% (4)	4.00%	4056285.60	– 2.06% (5)	16.05%	398524	457.05 (3) 1.17% (2)	2.75%	11245	14.10 (5) –	1.6596	1.3160
	50	8.53%	5107583	936.28 (2) 0.20% (3)	3.23%	3092638.40	– 1.32% (5)	15.54%	808339	– 1.00% (5)	2.33%	6613	16.42 (5) –	1.6965	1.2791
10	10	54.78%	84244	45.59 (5) –	6.68%	9153.20	6.93 (5) –	5.74%	13015	16.18 (5) –	0.60%	1143	1.56 (5) –	0.8775	2.0981
	20	27.55%	1990243	– 1.75% (5)	5.02%	2416492.60	302.57 (3) 2.21% (2)	9.17%	1375110	– 0.69% (5)	1.46%	35447	72.35 (5) –	1.1675	1.8081
	40	13.89%	2911117	– 0.82% (5)	2.74%	4650015.20	– 1.51% (5)	11.19%	787648	789.20 (2) 0.38% (3)	1.39%	140599	47.49 (5) –	1.3071	1.6685
	50	11.42%	4263511	– 0.46% (5)	2.43%	4163282.20	– 1.37% (5)	11.05%	910859	33.84 (1) 0.35% (4)	1.27%	4725	6.71 (5) –	1.3361	1.6395

Table 4.2: **Common assortment under a fixed cost: Sample averages for problems with 100 products and 10 classes.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	63.49%	24006	– 20.43% (5)	13.84%	56930	– 7.37% (5)	12.08%	32424	– 5.38% (5)	2.53%	14519	136.95 (5) –	1.2894	1.6931
	20	30.83%	74079	– 12.96% (5)	8.89%	194446	– 6.29% (5)	18.61%	23402	– 11.04% (5)	4.10%	87780	– 0.71% (5)	1.6157	1.3668
	40	18.89%	290040	– 16.57% (5)	7.66%	505605	– 5.99% (5)	23.80%	36351	– 16.47% (5)	4.97%	60481	– 0.83% (5)	1.7766	1.2059
	50	17.56%	576997	– 16.48% (5)	8.38%	688000	– 6.75% (5)	24.19%	21636	– 12.60% (5)	5.94%	71899	– 0.97% (5)	1.7962	1.1863
	100	12.10%	90390	– 9.18% (5)	7.08%	1184026	– 5.76% (5)	23.92%	55008	– 14.63% (5)	6.44%	77100	– 0.87% (5)	1.8817	1.1008
10	10	67.84%	397488	– 5.12% (5)	12.71%	123624	1441 (1) 4.28% (4)	6.54%	50539	– 1.38% (5)	0.83%	31682	103.69 (5) –	0.8862	2.0963
	20	49.11%	26746	– 16.84% (5)	9.81%	98612	– 6.50% (5)	11.06%	33219	– 5.19% (5)	2.03%	134500	– 0.28% (5)	1.2251	1.7574
	40	23.49%	54352	– 9.68% (5)	5.55%	370883	– 4.39% (5)	16.23%	49432	– 8.67% (5)	2.63%	107671	– 0.37% (5)	1.4776	1.5049
	50	20.63%	45135	– 8.74% (5)	5.56%	557783	– 4.36% (5)	17.25%	34530	– 8.37% (5)	2.74%	106933	– 0.32% (5)	1.5126	1.4699
	100	15.72%	37180	– 6.68% (5)	6.36%	1229997	– 5.18% (5)	17.83%	65721	– 10.03% (5)	3.82%	119213	– 0.30% (5)	1.5758	1.4067

However in MILP+MC approach, with the addition of McCormick cuts instead of Big-M constraints, there occurs a significant decrease in the root gap. The maximum root gap decreases until 7.82% under a tight capacity constraint while it was 43.39% in the same instance in MILP approach. Still, the same number of instances remain unsolved.

With the introduction of the conic formulation, we observe the that root gaps decrease compared to MILP formulation as well as the number of nodes especially when the capacity is tight. We observe that CONIC model is able to solve three more instances in general. However, if the capacity is not binding, MILP approach may remain with smaller root and termination gaps. It is also hard to argue that CONIC formulation is better than MILP+MC approach as when the capacities are 10 and 20, we observe that it solves three instances which the CONIC approach cannot solve itself and the solution times are faster than CONIC. We obtain the best results over all approaches when we add McCormick inequalities to conic formulation. The CONIC+MC approach is able to solve all instances in the first setting within less than a minute on average. The root gaps decreases dramatically with a maximum gap being only 2.93% which leads to a limited exploration while branching. Consequently, the number of nodes discovered during branch and bound decreases drastically.

The second set is indeed harder than the previous one. In Table 4.2, we observe that the performance of MILP approach gets even poorer, not being able to solve any 50 of the instances. The root and termination gaps are incontrovertibly high. The performance of CONIC approach is no different than MILP except for the slight improvement in root gaps and number of nodes. The MILP+MC performs better compared to those two, especially when the capacity is not very tight. Unlike MILP and CONIC approaches, it is able to solve one instance but still, other instances remain with at least 4.28% optimality gap. Now, with the addition of McCormick estimators to conic formulation, the performance of the model again improves. We are able to solve ten instances within 2.5 minutes. With the joint effect of tightening the formulation with conic constraints as well as McCormick cuts, the root gaps decreases by 89% on the average compared to MILP and all the termination gaps drop below 1%.

It is observed that the second variant of our problem is more difficult to solve than the first one. The reason behind is the increase in number of binary variables introduced to the model since we have separate binary variables for each product and segment prior for carrying and offering. Although the root gaps are very close in both variants, the number of instances that are solved until optimality is less in the second one. In Table 4.3 and 4.4, we again observe that CONIC+MC model over-performs other approaches. In the first set, this approach is able to solve 37 of 40 instances while MILP+MC shows the second best performance by solving only 10 instances. The remaining three sets of instances that are unsolved only has a termination gap of 0.16%, which can be considered as negligible. MILP+MC works also quite efficient and on 75% of the instances, it dominates other two

Table 4.3: **Customized assortments under a fixed cost: Sample averages for problems with 50 products and 5 classes.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	43.39%	538799	– 2.25% (5)	7.90%	164478	390.16 (5) –	10.20%	249690	1021.40 (1) 1.50% (4)	1.73%	6403	20.05 (5) –	1.2842	1.6914
	20	19.21%	223041	– 3.74% (5)	5.49%	1026669	– 2.90% (5)	15.02%	181338	– 6.66% (5)	3.00%	57095	34.95 (4) 0.30% (1)	1.5448	1.4308
	40	10.86%	163026	– 2.37% (5)	4.28%	1035053	– 1.60% (5)	16.19%	192683	– 1.94% (5)	3.19%	260244	126.55 (4) 0.07% (1)	1.6608	1.3148
	50	8.40%	196874	– 1.54% (5)	3.45%	1084163	– 0.97% (5)	15.64%	277925	– 1.12% (5)	2.71%	33514	61.06 (5) –	1.6985	1.2771
10	10	54.78%	259883	425.99 (5) –	6.72%	15646	17.55 (5) –	5.74%	297993	587.97 (4) 0.42% (1)	0.60%	770	3.22 (5) –	0.8775	2.0981
	20	27.55%	549472	– 2.07% (5)	5.11%	2047823	– 1.73% (5)	9.17%	228552	– 2.76% (5)	1.46%	86379	11.27 (4) 0.11% (1)	1.1675	1.8081
	40	13.85%	468196	– 1.25% (5)	3.02%	3001578	– 1.72% (5)	11.26%	204184	– 0.70% (5)	1.51%	3441	17.18 (5) –	1.3076	1.6680
	50	11.34%	693390	– 1.05% (5)	2.67%	2508603	– 1.21% (5)	11.12%	373379	– 0.52% (5)	1.37%	3430	14.95 (5) –	1.3370	1.6386

Table 4.4: **Customized assortments under a fixed cost: Sample averages for problems with 100 products and 10 classes.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	61.94%	22993	– 23.09% (5)	12.84%	52015	– 7.68% (5)	12.07%	22778	– 21.80% (5)	2.53%	50411	820.73 (5) –	1.2894	1.6931
	20	29.81%	20661	– 11.41% (5)	8.24%	160243	– 6.24% (5)	18.61%	10310	– 31.58% (5)	4.11%	45271	– 0.99% (5)	1.6156	1.3669
	40	18.04%	22687	– 9.16% (5)	7.18%	83683	– 6.24% (5)	23.89%	6281	– 18.24% (5)	5.58%	21283	– 1.13% (5)	1.7766	1.2059
	50	16.78%	158612	– 14.62% (5)	7.96%	78897	– 6.70% (5)	24.44%	7136	– 15.28% (5)	6.83%	24564	– 1.34% (5)	1.7953	1.1872
	100	11.23%	31071	– 6.35% (5)	6.54%	63631	– 5.09% (5)	24.17%	19685	– 11.31% (5)	7.62%	15612	– 1.07% (5)	1.8841	1.0984
10	10	65.59%	91649	– 13.08% (5)	11.20%	85005	– 7.27% (5)	6.54%	36352	– 3.89% (5)	0.83%	51838	295.73 (5) –	0.8862	2.0963
	20	47.57%	22373	– 12.28% (5)	8.75%	87665	– 7.21% (5)	11.06%	18602	– 8.39% (5)	2.03%	72157	– 0.33% (5)	1.2250	1.7575
	40	22.44%	20824	– 6.80% (5)	4.98%	161186	– 4.77% (5)	16.23%	12081	– 17.82% (5)	2.66%	77441	– 0.43% (5)	1.4775	1.5050
	50	19.61%	24376	– 8.34% (5)	5.05%	141056	– 5.06% (5)	17.27%	10729	– 10.40% (5)	2.97%	42322	– 0.40% (5)	1.5128	1.4697
	100	14.73%	31614	– 4.76% (5)	5.84%	226312	– 5.62% (5)	17.97%	22400	– 5.62% (5)	4.33%	58849	– 0.54% (5)	1.5764	1.4061

approaches in terms of root gap, solution time and number of instances solved. In addition, the conic formulation without McCormick inequalities is strikingly better in terms of having a smaller root gap and discovered number of nodes especially when $\kappa = 10, 20$. However, one do not dominate the other in terms of remaining gap. In the second set, when there are 100 products with 10 customer classes, CONIC+MC still performs very well. Even though we deal with a harder variant with more number of variables and a harder parameter set which has more customer classes and products, this approach is able to solve ten of the instances within only 9.3 minutes. The root gap decreases by 87% on the average compared to traditional MILP formulation and the average termination gap is 0.78%.

All four tables prove that CONIC+MC approach works significantly better than the remaining approaches and the performance is dramatically better especially when the capacity constraint is binding. We also observe that the gap between integer and relaxation objective in MILP approach may even be ten times larger than CONIC and the gap is always smaller in CONIC under tight capacities $\kappa = 10, 20$. Another general observation is that the MILP+MC formulation always over-performs MILP approach in terms of root gap and total number of instances solved and solution times and 78% of the time dominates in terms of termination gap. Hence, these observations give an insight of how CONIC+MC approach will be efficient on this multi-locational assortment optimization problem and will provide convincing results with the joint effect of the tightening of the formulation using conic constraints and McCormick inequalities.

Next, we are going to analyze the effects of different parameters on results under the fixed cost setting. But before proceeding to this, we are going to discuss a new concept called "Degree of Commonality Index" (DCI), which helps understand the assortment selection behaviour of models under different settings. This term is originally presented by Collier [59] and used in manufacturing to understand and measure the degree of commonality within a product family.

DCI is formally defined as the following:

$$DCI = \sum_{j=i+1}^{i+d} \phi_j / d, \tag{4.1}$$

where ϕ_j is the number of immediate parents component j has over a set of end items, d is the total number of distinct components in the set of end items and i is the total number of end items [60]. In our case, a distribution center $i \in M$ stands for the end product and the assortments at each location are parent products. Now, d is the number of unique products in the overall assortment, i.e., how many distinct products are chosen to be included in the overall assortment and $\sum_{j=i+1}^{i+d} \phi_j$ stands for $\sum_{i \in M j \in N} o_{ij}$.

Figure 4.1 illustrates the DCI idea with an example for our setting. DC is used as an abbreviation for distribution center and the numbers within circles represent

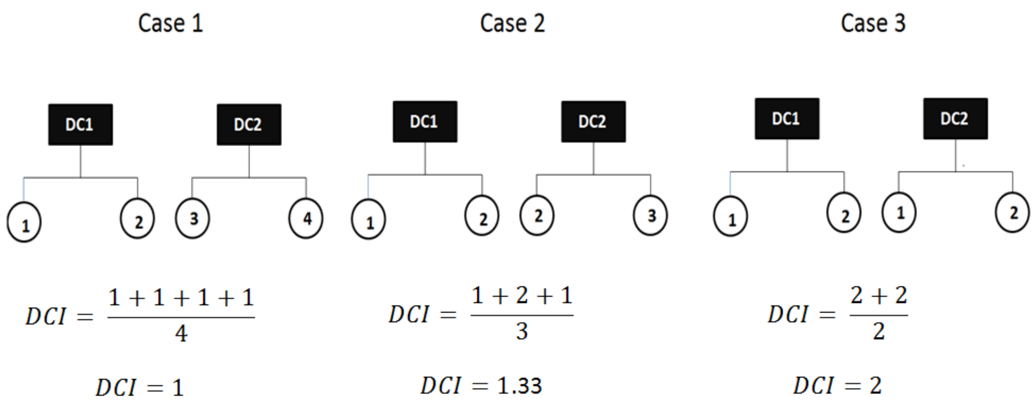

Figure 4.1: Computational Examples for DCI

different product types carried in the assortment.

In the first case, we see that the distribution centers (end products) do not share any common assortment (immediate parents), thus the DCI is 1. It is the minimum value that DCI can take and it represents that commonality is minimum and no products are shared in any of the locations. In the second case, we observe that DCI is higher as they now share one product in common and finally in case 3, DCI is the highest as they carry the same products in their assortment. Thus, DCI changes between 1 and the number of distribution centers and as it goes up from 1, we observe an increase in commonality.

We also calculate an additional index in customized assortments variant and call it DCI'. We now use m_{ij} variable rather than o_{ij}. We take $\sum_{i \in M j \in N} m_{ij}$ and divide it to number of unique products to find DCI'. By doing this, we aim to have an idea on the commonality of products that are shown to customers. Again, the increase in the index shows that the products on the overall assortment are shown to more number of customer class. However, this time, the reference point of the DCI' is 5, number of customer classes, rather than 1 and the decrease in that number stands for a decrease in the commonality of products that are shown to each location. Let us illustrate those concepts with an example. Assume we have 50 products and 5 locations each having a capacity of 2 products. If the model chooses ten different products to include in the assortment, the only way to do this is to put 2 of them into each location. There will be no common variant in any of the locations hence $DCI = (\sum_{i \in M j \in N} o_{ij})/(\#of\ unique\ products) = 10/10 = 1$. Moreover, if we are considering the second variant and if the model chooses to show each product to every customer class, then $DCI' = (\sum_{i \in M j \in N} m_{ij})/(\#of\ unique\ products) = 50/10 = 5$. If those ten products are not shown in every location, say half of them are shown only in a single location, then $DCI' = 30/10 = 3$.

Now, we are going to analyze the effects of no purchase option, capacities, shipping costs and the different problem variants (i.e., showing common or customized assortments to customers) on assortment selection and profit. In order to understand the effect of shipping cost change, we consider different values of fixed shipping costs $\tau = \{0,\ 0.25,\ 1\}$ in addition to $\tau = \{0.5\}$. We only consider the first set of parameters with 50 products and 5 customer classes and by taking advantage of the strength of our CONIC+MC approach, we can obtain the optimal results under different τ values. By this way, we intend to assess those results and gain useful insights.

We try to understand how the problem behaves in general when these parameters change. To that end, we examine the change in profit and commonalities of the assortments by using optimal minimization (conic) objective and DCI values of the instances. First, we present the CONIC+MC run results of both variants when $\tau = 0, 0.25, 0.5, 1$. Then, we present DCI and DCI' values for each variant to compare commonalities.

The run results are presented in Table 4.5- 4.6. For both variants of the problem, we also present the average DCI and DCI' for the changing values of $\tau = 0, 0.25, 0.5, 1$. These are presented in Table 4.7, 4.8, 4.9, 4.10 respectively. Comparison of the run results and DCI values of those five samples are given separately in Appendix B and C. Note that while constructing DCI tables below, again average values of five samples are used. Hence in order to come up with, for example, each DCI value under each capacity in Table 4.7, the average of five samples' DCI values under each capacity in Appendix C- Table C.1 is taken.

Table 4.5: **Common assortment under a fixed cost: Comparison of models with τ= 0, 0.25, 0.5, 1**

v_0	κ	Cost=0				Cost=0.25				Cost=0.5				Cost=1			
		Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.08%	0	0.42 (5) –	1.5067	0.91%	211	1.11 (5) –	1.5992	1.73%	3950	2.91 (5) –	1.6914	3.41%	6582	11.59 (5) –	1.8744
	20	0.08%	64	0.37 (5) –	1.2045	1.58%	78798	186.76 (5) –	1.3180	2.93%	71983	225.98 (5) –	1.4308	5.93%	257865	336.16 (3) 1.25% (2)	1.6510
	40	0.00%	0	0.15 (5) –	1.1347	1.42%	2989	3.78 (5) –	1.2298	2.75%	11245	14.10 (5) –	1.3160	5.01%	385333	238.28 (3) 0.21% (2)	1.4513
	50	0.00%	0	0.15 (5) –	1.1347	1.24%	1094	3.32 (5) –	1.2116	2.33%	6613	16.42 (5) –	1.2791	3.86%	204089	23.66 (4) 0.13% (1)	1.3786
10	10	0.03%	0	0.41 (5) –	1.9707	0.32%	30	0.77 (5) –	2.0345	0.60%	1143	1.56 (5) –	2.0981	1.21%	5341	4.93 (5) –	2.2245
	20	0.12%	38	0.58 (5) –	1.6353	0.80%	5495	5.46 (5) –	1.7219	1.46%	35447	72.35 (5) –	1.8081	2.76%	142989	27.52 (4) 0.35% (1)	1.9793
	40	0.00%	0	0.16 (5) –	1.4875	0.69%	4419	3.36 (5) –	1.5833	1.39%	140599	47.49 (5) –	1.6685	2.87%	380593	160.70 (4) 0.14% (1)	1.8157
	50	0.00%	0	0.16 (5) –	1.4875	0.63%	1137	2.44 (5) –	1.5686	1.27%	4725	6.71 (5) –	1.6395	2.49%	47975	45.24 (5) –	1.7587

Table 4.6: **Customized assortments under a fixed cost: Comparison of models with τ= 0, 0.25, 0.5, 1**

v_0	κ	Cost=0				Cost=0.25				Cost=0.5				Cost=1			
		Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.08%	0	0.57 (5) –	1.5067	0.91%	3053	3.80 (5) –	1.5992	1.73%	6403	20.05 (5) –	1.6914	3.50%	8877	31.19 (5) –	1.8744
	20	0.08%	0	0.57 (5) –	1.2045	1.58%	25820	84.08 (5) –	1.3180	3.00%	57095	34.95 (5) –	1.4308	7.06%	256787	– 1.49% (5)	1.6506
	40	0.00%	0	0.17 (5) –	1.1344	1.49%	1098	8.97 (5) –	1.2292	3.19%	260244	126.55 (4) 0.07% (1)	1.3148	5.98%	394455	– 0.53% (5)	1.4396
	50	0.00%	0	0.24 (5) –	1.1344	1.30%	1093	7.14 (5) –	1.2108	2.71%	33514	61.06 (5) –	1.2771	4.42%	240412	243.28 (4) 0.26% (1)	1.3618
10	10	0.03%	0	0.56 (5) –	1.9707	0.32%	90	1.27 (5) –	2.0345	0.60%	770	3.22 (5) –	2.0981	1.21%	6495	11.01 (5) –	2.2245
	20	0.12%	244	1.80 (5) –	1.6353	0.80%	41351	117.65 (5) –	1.7219	1.46%	86379	11.27 (4) 0.11% (1)	1.8081	3.00%	177716	519.99 (4) 0.74% (1)	1.9793
	40	0.00%	0	0.19 (5) –	1.4870	0.71%	5304	9.85 (5) –	1.5829	1.51%	3441	17.18 (5) –	1.6680	3.53%	445806	765.42 (3) 0.21% (2)	1.8131
	50	0.00%	0	0.17 (5) –	1.4870	0.66%	4324	7.36 (5) –	1.5680	1.37%	3430	14.95 (5) –	1.6386	3.04%	577551	264.12 (3) 0.12% (2)	1.7537

First, we intend to understand how the model behaves under two different variants: common or customized assortments. Note that we use the conic approach, which means we consider the results based on a minimization objective. This means that the less the objective value is, the more profit the company makes. First of all, when we compare Table 4.5 and 4.6 and check the optimal objective values, surprisingly, we observe that objectives are very close to each other and profit values differ only slightly. Still, we see that the objective values of customized assortments variant are always less than or equal to the common assortment variant. This means having customized assortments helps us increase profit. Although the change is minor for this set of parameters, the profit that company makes under large capacities (when $\kappa = 40, 50$) is always higher than common assortment variant. In addition, first variant's DCI values is greater than or equal to the second variant in 88% of the instances (31% higher, 57% same). But still, in some cases, the retailer may carry more number of unique

Table 4.7: **DCI analysis of the models with fixed cost, τ=0.**

v_0	κ	Common Assortment			Customized Assortments				
		Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	$\sum_{i\in Mj\in N} m_{ij}$	DCI'
5	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	29	29	1	30	39	1.3338	144	4.8760
	50	29	29	1	30	46	1.5692	144	4.8760
10	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	37	37	1.0051	38	40	1.0623	180	4.7936
	50	37	39	1.0624	38	49	1.2967	181	4.7567

Table 4.8: **DCI analysis of the models with fixed cost, τ=0.25.**

v_0	κ	Common Assortment			Customized Assortments				
		Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	$\sum_{i\in Mj\in N} m_{ij}$	DCI'
5	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	28	40	1.4513	28	40	1.4345	135	4.8468
	50	27	49	1.7895	28	50	1.7903	134	4.7839
10	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	33	40	1.2282	34	40	1.1920	163	4.8454
	50	33	50	1.5353	34	50	1.4811	162	4.7865

Table 4.9: **DCI analysis of the models with fixed cost, τ=0.5.**

v_0	κ	Common Assortment			Customized Assortments				
		Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	$\sum_{i\in Mj\in N} m_{ij}$	DCI'
5	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	24	40	1.6488	25	40	1.5839	120	4.7373
	50	26	50	1.9595	27	50	1.8879	119	4.6378
10	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	30	40	1.3548	31	40	1.3087	149	4.8595
	50	30	50	1.6704	31	50	1.6156	147	4.7537

Table 4.10: **DCI analysis of the models with fixed cost, τ=1.**

v_0	κ	Common Assortment			Customized Assortments				
		Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	Number of Unique Products	$\sum_{i\in Mj\in N} o_{ij}$	DCI	$\sum_{i\in Mj\in N} m_{ij}$	DCI'
5	10	10	10	1	10	10	1	50	5
	20	18	20	1	19	20	1.0786	89	4.7913
	40	18	40	2.2564	21	40	1.9229	78	3.7113
	50	19	50	2.6312	22	50	2.2624	77	3.4746
10	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	25	40	1.6210	27	40	1.5068	121	4.5517
	50	25	50	2.0113	27	50	1.8570	119	4.4192

products in the second variant, but at the same time, repeat more products in different locations which lead to a higher DCI for the second model.

For every problem variant and changing values of capacity and transportation cost, we evaluate the model on two different no-purchase preference values. In both problem variants, we observe that the objective values are less overall when $v_0 = 5$. This is reasonable because by lowering the value of no purchase, we increase the probability that customer makes a purchase among the products in the assortment. This somehow decreases the value of a wider assortment because the customer will be more willing to buy a product regardless of the product utilities. Hence, even if the company does not involve one's favourite product, perhaps because of a possible capacity constraint, this affects the customer less because of a lower no-purchase value.

When v_0 is low, the customer is likely to purchase from a smaller assortment even if the prices of the products in the assortment are high. This way, the retailer can focus on higher priced products in the assortment which it may repeat in many distribution center. The fact that the assortments are more common (higher DCI) also allows the retailer to avoid transshipments from one location to another. All of this is possible when no-purchase preference is not very high. If it is high, the company needs to offer a wider assortment (higher number of unique products) and also offer products that are locally popular to avoid transshipments.

Next, we analyze the behaviour of our models under different capacities. The optimal objective values show a monotonically decreasing behaviour when total capacity of the system increases. This property holds for each problem variant, no purchase option and cost. Thus, the company makes more profit when it has a chance to broaden its product variety. This is expected since increasing the capacity is simply increasing the feasible region of the problem. On the other hand, we observe that DCI increases monotonically while capacity is increasing. This is because under tight capacity constraints such as $\kappa = \{10,\ 20\}$, the company needs to store different products at different locations to be able to offer a wider assortment to attract more customers. This leads to a low DCI. Unlike DCI, in 98% of the instances, DCI' values decrease with an increase in capacity and the company chooses to show more specific products to customers according to their own preference values.

Lastly, following the analysis of the change in no-purchase preference and capacity, we also investigate the effect of shipping cost on profit and assortment selection. We run the models separately for each cost value $\tau = \{0,\ 0.25,\ 0.5,\ 1\}$. First of all, we observe that the problem somehow gets more difficult when τ is larger. Note that the minimum product revenue is 1, thus all shipping costs are chosen to be equal or less than that value. On the average, we observe a 22% increase in the objective value of both variants when we increase cost from 0 to 1, and conclude that shipping costs have a considerable effect on profit. In both variants, the number of unique products in the assortment decreases as the transportation cost increases. This is expected since the company wants to avoid shipping products from secondary locations and focus on a narrower product assortment that it can ship from the primary distribution center. However, the effect is less pronounced in the customized assortments variant as the company can pick and choose what to show to each region and avoid transshipments. The effect of transportation cost on commonality (DCI) for the common assortment variant can be explained the same way. As the cost goes up, the company reduces the overall assortment and repeats many products in many locations. As for DCI' in the second variant, the commonality of products shown to customers goes down as the cost goes up. This is also expected since when the transportation costs are high, the company would offer more customized products to locations, i.e., products that are mostly available in the primary distribution centers. This means that DCI' will be lowered.

Up to this point, we have studied the performances of four different modeling approaches under a fixed cost setting. We have presented the run results of the models under two sets of parameters each having five different samples. We have discussed how the root gaps, number of nodes and solution times change according to those approaches. We have also discussed the effects of no-purchase option, capacity, cost and customer interface on profitability and assortment commonalities. In the next section, we will consider both variants when the shipping costs vary according to shipping locations.

4.2 Different Costs Setting

Before presenting the results, we explain how the costs are determined in different costs setting. As it has been mentioned in the previous chapter, the shipping costs vary according to distances between locations. In this thesis, we assume that the locations are positioned linearly, in an ascending order. As an example, the order may be {1,2,3,4} while $|M| = 4$. Now, the shipping costs are the same across the products, i.e., $\tau_{ikm} = \tau_{ikn}$ for any $i, k \in M$ and any $m, n \in N$. The transportation cost between locations i and k is taken as $\frac{1}{3}\min_{j\in N}\{\pi_j\}$+0.05 $|i - k|$ if $i \neq k$ and 0 if $i = k$. Now, let us demonstrate this with an example with 4 locations. Let the revenue of the product with a minimum price is 1.945. Hence, the minimum the shipping cost of a chosen location to its nearest neighbor will be $\frac{1}{3}\min_{j\in N}\{\pi_j\}$= 1.945/3= 0.648. Then, the cost matrix of locations will be as follows:

$$\tau = \begin{bmatrix} 0.000 & 0.648 & 0.698 & 0.748 \\ 0.648 & 0.000 & 0.648 & 0.698 \\ 0.698 & 0.648 & 0.000 & 0.648 \\ 0.748 & 0.698 & 0.648 & 0.000 \end{bmatrix}$$

The preference matrix, Φ, is also demonstrated in accordance with the distances, hence the cost matrix. Again, preferences are same across the products. Each row in the preference matrix stands for a location and each column in a row represents that location's preference order of 'from where to ship'. Thus, for the above example, the preference matrix becomes:

$$\Phi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 2 & 4 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

A location is indifferent between shipping from its left or right neighbor as long as the distances are the same, so at all instances, we always put the preference order in such a way that the left neighbor preceded the right neighbor. The same data sets in the fixed cost model are also used in here, except for the cost. The run results of the first variant under both set of parameters are given in Table 4.11-4.12.

Table 4.11: **Common assortment under different costs: Sample averages for problems with 50 products and 5 classes.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	38.98%	529153	– 9.47% (5)	7.06%	185049.00	306.16 (5) –	11.46%	114660	– 3.59% (5)	1.86%	9901	36.53 (5) –	1.3250	1.6506
	20	15.46%	115110	– 9.84% (5)	4.48%	1522113.20	– 2.40% (5)	18.24%	99767	– 9.54% (5)	3.63%	144181	– 0.48% (5)	1.5949	1.3807
	40	8.25%	74459	– 7.66% (5)	3.19%	986265.60	– 2.29% (5)	19.18%	53779	– 12.64% (5)	3.51%	148953	– 1.72% (5)	1.7009	1.2747
	50	6.92%	61819	– 5.94% (5)	3.04%	1265396.60	– 1.80% (5)	18.31%	66056	– 9.75% (5)	3.46%	170835	– 1.23% (5)	1.7222	1.2534
10	10	49.93%	707211	– 8.40% (5)	6.23%	41751.40	55.95 (5) –	6.58%	158818	– 1.41% (5)	0.71%	8261	9.77 (5) –	0.9059	2.0697
	20	23.47%	106201	– 10.12% (5)	4.72%	1524349.80	– 1.78% (5)	11.16%	148660	– 4.45% (5)	1.75%	409407	409.83 (1) 0.28% (4)	1.2060	1.7696
	40	10.51%	122069	– 8.98% (5)	2.54%	1521846.40	– 1.87% (5)	14.20%	90469	– 3.82% (5)	1.81%	175973	– 0.69% (5)	1.3471	1.6285
	50	8.50%	44860	– 7.52% (5)	2.11%	1667166.60	– 1.58% (5)	13.26%	67655	– 4.09% (5)	1.52%	190525	– 0.57% (5)	1.3721	1.6035

Table 4.12: **Common assortment under different costs: Sample averages for problems with 100 products and 10 classes.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	61.62%	7612	– 48.69% (5)	13.48%	9399	– 10.61% (5)	14.65%	2349	– 28.66% (5)	3.12%	7576	– 2.21% (5)	1.3042	1.6783
	20	29.48%	70520	– 29.03% (5)	8.91%	24031	– 7.51% (5)	24.15%	2556	– 38.04% (5)	6.18%	8034	– 4.44% (5)	1.6324	1.3501
	40	17.24%	101756	– 17.32% (5)	7.81%	23271	– 6.68% (5)	34.60%	2446	– 41.61% (5)	9.59%	1636	– 7.66% (5)	1.8006	1.1819
	50	15.25%	92946	– 15.47% (5)	7.79%	22261	– 6.64% (5)	29.56%	2899	– 33.17% (5)	10.39%	1933	– 8.44% (5)	1.8310	1.1515
	100	9.69%	140753	– 10.73% (5)	5.82%	22133	– 5.01% (5)	30.03%	2391	– – (5)	8.91%	4384	– 7.57% (5)	1.9224	1.0601
10	10	65.86%	12073	– 39.01% (5)	12.45%	18058	– 7.67% (5)	8.27%	3674	– 12.93% (5)	1.20%	35659	– 0.75% (5)	0.8968	2.0857
	20	47.43%	13054	– 38.96% (5)	9.90%	23586	– 8.22% (5)	14.67%	5262	– 22.47% (5)	2.94%	13578	– 2.02% (5)	1.2387	1.7438
	40	22.31%	17479	– 20.86% (5)	6.41%	41644	– 5.42% (5)	23.83%	2486	– 30.38% (5)	4.95%	3531	– 3.76% (5)	1.4919	1.4906
	50	19.33%	22463	– 18.72% (5)	6.66%	27748	– 5.61% (5)	26.52%	2783	– 25.23% (5)	5.69%	1720	– 4.35% (5)	1.5293	1.4532
	100	12.45%	21810	– 14.67% (5)	5.43%	30367	– 4.87% (5)	23.75%	2536	– 29.44% (5)	5.50%	5073	– 4.40% (5)	1.6203	1.3622

We can confirm the strength of CONIC+MC approach in both sets. With the joint effect of conic formulation and McCormick inequalities, CONIC+MC approach is able to solve the highest number of instances by having the smallest root gaps. In the first set, even for the unsolved instances, the average remaining optimality gap is 0.85%, which is negligible especially when compared to other models. In the second set, it increases to 4.5%. However, one should note that in other approaches, there exist instances denoted as (–) which the solver is not even able to find an incumbent solution. The results of the second variant under both sets are provided in Table 4.13 and 4.14 below. The detailed run results of each five samples in both sets can be found in Appendix D.

Table 4.13: **Customized assortments under different costs: Sample averages for problems with 50 products and 5 classes.**

v_0	κ	MILP				MILP+MC				CONIC				CONIC+MC				Max Obj	Min Obj
		Gap	Nodes	Time		Gap	Nodes	Time		Gap	Nodes	Time		Gap	Nodes	Time			
5	10	38.98%	22003	–		7.10%	313293.60	837.89	(4)	11.46%	64058	–		1.86%	14339	49.85	(5)	1.3250	1.6506
				11.79%	(5)			0.85%	(1)			5.50%	(5)			–			
	20	15.51%	43241	–		4.60%	286914.80	–		18.28%	42559	–		3.67%	92924	–		1.5942	1.3814
				10.51%	(5)			3.37%	(5)			14.72%	(5)			0.62%	(5)		
	40	8.20%	622462	–		3.24%	357085.00	–		19.15%	23127	–		3.57%	73081	–		1.7016	1.2740
				8.11%	(5)			2.60%	(5)			13.24%	(5)			2.06%	(5)		
	50	6.28%	656872	–		2.53%	406126.60	–		17.67%	22583	–		2.83%	96574	–		1.7324	1.2432
				6.23%	(5)			2.03%	(5)			17.54%	(5)			1.54%	(5)		
10	10	49.93%	94082	–		6.28%	53531.00	118.57	(5)	6.57%	76310	–		0.71%	11377	23.90	(5)	0.9059	2.0697
				12.57%	(5)			–				2.05%	(5)			–			
	20	23.47%	17866	–		4.77%	714652.80	–		11.16%	53061	–		1.75%	165736	359.83	(2)	1.2060	1.7696
				6.95%	(5)			2.27%	(5)			10.07%	(5)			0.38%	(3)		
	40	10.42%	221086	–		2.52%	539599.40	–		14.15%	35032	–		1.78%	74235	–		1.3482	1.6274
				9.55%	(5)			2.13%	(5)			14.37%	(5)			0.72%	(5)		
	50	8.42%	245781	–		2.12%	577743.80	–		13.22%	26816	–		1.52%	91943	–		1.3731	1.6025
				8.31%	(5)			1.79%	(5)			10.47%	(5)			0.63%	(5)		

Table 4.14: **Customized assortments under different costs: Sample averages for problems with 100 products and 10 classes.**

v_0	κ	MILP				MILP+MC				CONIC				CONIC+MC				Max Obj	Min Obj
		Gap	Nodes	Time		Gap	Nodes	Time		Gap	Nodes	Time		Gap	Nodes	Time			
5	10	60.94%	3140	–		13.10%	6956	–		14.37%	2308	–		2.80%	10837	–		1.3110	1.6715
				58.91%	(5)			12.94%	(5)			26.47%	(5)			2.06%	(5)		
	20	29.43%	3422	–		9.01%	6332	–		24.13%	1992	–		6.14%	3777	–		1.6325	1.3500
				33.39%	(5)			9.02%	(5)			40.31%	(5)			4.34%	(5)		
	40	17.41%	34233	–		8.06%	8372	–		34.74%	2651	–		9.85%	1562	–		1.7979	1.1846
				17.57%	(5)			7.60%	(5)			42.63%	(5)			7.80%	(5)		
	50	15.12%	55973	–		7.76%	8606	–		34.68%	329	–		10.30%	2377	–		1.8331	1.1494
				16.14%	(5)			7.50%	(5)			39.20%	(5)			8.51%	(5)		
	100	9.98%	83786	–		6.20%	15219	–		24.96%	18	–		9.48%	2947	–		1.9172	1.0653
				12.07%	(5)			5.36%	(5)			38.01%	(5)			8.49%	(5)		
10	10	64.03%	18534	–		11.25%	17419	–		7.82%	2579	–		1.21%	37671	–		0.9093	2.0732
				43.57%	(5)			10.20%	(5)			15.99%	(5)			0.72%	(5)		
	20	47.11%	3699	–		9.71%	10883	–		14.53%	1938	–		2.79%	8657	–		1.2420	1.7405
				36.29%	(5)			9.73%	(5)			25.89%	(5)			1.93%	(5)		
	40	22.14%	8061	–		6.34%	10580	–		23.73%	2009	–		4.84%	1182	–		1.4944	1.4881
				21.44%	(5)			6.29%	(5)			31.34%	(5)			3.74%	(5)		
	50	19.01%	10843	–		6.45%	6925	–		26.32%	2706	–		5.47%	1871	–		1.5331	1.4494
				18.83%	(5)			6.45%	(5)			29.83%	(5)			4.31%	(5)		
	100	12.22%	58196	–		5.28%	9792	–		23.58%	12	–		5.33%	3710	–		1.6242	1.3583
				15.06%	(5)			5.41%	(5)			28.61%	(5)			4.57%	(5)		

We can observe that with the incorporation of different costs (addition of p_{ikj} variable), the problem gets harder. This causes an overall decrease in performances of each approach. To begin with, the number of instances solved decreases in each approach. The termination gaps get higher as well as the solution times for the instances that can be solved in different costs setting. In CONIC approach itself, we even observe instances that the solver cannot come up with any incumbent solution within half an hour. Next, we will go through a similar analysis with the one we did in the fixed cost setting. We will compare how the model reacts to change in no-purchase, capacity and variant. The detailed version which shows DCI values separately for all samples is available in Appendix E. By taking averages of each column separately in Appendix E, we obtain Table 4.15.

Table 4.15: **DCI analysis of the models with different costs.**

v_0	κ	Common Assortment			Customized Assortments				
		Number of Unique Products	$\sum_{i\in M j\in N} o_{ij}$	DCI	Number of Unique Products	$\sum_{i\in M j\in N} o_{ij}$	DCI	$\sum_{i\in M j\in N} m_{ij}$	DCI'
5	10	10	10	1	10	10	1	50	5
	20	20	20	1	20	20	1	100	5
	40	25	40	1.5938	27	40	1.4733	128	4.7098
	50	26	50	1.9121	27	50	1.8273	128	4.6831
10	10	10	10	1.0000	10	10	1.0000	50	5.0000
	20	20	20	1.0000	20	20	1.0000	100	5.0000
	40	31	40	1.3109	31	40	1.2844	153	4.8909
	50	31	50	1.6387	32	50	1.5631	154	4.7981

We observe that the company makes less profit when the no-purchase option value is higher. Moreover, it is observed that product commonalities also decreases with higher no-purchase options. On the other hand, the optimal strategy for the company becomes showing the available products on more locations, thus DCI' values increase at all instances when $v_0 = 10$. With the increase in available capacity, the expected profit gets higher as now the company is able to increase its product assortment. Commonalities of products among locations also increases while DCI' decreases except one instance. While the expected profit is higher under the customized assortments, the difference is rather small. When we look at the commonality indices, this time we see that DCI of the common assortment variant is always greater than or equal to the customized assortments variant.

Below table summarizes how the model changes in our numerical study via the change in parameters under fixed cost and different costs settings.

Table 4.16: **Profit and Commonality Comparisons of Models**

		Fixed Cost		Different Costs	
		Profit	Commonality	Profit	Commonality
v_0	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
κ	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
τ	$\uparrow$	$\downarrow$	$\uparrow$	NA	NA
Variant 1	$\rightarrow$ 2	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$

The objective of this chapter was to present the results of four different solution approaches under two different data sets, different problem variants and different settings. We tried to assess and understand how the models behave under different problem structures and parameters. We attempted to quantify it by using the objective values of our run results and using a commonality index. Finally, we investigated the change in those results.

Chapter 5

Conclusion

Online retailing is an increasing trend all over the world and its value in Turkey is almost doubling every year. Both local and global companies are trying to manage their online operations better and making correct fulfillment decisions is key for this. However, online retailing is different than traditional retailing in terms of customer and product variety and demand fulfillment. Retailers own a network of fulfillment centers, customers are able to see and purchase any item in the network and retailers have to decide from where to satisfy this demand.

The purchasing behaviour of customers highly depends on assortment decisions, i.e., determining which products to offer to customers. Thus, it is important for an online retailer to plan the assortment at each location correctly. Furthermore, as each center in a given location may be used to satisfy the demand of another location by shipping the product in online setting, the company should also consider the additional cost of this. Under such setting, this study deals with the problem of maximizing the profit of an online retailing company by choosing the assortments in each location. Customers that reside in a geographical area are assumed to form a segment whose choice follows a separate MNL model. It is also assumed that a distribution center primarily serves a geographical area. As there are more than one location, the overall consumer choice may be considered as MMNL. We also consider a predetermined capacity/cardinality constraint on the number of products that can be selected at each distribution center.

With the work of van Ryzin and Mahajan [10], the single location assortment planning problem under MNL model has received considerable interest in the literature. Assortment planning under MMNL model has also been quite popular in the last few years. There has only been two studies in this area that study multi-store, multi-echelon setting, but in both of these papers, they only consider two-tier supply chains. We, on the other hand, put our emphasis on a multi-locational setting.

We consider two variants of the problem. In the first variant, the customers visiting e-retailer's online store have an access to the entire assortment in each location. In the second variant, the retailer decides which product to show to customer at each location separately. The company may even prefer not to show a specific product in a distribution center even though it is in the same location where the demand originates from. The demand can be satisfied from any of the stores with an extra cost. Considering this, the extra shipping costs are handled in two different settings. The first one considers a fixed shipping cost that is regardless of the shipping location. In the second setting, the costs varies across different locations depending on the distance.

The first variant of the problem under fixed shipping costs is chosen as the baseline model. We first provide a non-linear formulation for the baseline model, linearize and formulate it as an MILP, then by constructing on Sen et al. [26], we are able to reformulate it as a conic quadratic mixed integer program. We also further strengthen both formulations by introducing valid McCormick inequalities. Then, we provide linear and conic programs of each variant under each setting by going through the same transformation steps as the baseline model.

We provide a numerical study under two sets of parameters each having five different samples in it. The sets are created randomly by generating instances first with fifty products and five customer classes, and second with hundred products and ten customer classes. We run each set under four different approaches, which are MILP, MILP+MC, CONIC and CONIC+MC and observe that conic formulation together with McCormick cuts improves the solution times significantly at each instance. MILP itself cannot handle the binding capacities and CONIC formulation is not enough to solve instances although it is able decrease root gaps. When we use MILP+MC, we see that the valid inequalities considerably reduce the solution times and termination gaps. We get the strongest results when we combine conic formulation with the valid McCormick inequalities.

We also compare how profit and commonalities of assortments among locations change by changing values of no-purchase preference, capacity, shipping cost and by changing the variant. The numerical study shows that the online retailer manages to increase profit under each setting and each variant if it is able to engage with customers having low no-purchase preferences. Moreover, it makes more profit if the distribution centers' capacity constraints are not binding. Whenever the company is facing such cases, the optimal strategy for them would be to increase commonality of the products in each warehouse. Under the fixed cost setting, we observe that the increase in cost effects profits in a negative way and in that case, the commonalities should be increased. We see that customizing assortments in the second variant leads to minor increases in profit. We also see that the overall assortment is wider in the second variant.

Since the effects of making better assortment decisions are obvious, one immediate further research direction can be to integrate stocking and pricing decisions to multi-locational assortment optimization problem. It is also important to conduct a numerical study using real data from industry. For further research, this will certainly be helpful to support our results and come up with consistent insights. Finally, another research direction can be to expand this study using other consumer choice models.

Bibliography

[1] B. E. Kahn, "Consumer variety-seeking among goods and services: An integrative review," *Journal of Retailing and Consumer Services*, vol. 2, no. 3, pp. 139–148, 1995.

[2] B. P. Pashigian, "Demand uncertainty and sales: A study of fashion and markdown pricing," *The American Economic Review*, pp. 936–953, 1988.

[3] B. L. Bayus and W. P. Putsis Jr, "Product proliferation: An empirical analysis of product line determinants and market outcomes," *Marketing Science*, vol. 18, no. 2, pp. 137–153, 1999.

[4] S. A. Smith, "Optimizing retail assortments for diverse customer preferences," in *Retail Supply Chain Management*, pp. 183–205, Springer, 2008.

[5] D. Brandes and N. Brandes, "In retailing, assortment is job one." https://hbr.org/2011/12/in-retailing-assortment-is-job, 2011. Accessed May, 2016.

[6] A. G. Kök, M. L. Fisher, and R. Vaidyanathan, "Assortment planning: Review of literature and industry practice," in *Retail supply chain management*, pp. 99–153, Springer, 2008.

[7] O. Besbes and D. Saure, "Product assortment and price competition under multinomial logit demand," *Production and Operations Management*, vol. 25, no. 1, pp. 114–127, 2016.

[8] S. P. Anderson, A. De Palma, and J. F. Thisse, *Discrete choice theory of product differentiation.* MIT press, 1992.

[9] M. E. Ben-Akiva and S. R. Lerman, *Discrete choice analysis: theory and application to travel demand*, vol. 9. MIT press, 1985.

[10] G. van Ryzin and S. Mahajan, “On the relationship between inventory costs and variety benefits in retail assortments,” *Management Science*, vol. 45, no. 11, pp. 1496–1509, 1999.

[11] P. M. Guadagni and J. D. Little, “A logit model of brand choice calibrated on scanner data,” *Marketing science*, vol. 2, no. 3, pp. 203–238, 1983.

[12] F. S. Koppelman and C. Bhat, *A self instructing course in mode choice modeling: multinomial and nested logit models*. FTA, 2006.

[13] J. J. Louviere and G. Woodworth, “Design and analysis of simulated consumer choice or allocation experiments: an approach based on aggregate data,” *Journal of marketing research*, pp. 350–367, 1983.

[14] S. A. Smith and N. Agrawal, “Management of multi-item retail inventory systems with demand substitution,” *Operations Research*, vol. 48, no. 1, pp. 50–64, 2000.

[15] A. G. Kök and M. L. Fisher, “Demand estimation and assortment optimization under substitution: Methodology and application,” *Operations Research*, vol. 55, no. 6, pp. 1001–1021, 2007.

[16] K. J. Lancaster, “A new approach to consumer theory,” *The journal of political economy*, pp. 132–157, 1966.

[17] H. Hotelling, “Stability in competition,” in *The Collected Economics Articles of Harold Hotelling*, pp. 50–63, Springer, 1990.

[18] R. D. McBride and F. S. Zufryden, “An integer programming approach to the optimal product line selection problem,” *Marketing Science*, vol. 7, no. 2, pp. 126–140, 1988.

[19] R. Kohli and R. Sukumar, “Heuristics for product-line design using conjoint analysis,” *Management Science*, vol. 36, no. 12, pp. 1464–1478, 1990.

[20] V. Gaur and D. Honhon, “Assortment planning and inventory decisions under a locational choice model,” *Management Science*, vol. 52, no. 10, pp. 1528–1543, 2006.

[21] Forrester, “Us retail ecommerce: Readjust continually to combat amazon.” https://www.forrester.com/report/Brief US Online Retail Forecast 2015 To 2020/-/E-RES122904, 2016. Accessed May, 2016.

[22] J. A. Acimovic, *Lowering outbound shipping costs in an online retail environment by making better fulfillment and replenishment decisions*. PhD thesis, Massachusetts Institute of Technology, 2012.

[23] http://ecommercenews.eu/, “Ecommerce in turkey reaches 6.34 billion.” http://ecommercenews.eu/ecommerce-in-turkey-reaches-e6-34-billion, 2015. Accessed June, 2016.

[24] J. J. M. Bront, I. Méndez-Díaz, and G. Vulcano, “A column generation algorithm for choice-based network revenue management,” *Operations Research*, vol. 57, no. 3, pp. 769–784, 2009.

[25] P. Rusmevichientong, D. Shmoys, C. Tong, and H. Topaloglu, “Assortment optimization under the multinomial logit model with random choice parameters,” *Production and Operations Management*, vol. 23, no. 11, pp. 2023–2039, 2014.

[26] A. Şen, A. Atamtürk, and P. Kaminsky, “A conic integer programming approach to constrained assortment optimization under the mixed multinomial logit model,” *Research Report*, October 2015.

[27] G. P. Cachon, C. Terwiesch, and Y. Xu, “Retail assortment planning in the presence of consumer search,” *Manufacturing & Service Operations Management*, vol. 7, no. 4, pp. 330–346, 2005.

[28] S. Mahajan and G. van Ryzin, “Stocking retail assortments under dynamic consumer substitution,” *Operations Research*, vol. 49, no. 3, pp. 334–351, 2001.

[29] J.-K. Chong, T.-H. Ho, and C. S. Tang, "A modeling framework for category assortment planning," *Manufacturing & Service Operations Management*, vol. 3, no. 3, pp. 191–210, 2001.

[30] K. Talluri and G. Van Ryzin, "Revenue management under a general discrete choice model of consumer behavior," *Management Science*, vol. 50, no. 1, pp. 15–33, 2004.

[31] Z. Li, "A single-period assortment optimization model," *Production and Operations Management*, vol. 16, no. 3, p. 369, 2007.

[32] B. Maddah and E. K. Bish, "Joint pricing, assortment, and inventory decisions for a retailer's product line," *Naval Research Logistics (NRL)*, vol. 54, no. 3, pp. 315–330, 2007.

[33] W. J. Hopp and X. Xu, "A static approximation for dynamic demand substitution with applications in a competitive market," *Operations Research*, vol. 56, no. 3, pp. 630–645, 2008.

[34] C. M. Miller, S. A. Smith, S. H. McIntyre, and D. D. Achabal, "Optimizing and evaluating retail assortments for infrequently purchased products," *Journal of Retailing*, vol. 86, no. 2, pp. 159–171, 2010.

[35] P. Rusmevichientong, Z.-J. M. Shen, and D. B. Shmoys, "Dynamic assortment optimization with a multinomial logit choice model and capacity constraint," *Operations research*, vol. 58, no. 6, pp. 1666–1680, 2010.

[36] P. Rusmevichientong and H. Topaloglu, "Robust assortment optimization in revenue management under the multinomial logit choice model," *Operations Research*, vol. 60, no. 4, pp. 865–882, 2012.

[37] R. Wang, "Capacitated assortment and price optimization under the multinomial logit model," *Operations Research Letters*, vol. 40, no. 6, pp. 492–497, 2012.

[38] J. Davis, G. Gallego, and H. Topaloglu, "Assortment planning under the multinomial logit model with totally unimodular constraint structures," *Department of IEOR, Columbia University. Available at http://www. columbia. edu/ gmg2/logit_const. pdf*, 2013.

[39] H. Topaloglu, "Joint stocking and product offer decisions under the multinomial logit model," *Production and Operations Management*, vol. 22, no. 5, pp. 1182–1199, 2013.

[40] V. Goyal, R. Levi, and D. Segev, "Near-optimal algorithms for the assortment planning problem under dynamic substitution and stochastic demand," *Operations Research*, 2016.

[41] A. G. Kök and Y. Xu, "Optimal and competitive assortments with endogenous pricing under hierarchical consumer choice models," *Management Science*, vol. 57, no. 9, pp. 1546–1563, 2011.

[42] J. M. Davis, G. Gallego, and H. Topaloglu, "Assortment optimization under variants of the nested logit model," *Operations Research*, vol. 62, no. 2, pp. 250–273, 2014.

[43] G. Gallego and H. Topaloglu, "Constrained assortment optimization for the nested logit model," *Management Science*, vol. 60, no. 10, pp. 2583–2601, 2014.

[44] J. Feldman and H. Topaloglu, "Bounding optimal expected revenues for assortment optimization under mixtures of multinomial logits," *Production and Operations Management*, vol. 24, no. 10, pp. 1598–1620, 2015.

[45] G. Li and P. Rusmevichientong, "A greedy algorithm for the two-level nested logit model," *Operations Research Letters*, vol. 42, no. 5, pp. 319–324, 2014.

[46] G. Li, P. Rusmevichientong, and H. Topaloglu, "The d-level nested logit model: Assortment and price optimization problems," *Operations Research*, vol. 63, no. 2, pp. 325–342, 2015.

[47] W. Z. Rayfield, P. Rusmevichientong, and H. Topaloglu, "Approximation methods for pricing problems under the nested logit model with price

bounds," *INFORMS Journal on Computing*, vol. 27, no. 2, pp. 335–357, 2015.

[48] N. Cardell and F. C. Dunbar, "Measuring the societal impacts of automobile downsizing," *Transportation Research Part A: General*, vol. 14, no. 5, pp. 423 – 434, 1980.

[49] J. Boyd and R. E. Mellman, "The effect of fuel economy standards on the u.s. automotive market: An hedonic demand analysis," *Transportation Research Part A: General*, vol. 14, no. 5, pp. 367 – 378, 1980.

[50] I. Méndez-Díaz, J. J. Miranda-Bront, G. Vulcano, and P. Zabala, "A branch-and-cut algorithm for the latent-class logit assortment problem," *Discrete Applied Mathematics*, vol. 164, pp. 246–263, 2014.

[51] D. McFadden, K. Train, *et al.*, "Mixed mnl models for discrete response," *Journal of applied Econometrics*, vol. 15, no. 5, pp. 447–470, 2000.

[52] G. Gallego, G. Iyengar, R. Phillips, and A. Dubey, "Managing flexible products on a network," 2004.

[53] Q. Liu and G. Van Ryzin, "On the choice-based linear programming model for network revenue management," *Manufacturing & Service Operations Management*, vol. 10, no. 2, pp. 288–310, 2008.

[54] P. Singh, H. Groenevelt, and N. Rudi, "Product variety and supply chain structures," *Working paper. University of Rochester*, 2005.

[55] G. Aydın and W. H. Hausman, "The role of slotting fees in the coordination of assortment decisions," *Production and Operations Management*, vol. 18, no. 6, pp. 635–652, 2009.

[56] T.-H. Wu, "A note on a global approach for general 0–1 fractional programming," *European Journal of Operational Research*, vol. 101, no. 1, pp. 220–223, 1997.

[57] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, "Applications of second-order cone programming," *Linear algebra and its applications*, vol. 284, no. 1, pp. 193–228, 1998.

[58] G. P. McCormick, "Computability of global solutions to factorable nonconvex programs: Part iconvex underestimating problems," *Mathematical programming*, vol. 10, no. 1, pp. 147–175, 1976.

[59] D. A. Collier, "The measurement and operating benefits of component part commonality," *Decision Sciences*, vol. 12, no. 1, pp. 85–96, 1981.

[60] H. J. Thevenot and T. W. Simpson, "Commonality indices for assessing product families," in *Product platform and product family design*, pp. 107–129, Springer, 2006.

Appendix A

Performance Comparison of the Models Under Fixed Shipping Cost Setting

A.1 Common assortment under a fixed cost

A.1.1 Results of problems with 50 products and 5 customer classes

Table A.1: **Common assortment under a fixed cost- Sample 1 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	40.86%	184262	148.39 –	7.25%	14863	12.00 –	9.50%	22809	33.53 –	1.55%	2126	1.72 –	1.2744	1.6806
	20	18.23%	708249	– 2.37%	4.33%	2285714	– 1.13%	15.04%	668641	– 0.76%	2.66%	5890	11.01 –	1.5183	1.4367
	40	10.19%	656010	991.18 –	3.79%	3325304	– 2.07%	16.42%	232521	282.71 –	2.70%	2465	6.61 –	1.6292	1.3258
	50	7.79%	208935	85.99 –	3.00%	4506398	– 1.38%	15.95%	1013154	– 0.60%	2.26%	2001	8.02 –	1.6653	1.2897
10	10	51.73%	24158	17.20 –	5.85%	4046	3.74 –	5.52%	5663	7 –	0.51%	0	1.63 –	0.8630	2.0920
	20	27.26%	1560138	– 1.11%	3.85%	1742714	496.10 –	9.45%	1444205	– 0.45%	1.21%	4989	5.90 –	1.1358	1.8192
	40	14.51%	2765580	– 1.15%	3.06%	5595171	– 1.43%	11.96%	601442	– 0.65%	1.30%	8733	9.50 –	1.2623	1.6927
	50	11.96%	2907416	– 0.67%	2.67%	5417467	– 1.30%	11.77%	838285	– 0.45%	1.18%	6708	8.07 –	1.2910	1.6640

Table A.2: **Common assortment under a fixed cost- Sample 2 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC				
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	41.17%	588231	391.73 –	6.82%	33904	24.59 –	11.62%	70691	98.46 –	1.67%	4365	2.72 –	1.2905	1.7075
	20	18.89%	426740	– 1.97%	4.26%	3594959	– 1.30%	16.72%	400332	– 2.13%	2.56%	3544	7.52 –	1.5323	1.4657
	40	11.59%	2050186	– 0.70%	4.71%	4475724	– 2.60%	17.49%	658812	– 1.15%	3.35%	30692	33.77 –	1.6326	1.3654
	50	8.91%	8287699	1786.56 –	3.71%	2957550	– 1.55%	16.76%	840855	– 1.38%	2.66%	26499	46.47 –	1.6728	1.3252
10	10	54.85%	79015	40.89 –	5.76%	6669	5.75 –	6.57%	17879	23.15 –	0.61%	930	1.58 –	0.8884	2.1096
	20	26.45%	2528303	– 1.95%	4.01%	6584494	– 0.24%	10.36%	1476778	– 0.68%	1.30%	10276	9.16 –	1.1665	1.8315
	40	13.92%	2652112	– 1.72%	3.07%	4602004	– 1.93%	11.69%	1129044	– 0.10%	1.69%	684228	215.62 –	1.2948	1.7032
	50	11.32%	5277392	– 0.40%	2.60%	4721428	– 1.61%	11.50%	805985	– 0.57%	1.48%	1100	7.13 –	1.3250	1.6730

Table A.3: **Common assortment under a fixed cost- Sample 3 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	43.53%	390249	448.15 –	7.48%	51275	33.80 –	8.92%	46824	85.20 –	1.75%	3231	2.69 –	1.3171	1.6489
	20	17.89%	587032	– 1.49%	3.44%	4071195	– 0.67%	12.59%	1128639	– 1.08%	2.44%	1085	5.05 –	1.6036	1.3624
	40	10.50%	2438618	– 0.38%	3.73%	2922840	– 1.15%	14.95%	523799	902.25 –	2.70%	8692	9.94 –	1.7109	1.2551
	50	8.17%	5188126	– 0.28%	3.07%	2225971	– 1.10%	14.61%	876268	– 1.33%	2.29%	1116	6.73 –	1.7476	1.2184
10	10	55.54%	80655	39.53 –	6.09%	6020	4.85 –	4.73%	7648	10.08 –	0.62%	798	1.28 –	0.9089	2.0571
	20	25.36%	2715580	– 0.93%	3.13%	548454	160.82 –	7.24%	1676802	– 0.43%	1.10%	2120	3.64 –	1.2209	1.7451
	40	13.46%	3709739	– 0.60%	2.56%	4610514	– 1.43%	10.68%	953371	1244.97 –	1.40%	1084	3.44 –	1.3490	1.6170
	50	11.07%	4496535	– 0.53%	2.32%	3737982	– 1.33%	10.61%	1757165	– 0.05%	1.30%	11533	4.83 –	1.3779	1.5881

Table A.4: **Common assortment under a fixed cost- Sample 4 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	47.54%	1075565	859.82 –	9.38%	84279	81.28 –	10.88%	58022	101.45 –	1.85%	5358	4.39 –	1.2527	1.7283
	20	21.93%	897243	– 4.02%	9.47%	2170552	– 5.41%	15.57%	460458	– 2.46%	4.50%	347346	1099.73 –	1.5159	1.4651
	40	11.31%	3519498	– 0.47%	3.61%	4969246	– 2.01%	15.05%	75069	186.18 –	2.21%	1102	5.74 –	1.6604	1.3206
	50	9.04%	6049656	– 0.21%	3.02%	3286481	– 1.43%	14.63%	757243	– 0.58%	2.03%	1081	8.67 –	1.6951	1.2860
10	10	56.97%	184186	93.91 –	8.64%	22656	14.60 –	6.30%	21630	22.14 –	0.72%	3555	2.13 –	0.8494	2.1316
	20	31.62%	900708	– 3.08%	10.19%	2344249	– 4.18%	9.61%	942118	– 1.31%	2.48%	158156	338.89 –	1.1399	1.8411
	40	14.19%	1685882	– 0.42%	2.42%	4041969	– 1.25%	10.34%	248655	333.43 –	1.16%	1088	3.58 –	1.3139	1.6671
	50	11.76%	3396746	– 0.35%	2.19%	3313372	– 1.22%	10.19%	27074	33.84 –	1.09%	3201	5.75 –	1.3425	1.6385

Table A.5: **Common assortment under a fixed cost- Sample 5 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC				
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	43.87%	630018	406.71 –	8.17%	42482	39.99 –	10.09%	82365	193.27 –	1.84%	4672	3.05 –	1.2861	1.6919
	20	19.09%	676153	– 2.22%	4.35%	3621990	– 1.01%	15.18%	854488	– 1.76%	2.48%	2051	6.61 –	1.5537	1.4243
	40	11.13%	1803951	– 0.47%	4.15%	4588314	– 2.46%	16.32%	502419	– 1.20%	2.80%	13276	14.45 –	1.6650	1.3130
	50	8.74%	5803501	– 0.11%	3.38%	2486792	– 1.15%	15.77%	554176	– 1.13%	2.40%	2367	12.22 –	1.7016	1.2764
10	10	54.80%	53206	36.41 –	7.04%	6375	5.69 –	5.59%	12254	18.09 –	0.56%	430	1.17 –	0.8778	2.1002
	20	27.04%	2246486	– 1.70%	3.94%	862552	250.78 –	9.18%	1335645	– 0.57%	1.21%	1695	4.14 –	1.1743	1.8037
	40	13.39%	3742274	– 0.23%	2.58%	4400418	– 1.49%	11.30%	1005727	– 0.40%	1.40%	7861	5.33 –	1.3156	1.6624
	50	11.00%	5239465	– 0.33%	2.34%	3626162	– 1.41%	11.16%	1125786	– 0.34%	1.30%	1082	7.77 –	1.3440	1.6340

A.1.2 Results of problems with 100 products and 10 customer classes

Table A.6: **Common assortment under a fixed cost- Sample 1 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC				
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	63.22%	25713	– 21.03%	14.12%	83059	– 7.93%	11.10%	28783	– 4.80%	2.52%	21683	116.41 –	1.3240	1.6690
	20	30.13%	105365	– 13.57%	8.57%	262440	– 5.91%	16.89%	25867	– 10.11%	4.40%	112283	– 1.17%	1.6606	1.3324
	40	18.17%	138635	– 15.48%	7.30%	737767	– 5.66%	22.09%	24807	– 13.87%	5.06%	77325	– 0.77%	1.8286	1.1644
	50	16.96%	715287	– 15.81%	8.06%	1138070	– 6.43%	22.57%	10112	– 7.55%	6.15%	112158	– 0.86%	1.8476	1.1454
	100	11.80%	96639	– 8.78%	6.94%	1673092	– 5.70%	22.57%	104966	– 15.84%	6.87%	67060	– 0.80%	1.9329	1.0601
10	10	69.25%	469676	– 7.67%	13.85%	141766	– 5.19%	6.12%	66684	– 0.89%	0.86%	92830	253.61 –	0.9050	2.0880
	20	47.84%	26223	– 15.11%	9.75%	117001	– 6.50%	10.14%	32445	– 4.09%	2.15%	169118	– 0.36%	1.2588	1.7342
	40	22.05%	59916	– 9.15%	5.10%	520334	– 3.91%	15.07%	32982	– 6.94%	2.62%	119138	– 0.36%	1.5247	1.4683
	50	19.51%	51961	– 8.51%	5.27%	855968	– 4.00%	16.47%	20843	– 5.30%	2.78%	172213	– 0.25%	1.5571	1.4359
	100	14.95%	47199	– 6.94%	6.19%	1789951	– 4.92%	17.24%	72998	– 7.69%	4.00%	177965	– 0.20%	1.6188	1.3742

Table A.7: **Common assortment under a fixed cost- Sample 2 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC				
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	63.44%	13938	–	13.85%	63185	–	12.44%	32936	–	2.63%	15480	149.85	1.2764	1.6946
				22.52%			6.47%			5.37%			–		
	20	31.07%	50815	–	8.98%	261194	–	19.30%	19920	–	3.90%	79992	–	1.5916	1.3794
				13.69%			6.09%			12.18%			0.58%		
	40	19.22%	404811	–	7.86%	638411	–	24.40%	40174	–	5.00%	55209	–	1.7497	1.2213
				17.92%			6.00%			16.64%			0.91%		
	50	17.91%	395876	–	8.66%	901514	–	24.84%	23758	–	6.03%	59515	–	1.7693	1.2017
				16.88%			6.81%			14.72%			1.12%		
	100	12.55%	78538	–	7.51%	1498861	–	24.68%	39722	–	6.72%	104137	–	1.8535	1.1175
				10.69%			5.98%			14.36%			1.01%		
10	10	68.02%	212397	–	13.11%	179465	–	6.95%	42636	–	0.91%	15056	71.92	0.8688	2.1022
				7.90%			3.06%			1.34%			–		
	20	49.03%	26980	–	9.92%	140832	–	11.44%	31967	–	1.94%	119099	–	1.2080	1.7630
				22.02%			6.13%			5.47%			0.27%		
	40	23.77%	46493	–	5.78%	480962	–	16.79%	70165	–	2.76%	94450	–	1.4545	1.5165
				12.51%			4.53%			9.42%			0.41%		
	50	20.65%	39080	–	5.61%	666662	–	17.50%	49454	–	2.75%	99762	–	1.4921	1.4789
				9.30%			4.39%			10.80%			0.33%		
	100	15.66%	22260	–	6.37%	1490846	–	17.92%	53398	–	3.79%	123548	–	1.5565	1.4145
				7.15%			5.15%			10.56%			0.29%		

Table A.8: **Common assortment under a fixed cost- Sample 3 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	62.80%	23673	– 19.13%	13.41%	32319	– 7.47%	12.44%	36128	– 5.25%	2.63%	15480	142.17 –	1.2814	1.6946
	20	30.66%	87049	– 13.07%	8.64%	113808	– 6.42%	19.30%	20789	– 12.11%	3.90%	80718	– 0.58%	1.5966	1.3794
	40	18.88%	174246	– 16.58%	7.55%	288043	– 6.17%	24.40%	40609	– 16.59%	5.00%	55641	– 0.91%	1.7547	1.2213
	50	17.57%	584704	– 16.62%	8.36%	445093	– 6.94%	24.84%	23863	– 14.72%	6.03%	59866	– 1.12%	1.7743	1.2017
	100	12.25%	153426	– 9.94%	7.22%	777146	– 6.05%	24.68%	39306	– 14.37%	6.72%	102970	– 1.01%	1.8585	1.1175
10	10	67.02%	501009	– 3.80%	11.25%	75513	– 3.61%	6.41%	55081	– 1.27%	0.80%	21469	43.00 –	0.9004	2.0756
	20	48.41%	27639	– 16.16%	9.47%	62323	– 6.49%	11.44%	34622	– 5.41%	1.94%	120336	– 0.27%	1.2130	1.7630
	40	23.35%	67639	– 10.82%	5.41%	212449	– 4.70%	16.79%	71333	– 9.40%	2.76%	95222	– 0.41%	1.4595	1.5165
	50	20.25%	42478	– 7.90%	5.26%	326514	– 4.49%	17.50%	50162	– 10.80%	2.75%	99984	– 0.33%	1.4971	1.4789
	100	15.29%	44833	– 6.63%	6.03%	783215	– 5.23%	17.92%	58322	– 10.53%	3.79%	123881	– 0.29%	1.5615	1.4145

Table A.9: **Common assortment under a fixed cost- Sample 4 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	64.74%	27959	– 21.70%	14.12%	36742	– 8.61%	11.97%	28804	– 4.49%	2.60%	16621	139.36 –	1.2757	1.7143
	20	30.22%	60511	– 11.12%	8.51%	100934	– 6.03%	17.19%	21696	– 9.31%	4.08%	101176	– 0.52%	1.6139	1.3761
	40	18.51%	579249	– 17.17%	7.37%	310225	– 5.86%	22.85%	32767	– 12.86%	4.79%	64471	– 0.74%	1.7734	1.2166
	50	17.18%	551244	– 15.94%	8.04%	367646	– 6.51%	23.20%	27787	– 12.23%	5.72%	66733	– 0.80%	1.7935	1.1965
	100	11.67%	90887	– 9.09%	6.62%	667222	– 5.35%	22.96%	42822	– 13.14%	6.05%	54148	– 0.67%	1.8819	1.1081
10	10	70.36%	407879	– 6.21%	13.15%	59242	– 5.27%	6.54%	41603	– 1.06%	0.81%	16377	46.23 –	0.8707	2.1193
	20	48.82%	25953	– 14.74%	9.28%	66392	– 6.17%	10.06%	33877	– 4.48%	2.02%	152687	– 0.22%	1.2205	1.7695
	40	23.41%	48889	– 7.61%	5.33%	216625	– 4.25%	15.22%	34271	– 8.85%	2.40%	97422	– 0.29%	1.4718	1.5182
	50	20.76%	38614	– 7.03%	5.47%	330794	– 4.27%	16.66%	29732	– 7.48%	2.63%	96335	– 0.36%	1.5042	1.4858
	100	15.98%	23592	– 6.19%	6.31%	783872	– 5.09%	17.54%	81024	– 10.87%	3.75%	86530	– 0.39%	1.5662	1.4238

Table A.10: **Common assortment under a fixed cost- Sample 5 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	63.27%	28748	– 17.78%	13.71%	69347	– 6.40%	12.43%	35469	– 6.98%	2.24%	3331	47.06 –	1.2198	1.6952
	20	32.08%	66656	– 13.35%	9.73%	233853	– 7.00%	20.34%	28739	– 11.47%	4.22%	64732	– 0.92%	1.5078	1.4072
	40	19.68%	153260	– 15.67%	8.21%	553581	– 6.27%	25.26%	43397	– 22.38%	4.98%	49760	– 1.01%	1.6640	1.2510
	50	18.16%	637875	– 17.14%	8.79%	587675	– 7.05%	25.48%	22662	– 13.77%	5.75%	61223	– 0.99%	1.6855	1.2295
	100	12.24%	32458	– 7.42%	7.13%	1303808	– 5.72%	24.70%	48223	– 15.42%	5.87%	57183	– 0.71%	1.7744	1.1406
10	10	64.54%	396479	– 0.01%	12.20%	162133	1440.97 –	6.69%	46693	– 2.34%	0.76%	12680	62.24 –	0.8306	2.0844
	20	51.45%	26937	– 16.20%	10.63%	106512	– 7.24%	12.20%	33184	– 6.49%	2.08%	111259	– 0.43%	1.1361	1.7789
	40	24.87%	48822	– 8.32%	6.13%	424046	– 4.57%	17.27%	38409	– 8.73%	2.63%	132121	– 0.39%	1.3778	1.5372
	50	21.98%	53541	– 10.97%	6.18%	608976	– 4.64%	18.14%	22460	– 7.45%	2.79%	66369	– 0.37%	1.4106	1.5044
	100	16.73%	48014	– 6.51%	6.90%	1302100	– 5.50%	18.52%	62863	– 10.52%	3.75%	84139	– 0.47%	1.4740	1.4410

A.2 Customized assortments under a fixed cost

A.2.1 Results of problems with 50 products and 5 customer classes

Table A.11: **Customized assortments under a fixed cost- Sample 1 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	40.86%	449626	– 0.01%	7.31%	86572	160.28 –	9.50%	196538	1021.40 –	1.55%	1875	3.78 –	1.2744	1.6806
	20	18.23%	428051	– 5.06%	4.69%	1291818	– 2.37%	15.04%	208753	– 7.28%	2.72%	15935	62.54 –	1.5183	1.4367
	40	9.95%	156136	– 1.91%	3.96%	903858	– 1.20%	16.37%	288205	– 2.17%	3.00%	1209	24.59 –	1.6327	1.3223
	50	7.54%	159941	– 1.87%	3.11%	1308617	– 0.58%	15.88%	221169	– 0.79%	2.54%	1176	19.07 –	1.6694	1.2856
10	10	51.73%	49984	82.75 –	5.89%	4855	6.79 –	5.52%	25582	86.17 –	0.51%	104	2.39 –	0.8630	2.0920
	20	27.26%	530128	– 1.94%	3.89%	2123988	– 0.88%	9.45%	226161	– 3.18%	1.21%	2590	11.01 –	1.1358	1.8192
	40	14.36%	693834	– 2.06%	3.31%	1471245	– 1.03%	11.94%	222741	– 1.17%	1.43%	2690	20.75 –	1.2640	1.6910
	50	11.78%	498906	– 2.42%	2.88%	2025890	– 1.12%	11.77%	238206	– 1.04%	1.28%	3755	15.98 –	1.2931	1.6619

Table A.12: **Customized assortments under a fixed cost- Sample 2 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	41.17%	660508	– 1.19%	6.87%	165137	384.32 –	11.62%	218267	– 2.06%	1.67%	5539	18.87 –	1.2905	1.7075
	20	18.89%	183524	– 4.09%	4.52%	967920	– 1.93%	16.72%	223869	– 7.86%	2.68%	7755	29.85 –	1.5323	1.4657
	40	11.52%	204385	– 3.11%	4.93%	1143385	– 2.02%	17.60%	219491	– 4.22%	3.73%	1113408	– 0.07%	1.6337	1.3643
	50	8.82%	212802	– 1.70%	3.91%	1278992	– 1.52%	16.86%	256064	– 2.06%	3.05%	21778	48.64 –	1.6742	1.3238
10	10	54.85%	159157	215.38 –	5.78%	12050	13.36 –	6.57%	759008	– 0.42%	0.61%	1231	3.25 –	0.8884	2.1096
	20	26.45%	676362	– 1.97%	4.09%	2246569	– 1.03%	10.36%	275320	– 3.55%	1.30%	3271	13.62 –	1.1665	1.8315
	40	13.89%	266833	– 1.06%	3.35%	2672030	– 1.88%	11.76%	223877	– 0.54%	1.81%	1644	16.70 –	1.2952	1.7028
	50	11.25%	337300	– 1.16%	2.85%	2072803	– 0.60%	11.57%	403130	– 0.32%	1.59%	1257	11.74 –	1.3258	1.6722

Table A.13: **Customized assortments under a fixed cost- Sample 3 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	43.53%	530135	– 4.06%	7.56%	178087	438.75 –	8.92%	374079	– 1.09%	1.75%	6131	27.87 –	1.3171	1.6489
	20	17.89%	146479	– 1.98%	3.78%	1266652	– 1.95%	12.59%	200938	– 5.58%	2.50%	1344	15.89 –	1.6036	1.3624
	40	10.50%	165815	– 1.95%	4.05%	1148618	– 1.62%	15.16%	140459	– 0.64%	3.09%	1152	35.09 –	1.7109	1.2551
	50	8.13%	232680	– 0.98%	3.30%	969599	– 0.90%	14.74%	389333	– 1.15%	2.62%	1167	26.16 –	1.7484	1.2176
10	10	55.54%	321170	444.71 –	6.15%	13302	11.12 –	4.73%	86759	293.83 –	0.62%	494	3.40 –	0.9089	2.0571
	20	25.36%	585381	– 1.28%	3.18%	2372057	– 0.33%	7.24%	254340	– 1.56%	1.10%	1057	7.84 –	1.2209	1.7451
	40	13.44%	656490	– 0.53%	2.74%	4097207	– 1.87%	10.72%	271488	– 0.27%	1.52%	1115	15.98 –	1.3492	1.6168
	50	11.04%	1035365	– 0.41%	2.51%	2540643	– 1.50%	10.67%	565603	– 0.66%	1.41%	1433	18.26 –	1.3783	1.5877

Table A.14: **Customized assortments under a fixed cost- Sample 4 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	47.54%	372936	– 4.08%	9.45%	226123	570.10 –	10.88%	223073	– 1.42%	1.85%	7650	18.29 –	1.2527	1.7283
	20	21.93%	236890	– 5.26%	9.79%	710400	– 6.00%	15.57%	146515	– 7.89%	4.52%	254858	– 0.30%	1.5159	1.4651
	40	11.29%	141127	– 2.66%	3.97%	1274149	– 1.43%	15.31%	199907	– 0.77%	2.80%	1110	16.54 –	1.6607	1.3203
	50	8.95%	197582	– 1.55%	3.29%	908045	– 0.71%	14.82%	262072	– 0.77%	2.49%	1087	13.13 –	1.6964	1.2846
10	10	56.97%	280886	353.40 –	8.70%	38796	42.88 –	6.30%	184100	762.78 –	0.71%	1464	4.12 –	0.8494	2.1316
	20	31.62%	403932	– 3.48%	10.41%	1043375	– 5.93%	9.61%	194843	– 3.15%	2.49%	423724	– 0.11%	1.1399	1.8411
	40	14.19%	308947	– 1.95%	2.76%	3997552	– 1.84%	10.47%	144178	– 0.50%	1.26%	1108	10.57 –	1.3139	1.6671
	50	11.72%	1056039	– 0.56%	2.47%	3675644	– 1.70%	10.33%	371886	– 0.27%	1.18%	5427	12.42 –	1.3429	1.6381

Table A.15: **Customized assortments under a fixed cost- Sample 5 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	43.87%	680791	–	8.28%	166471	397.33	10.09%	236493	–	1.84%	10822	31.46	1.2861	1.6919
				1.91%			–			1.41%			–		
	20	19.09%	120260	–	4.67%	896557	–	15.18%	126617	–	2.55%	5582	31.51	1.5537	1.4243
				2.31%			2.22%			4.68%			–		
	40	11.07%	147668	–	4.48%	705254	–	16.51%	115352	–	3.31%	184342	429.96	1.6660	1.3120
				2.23%			1.76%			1.89%			–		
	50	8.59%	181363	–	3.63%	955560	–	15.90%	260986	–	2.85%	142364	198.28	1.7040	1.2740
				1.59%			1.12%			0.85%			–		
10	10	54.80%	488219	1033.72	7.10%	9225	13.58	5.59%	434515	1209.10	0.56%	556	2.95	0.8778	2.1002
				–			–			–			–		
	20	27.04%	551556	–	4.00%	2453125	–	9.18%	192098	–	1.21%	1252	12.62	1.1743	1.8037
				1.66%			0.50%			2.38%			–		
	40	13.37%	414877	–	2.93%	2769854	–	11.41%	158635	–	1.53%	10650	21.89	1.3159	1.6621
				0.67%			1.97%			1.02%			–		
	50	10.92%	539340	–	2.64%	2228035	–	11.25%	288068	–	1.41%	5278	16.36	1.3449	1.6331
				0.73%			1.12%			0.30%			–		

A.2.2 Results of problems with 100 products and 10 customer classes

Table A.16: **Customized assortments under a fixed cost- Sample 1 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	63.22%	21556	– 23.72%	14.18%	50417	– 8.79%	11.10%	21928	– 26.98%	2.52%	99642	1258.26 –	1.3240	1.6690
	20	30.15%	24801	– 11.54%	8.74%	223016	– 6.42%	16.90%	9721	– 30.49%	4.42%	62296	– 1.38%	1.6604	1.3326
	40	18.17%	21796	– 8.69%	7.59%	85693	– 5.91%	22.18%	4206	– 13.49%	5.72%	26472	– 1.06%	1.8287	1.1643
	50	16.93%	26819	– 10.01%	8.32%	61534	– 6.00%	22.80%	6381	– 12.27%	7.00%	35290	– 1.16%	1.8480	1.1450
	100	11.70%	29121	– 5.74%	7.11%	76438	– 5.00%	22.95%	30379	– 12.09%	8.14%	21948	– 1.05%	1.9347	1.0583
10	10	69.25%	106221	– 13.91%	13.87%	77836	– 8.60%	6.12%	44758	– 2.80%	0.86%	156221	777.02 –	0.9050	2.0880
	20	47.85%	23067	– 12.43%	9.81%	81469	– 7.16%	10.15%	17081	– 9.00%	2.15%	77978	– 0.45%	1.2586	1.7344
	40	22.05%	22633	– 6.40%	5.45%	187871	– 4.41%	15.07%	13129	– 15.14%	2.64%	102154	– 0.43%	1.5247	1.4683
	50	19.48%	33405	– 6.27%	5.62%	143138	– 4.55%	16.46%	12253	– 8.39%	3.01%	51340	– 0.34%	1.5575	1.4355
	100	14.89%	28202	– 3.85%	6.52%	285438	– 5.40%	17.31%	26910	– 4.44%	4.49%	147191	– 0.39%	1.6198	1.3732

Table A.17: **Customized assortments under a fixed cost- Sample 2 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	63.44%	25886	– 22.38%	13.93%	55909	– 7.29%	12.44%	25617	– 24.30%	2.63%	26313	326.49 –	1.2764	1.6946
	20	31.07%	19514	– 11.15%	9.23%	136361	– 6.07%	19.30%	10638	– 31.92%	3.91%	45649	– 0.85%	1.5916	1.3794
	40	19.22%	22343	– 9.35%	8.15%	87154	– 6.20%	24.50%	5038	– 18.59%	5.59%	24354	– 1.23%	1.7498	1.2212
	50	18.04%	268275	– 17.82%	9.07%	102291	– 7.04%	25.13%	6882	– 16.54%	6.98%	25708	– 1.56%	1.7673	1.2037
	100	12.37%	30288	– 6.21%	7.63%	64236	– 5.28%	24.80%	19076	– 10.88%	7.79%	16323	– 1.16%	1.8565	1.1145
10	10	68.02%	90072	– 13.45%	13.14%	92768	– 7.48%	6.95%	31353	– 3.72%	0.91%	22316	117.12 –	0.8688	2.1022
	20	49.03%	26039	– 11.63%	10.00%	99922	– 7.11%	11.44%	20269	– 8.43%	1.94%	75011	– 0.30%	1.2080	1.7630
	40	23.79%	22499	– 6.97%	6.08%	181498	– 5.00%	16.80%	9745	– 14.80%	2.79%	83565	– 0.48%	1.4543	1.5167
	50	20.64%	24973	– 8.80%	5.97%	160619	– 5.25%	17.52%	11289	– 10.66%	2.99%	49968	– 0.40%	1.4922	1.4788
	100	15.62%	28555	– 4.40%	6.73%	246314	– 5.74%	18.11%	24705	– 5.52%	4.37%	33823	– 0.55%	1.5570	1.4140

Table A.18: **Customized assortments under a fixed cost- Sample 3 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	62.80%	22153	– 23.75%	13.49%	51287	– 7.35%	12.44%	25514	– 24.30%	2.63%	26313	360.36 –	1.2814	1.6946
	20	30.66%	17103	– 11.24%	8.89%	122878	– 6.09%	19.30%	10623	– 31.92%	3.91%	38813	– 0.87%	1.5966	1.3794
	40	18.88%	22323	– 9.36%	7.85%	73355	– 6.22%	24.50%	4091	– 18.59%	5.59%	19592	– 1.25%	1.7548	1.2212
	50	17.70%	251956	– 17.82%	8.76%	89223	– 7.04%	25.13%	5559	– 16.54%	6.98%	20444	– 1.56%	1.7723	1.2037
	100	12.07%	30292	– 6.21%	7.34%	55034	– 5.28%	24.80%	14933	– 10.88%	7.79%	13450	– 1.16%	1.8615	1.1145
10	10	67.02%	88800	– 14.05%	11.27%	83492	– 5.92%	6.41%	40476	– 4.26%	0.80%	34157	167.16 –	0.9004	2.0756
	20	48.42%	21006	– 11.98%	9.55%	86954	– 7.16%	11.44%	20144	– 8.43%	1.94%	61889	– 0.32%	1.2130	1.7630
	40	23.36%	22480	– 7.11%	5.72%	160980	– 5.01%	16.80%	7504	– 14.83%	2.79%	66456	– 0.48%	1.4593	1.5167
	50	20.24%	21620	– 8.84%	5.61%	142702	– 5.25%	17.52%	8494	– 10.66%	2.99%	40464	– 0.40%	1.4972	1.4788
	100	15.25%	28543	– 4.40%	6.39%	221372	– 5.74%	18.11%	20747	– 5.52%	4.37%	28177	– 0.56%	1.5620	1.4140

Table A.19: **Customized assortments under a fixed cost- Sample 4 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	64.74%	22387	– 24.04%	14.17%	49287	– 8.68%	11.97%	22159	– 20.91%	2.60%	80647	1337.82 –	1.2757	1.7143
	20	30.23%	21072	– 11.23%	8.64%	184986	– 6.02%	17.20%	9351	– 28.65%	4.09%	48944	– 0.86%	1.6138	1.3762
	40	18.53%	23141	– 8.69%	7.69%	82449	– 6.12%	22.97%	12497	– 21.34%	5.40%	18264	– 0.96%	1.7730	1.2170
	50	17.17%	220493	– 17.04%	8.32%	48961	– 6.09%	23.43%	9571	– 15.55%	6.48%	21768	– 1.07%	1.7936	1.1964
	100	11.57%	40542	– 6.35%	6.80%	49079	– 4.62%	23.28%	20829	– 10.95%	7.19%	13923	– 0.90%	1.8838	1.1062
10	10	70.36%	83517	– 14.58%	13.18%	79432	– 8.35%	6.54%	37309	– 4.05%	0.81%	29607	121.60 –	0.8707	2.1193
	20	48.82%	20924	– 12.00%	9.33%	96059	– 6.68%	10.06%	20305	– 7.12%	2.02%	89584	– 0.27%	1.2205	1.7695
	40	23.41%	19529	– 6.10%	5.66%	125763	– 4.52%	15.22%	15381	– 14.81%	2.42%	66228	– 0.33%	1.4718	1.5182
	50	20.76%	17918	– 8.72%	5.84%	130605	– 4.94%	16.68%	12164	– 9.49%	2.81%	39231	– 0.44%	1.5042	1.4858
	100	15.93%	39277	– 5.20%	6.64%	206720	– 5.47%	17.65%	23910	– 5.71%	4.20%	44190	– 0.64%	1.5668	1.4232

Table A.20: **Customized assortments under a fixed cost- Sample 5 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	55.49%	22982	– 21.59%	8.41%	53175	– 6.27%	12.43%	18673	– 12.52%	2.24%	19139	237.30 –	1.2808	1.6952
	20	26.94%	20814	– 11.90%	5.70%	133973	– 6.59%	20.34%	11217	– 34.90%	4.23%	30651	– 1.18%	1.5688	1.4072
	40	15.38%	23834	– 9.72%	4.64%	89764	– 6.76%	25.29%	5575	– 19.16%	5.61%	17735	– 1.03%	1.7260	1.2500
	50	14.06%	25515	– 10.39%	5.32%	92477	– 7.33%	25.70%	7288	– 15.51%	6.72%	19611	– 1.25%	1.7461	1.2299
	100	8.44%	25112	– 7.25%	3.83%	73366	– 5.25%	25.01%	13208	– 11.78%	7.21%	12414	– 1.11%	1.8366	1.1394
10	10	53.29%	89635	– 9.40%	4.56%	91497	– 5.99%	6.68%	27866	– 4.62%	0.76%	16891	85.22 –	0.8916	2.0844
	20	43.74%	20831	– 13.35%	5.07%	73920	– 7.92%	12.20%	15213	– 8.95%	2.08%	56323	– 0.49%	1.1970	1.7790
	40	19.58%	16978	– 7.39%	2.00%	149818	– 4.89%	17.27%	14644	– 29.51%	2.68%	68803	– 0.44%	1.4388	1.5372
	50	16.92%	23963	– 9.06%	2.18%	128218	– 5.33%	18.18%	9446	– 12.78%	3.08%	30605	– 0.53%	1.4715	1.5045
	100	11.97%	33494	– 5.96%	2.94%	171716	– 5.76%	18.65%	15728	– 6.92%	4.24%	40863	– 0.70%	1.5367	1.4393

Appendix B

Comparison of Fixed Cost Models with $\tau = \{0, 0.25, 0.5, 1\}$

B.1 Comparison under common assortment

Table B.1: **Sample 1 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.00%	0	0.30	1.5023	0.76%	0	0.92	1.5914	1.55%	2126	1.72	1.6806	3.35%	4106	8.23	1.8588
				–				–				–				–	
	20	0.00%	0	0.25	1.2135	1.38%	1097	4.16	1.3251	2.66%	5890	11.01	1.4367	5.09%	216615	596.30	1.6473
				–				–				–				–	
	40	0.01%	0	0.20	1.1606	1.38%	1109	2.63	1.2456	2.70%	2465	6.61	1.3258	4.78%	178963	293.68	1.4534
				–				–				–				–	
	50	0.01%	0	0.20	1.1606	1.19%	1082	3.19	1.2278	2.26%	2001	8.02	1.2897	3.72%	21769	36.73	1.3855
				–				–				–				–	
10	10	0.01%	0	0.30	1.9712	0.26%	0	0.71	2.0316	0.51%	0	1.63	2.0920	1.07%	2326	2.22	2.2129
				–				–				–				–	
	20	0.07%	0	0.63	1.6501	0.66%	2987	3.05	1.7352	1.21%	4989	5.90	1.8192	2.36%	23084	35.92	1.9863
				–				–				–				–	
	40	0.00%	0	0.17	1.5101	0.65%	16113	6.41	1.6111	1.30%	8733	9.50	1.6927	2.76%	608055	518.65	1.8268
				–				–				–				–	
	50	0.00%	0	0.18	1.5101	0.58%	1327	3.11	1.5957	1.18%	6708	8.07	1.6640	2.39%	57412	41.49	1.7718
		–				–				–				–			

Table B.2: **Sample 2 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.04%	0	0.39 –	1.5137	0.88%	267	1.13 –	1.6109	1.67%	4365	2.72 –	1.7075	3.29%	4771	8.2 –	1.8978
	20	0.02%	0	0.28 –	1.2323	1.39%	1411	3.14 –	1.3494	2.56%	3544	7.52 –	1.4657	5.73%	668303	– 0.28%	1.6949
	40	0.00%	0	0.14 –	1.1764	1.66%	10613	6.05 –	1.2746	3.35%	30692	33.77 –	1.3654	5.22%	979805	– 0.16%	1.4909
	50	0.00%	0	0.12 –	1.1764	1.42%	1118	5.03 –	1.2556	2.66%	26499	46.47 –	1.3252	3.50%	8797	19.72 –	1.4101
10	10	0.06%	0	0.53 –	1.9749	0.35%	106	0.66 –	2.0425	0.61%	930	1.58 –	2.1096	1.21%	6258	3.69 –	2.2422
	20	0.01%	0	0.27 –	1.6516	0.67%	5206	2.53 –	1.7416	1.30%	10276	9.16 –	1.8315	2.43%	42866	55.5 –	2.0099
	40	0.00%	0	0.14 –	1.5233	0.83%	1094	3.02 –	1.6174	1.69%	684228	215.62 –	1.7032	3.50%	1197769	– 0.14%	1.8566
	50	0.00% –	0	0.13	1.5233	0.76% –	1099	2.66	1.6026	1.48% –	1100	7.13	1.6730	2.88% –	153320	142.37	1.7951

Table B.3: **Sample 3 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.32%	0	0.59	1.4599	1.00%	465	1.22	1.5546	1.75%	3231	2.69	1.6489	3.56%	13398	19.06	1.8374
				–				–				–				–	
	20	0.01%	0	0.27	1.1304	1.31%	2085	2.81	1.2464	2.44%	1085	5.05	1.3624	5.06%	90360	216.14	1.5912
				–				–				–				–	
	40	0.00%	0	0.12	1.0756	1.46%	1081	4.02	1.1728	2.70%	8692	9.94	1.2551	4.89%	38013	61.98	1.3907
				–				–				–				–	
	50	0.00%	0	0.12	1.0756	1.29%	1114	2.83	1.1547	2.29%	1116	6.73	1.2184	3.88%	10217	20.24	1.3173
				–				–				–				–	
10	10	0.00%	0	0.23	1.9219	0.31%	0	0.65	1.9895	0.62%	798	1.28	2.0571	1.24%	6489	4.36	2.1899
				–				–				–				–	
	20	0.00%	0	0.26	1.5683	0.58%	2067	2.27	1.6567	1.10%	2120	3.64	1.7451	2.08%	2605	7.80	1.9219
				–				–				–				–	
	40	0.00%	0	0.11	1.4357	0.68%	1103	2.81	1.5310	1.40%	1084	3.44	1.6170	2.84%	49001	62.30	1.7652
				–				–				–				–	
	50	0.00%	0	0.13	1.4357	0.64%	1069	2.50	1.5165	1.30%	11533	4.83	1.5881	2.54%	21155	23.55	1.7079
		–				–				–				–			

Table B.4: **Sample 4 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.01%	0	0.45 –	1.5468	0.96%	122	1.02 –	1.6375	1.85%	5358	4.39 –	1.7283	3.48%	4066	9.89 –	1.9098
	20	0.36%	320	0.80 –	1.2455	2.50%	388322	920.66 –	1.3562	4.50%	347346	1099.73 –	1.4651	8.70%	237660	– 2.23%	1.6826
	40	0.00%	0	0.17 –	1.1332	1.19%	1082	3.44 –	1.2323	2.21%	1102	5.74 –	1.3206	5.34%	540672	– 0.26%	1.4772
	50	0.00%	0	0.18 –	1.1332	1.07%	1052	2.44 –	1.2144	2.03%	1081	8.67 –	1.2860	4.57%	970562	– 0.13%	1.4097
10	10	0.06%	0	0.48 –	2.0079	0.39%	42	0.95 –	2.0700	0.72%	3555	2.13 –	2.1316	1.33%	6376	8.95 –	2.2547
	20	0.46%	190	1.25 –	1.6738	1.46%	15120	16.73 –	1.7576	2.48%	158156	338.89 –	1.8411	4.64%	640481	– 0.35%	2.0063
	40	0.00%	0	0.19 –	1.4814	0.58%	1126	2.48 –	1.5790	1.16%	1088	3.58 –	1.6671	2.41%	3281	8.34 –	1.8189
	50	0.00% –	0	0.17	1.4814	0.54% –	1079	1.91	1.5643	1.09% –	3201	5.75	1.6385	2.19% –	2819	7.92	1.7644

Table B.5: **Sample 5 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.04%	0	0.38	1.5111	0.94%	203	1.26	1.6016	1.84%	4672	3.05	1.6919	3.38%	6570	12.59	1.8684
				–				–				–				–	
	20	0.02%	0	0.27	1.2009	1.31%	1077	3.02	1.3129	2.48%	2051	6.61	1.4243	5.04%	76385	196.03	1.6388
				–				–				–				–	
	40	0.00%	0	0.11	1.1279	1.41%	1059	2.74	1.2237	2.80%	13276	14.45	1.3130	4.84%	189214	359.17	1.4445
				–				–				–				–	
	50	0.00%	0	0.11	1.1279	1.23%	1102	3.11	1.2056	2.40%	2367	12.22	1.2764	3.62%	9102	17.96	1.3703
				–				–				–				–	
10	10	0.02%	0	0.50	1.9777	0.28%	0	0.86	2.0389	0.56%	430	1.17	2.1002	1.17%	5258	5.42	2.2227
				–				–				–				–	
	20	0.05%	0	0.47	1.6327	0.63%	2096	2.71	1.7182	1.21%	1695	4.14	1.8037	2.27%	5911	10.84	1.9724
				–				–				–				–	
	40	0.01%	0	0.18	1.4871	0.68%	2660	2.06	1.5779	1.40%	7861	5.33	1.6624	2.82%	44857	53.50	1.8112
				–				–				–				–	
	50	0.01%	0	0.17	1.4871	0.65%	1110	2.00	1.5639	1.30%	1082	7.77	1.6340	2.45%	5167	10.86	1.7540
		–				–				–				–			

B.2 Comparison under customized assortments

Table B.6: **Sample 1 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.00%	0	0.36	1.5023	0.76%	524	1.39	1.5914	1.55%	1875	3.78	1.6806	3.43%	3506	13.73	1.8588
				–				–				–				–	
	20	0.00%	0	0.34	1.2135	1.38%	1096	5.91	1.3251	2.72%	15935	62.54	1.4367	6.27%	265478	–	1.6476
				–				–				–				1.16%	
	40	0.00%	0	0.17	1.1598	1.41%	1072	7.06	1.2438	3.00%	1209	24.59	1.3223	5.74%	529192	–	1.4422
				–				–				–				0.19%	
	50	0.00%	0	0.16	1.1598	1.21%	1116	5.38	1.2258	2.54%	1176	19.07	1.2856	4.42%	30754	54.79	1.3698
				–				–				–				–	
10	10	0.01%	0	0.75	1.9712	0.26%	0	1.38	2.0316	0.51%	104	2.39	2.0920	1.07%	3378	3.83	2.2129
				–				–				–				–	
	20	0.08%	0	0.75	1.6501	0.66%	2514	7.36	1.7352	1.21%	2590	11.01	1.8192	2.59%	326495	1073.98	1.9863
				–				–				–				–	
	40	0.00%	0	0.22	1.5097	0.69%	20399	22.44	1.6102	1.43%	2690	20.75	1.6910	3.27%	160406	382.80	1.8208
				–				–				–				–	
	50	0.00%	0	0.17	1.5095	0.60%	13398	14.33	1.5943	1.28%	3755	15.98	1.6619	2.83%	408373	518.71	1.7643
				–				–				–				–	

Table B.7: **Sample 2 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.04%	0	0.41 –	1.5137	0.88%	5996	7.56 –	1.6109	1.67%	5539	18.87 –	1.7075	3.35%	9498	27.96 –	1.8978
	20	0.02%	0	0.35 –	1.2323	1.39%	3360	7.77 –	1.3494	2.68%	7755	29.85 –	1.4657	6.72%	298974	– 1.73%	1.6955
	40	0.00%	0	0.16 –	1.1762	1.73%	1095	9.75 –	1.2739	3.73%	1113408	– 0.07%	1.3643	6.19%	343765	– 0.85%	1.4838
	50	0.00%	0	0.17 –	1.1762	1.49%	1124	8.00 –	1.2549	3.05%	21778	48.64 –	1.3238	4.03%	85061	202.87 –	1.3964
10	10	0.06%	0	0.73 –	1.9749	0.35%	64	1.15 –	2.0425	0.61%	1231	3.25 –	2.1096	1.21%	8438	12.16 –	2.2422
	20	0.01%	0	0.33 –	1.6516	0.67%	1451	5.61 –	1.7416	1.30%	3271	13.62 –	1.8315	2.69%	233051	858.89 –	2.0099
	40	0.00%	0	0.18 –	1.5230	0.85%	2855	7.19 –	1.6170	1.81%	1644	16.70 –	1.7028	3.97%	591496	– 0.26%	1.8540
	50	0.00%	0	0.16 –	1.5230	0.78%	4942	7.82 –	1.6019	1.59%	1257	11.74 –	1.6722	3.32%	1073644	– 0.20%	1.7911

Table B.8: **Sample 3 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.32%	0	0.80 .	1.4599	1.00%	946	1.87 –	1.5546	1.75%	6131	27.87 –	1.6489	3.60%	15753	62.08 –	1.8374
	20	0.01%	0	0.35 –	1.1304	1.31%	1080	4.10 –	1.2465	2.50%	1344	15.89 –	1.3624	6.44%	281462	– 0.87%	1.5904
	40	0.00%	0	0.16 –	1.0755	1.56%	1120	9.78 –	1.1726	3.09%	1152	35.09 –	1.2551	5.86%	353710	– 0.21%	1.3789
	50	0.00%	0	0.16 –	1.0755	1.38%	1088	8.75 –	1.1542	2.62%	1167	26.16 –	1.2176	4.69%	335341	644.31 –	1.3053
10	10	0.00%	0	0.28 –	1.9219	0.30%	0	0.88 –	1.9895	0.62%	494	3.40 –	2.0571	1.24%	7529	18.48 –	2.1899
	20	0.00%	0	0.31 –	1.5683	0.58%	1065	3.15 –	1.6567	1.10%	1057	7.84 –	1.7451	2.40%	4976	22.63 –	1.9219
	40	0.00%	0	0.18 –	1.4355	0.70%	1078	6.78 –	1.5308	1.52%	1115	15.98 –	1.6168	3.42%	475676	1069.21 –	1.7633
	50	0.00%	0	0.17 –	1.4355	0.68%	1105	5.82 –	1.5164	1.41%	1433	18.26 –	1.5877	2.95%	79780	144.13 –	1.7030

Table B.9: **Sample 4 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.01%	0	0.63 –	1.5468	0.96%	3142	3.19 –	1.6375	1.85%	7650	18.29 –	1.7283	3.63%	7309	27.43 –	1.9098
	20	0.37%	0	1.39 –	1.2455	2.51%	122471	396.47 –	1.3562	4.52%	254858	– 0.30%	1.4651	9.70%	180748	– 2.68%	1.6817
	40	0.00%	0	0.19 –	1.1328	1.26%	1121	9.04 –	1.2324	2.80%	1110	16.54 –	1.3203	6.12%	335566	– 0.89%	1.4611
	50	0.00%	0	0.16 –	1.1328	1.11%	1062	6.16 –	1.2141	2.49%	1087	13.13 –	1.2846	4.75%	722444	– 0.26%	1.3856
10	10	0.06%	0	0.68 –	2.0079	0.39%	239	1.54 –	2.0700	0.71%	1464	4.12 –	2.1316	1.33%	7171	12.49 –	2.2547
	20	0.47%	1219	6.86 –	1.6738	1.46%	200568	567.46 –	1.7576	2.49%	423724	– 0.11%	1.8411	4.79%	283118	– 0.74%	2.0063
	40	0.00%	0	0.17 –	1.4807	0.62%	1105	6.87 –	1.5789	1.26%	1108	10.57 –	1.6671	3.32%	382402	844.26 –	1.8185
	50	0.00%	0	0.19 –	1.4807	0.57%	1072	4.75 –	1.5640	1.18%	5427	12.42 –	1.6381	2.91%	94363	129.52 –	1.7606

Table B.10: **Sample 5 results for problems with 50 products and 5 classes.**

		τ=0				τ=0.25				τ=0.5				τ=1			
v_0	κ	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj	Gap	Nodes	Time	Min Obj
5	10	0.04%	0	0.67 –	1.5111	0.94%	4655	4.98 –	1.6016	1.84%	10822	31.46 –	1.6919	3.52%	8319	24.74 –	1.8684
	20	0.02%	0	0.42 –	1.2009	1.31%	1094	6.14 –	1.3129	2.55%	5582	31.51 –	1.4243	6.19%	257271	– 1.02%	1.6379
	40	0.00%	0	0.17 –	1.1277	1.47%	1080	9.24 –	1.2234	3.31%	184342	429.96 –	1.3120	5.96%	410044	– 0.51%	1.4319
	50	0.00%	0	0.54 –	1.1277	1.31%	1074	7.42 –	1.2051	2.85%	142364	198.28 –	1.2740	4.23%	28460	71.15 –	1.3520
10	10	0.02%	0	0.34 –	1.9777	0.28%	145	1.38 –	2.0389	0.56%	556	2.95 –	2.1002	1.17%	5958	8.09 –	2.2227
	20	0.05%	0	0.77 –	1.6327	0.63%	1159	4.68 –	1.7182	1.21%	1252	12.62 –	1.8037	2.55%	40942	124.46 –	1.9724
	40	0.00%	0	0.19 –	1.4862	0.70%	1081	5.98 –	1.5776	1.53%	10650	21.89 –	1.6621	3.66%	619050	– 0.17%	1.8090
	50	0.00%	0	0.16 –	1.4862	0.66%	1104	4.09 –	1.5633	1.41%	5278	16.36 –	1.6331	3.20%	1231597	– 0.04%	1.7497

Appendix C

DCI Analysis of Fixed Cost Setting

Table C.1: **DCI analysis of the models with fixed cost, τ=0.**

v_0	κ	Sample	Common Assortment			Customized Assortments				
			Number of Unique Products	$\sum_{i\in M j\in N} O_{ij}$	DCI	Number of Unique Products	$\sum_{i\in M j\in N} O_{ij}$	DCI	$\sum_{i\in M j\in N} m_{ij}$	DCI'
5	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	27	27	1	28	39	1.3929	133	4.7500
		2	27	27	1	28	39	1.3929	136	4.8571
		3	30	30	1	30	39	1.3000	148	4.9333
		4	32	32	1	32	40	1.2500	157	4.9063
		5	30	30	1	30	40	1.3333	148	4.9333
	50	1	27	27	1	28	42	1.5000	133	4.7500
		2	27	27	1	28	49	1.7500	136	4.8571
		3	30	30	1	30	41	1.3667	148	4.9333
		4	32	32	1	32	50	1.5625	157	4.9063
		5	30	30	1	30	50	1.6667	148	4.9333
10	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	39	40	1.0256	40	40	1	195	4.8750
		2	33	33	1	34	40	1.1765	164	4.8235
		3	36	36	1	37	39	1.0541	180	4.8649
		4	40	40	1	40	40	1	187	4.6750
		5	36	36	1	37	40	1.0811	175	4.7297
	50	1	39	49	1.2564	42	49	1.1667	197	4.6905
		2	33	33	1	34	49	1.4412	164	4.8235
		3	36	36	1	37	50	1.3514	180	4.8649
		4	40	40	1	40	48	1.2000	187	4.6750
		5	36	38	1.0556	37	49	1.3243	175	4.7297

Table C.2: **DCI analysis of the models with fixed cost, τ=0.25.**

v_0	κ	Sample	Common Assortment			Customized Assortments				
			Number of Unique Products	$\sum_{i\in M j\in N} O_{ij}$	DCI	Number of Unique Products	$\sum_{i\in M j\in N} O_{ij}$	DCI	$\sum_{i\in M j\in N} m_{ij}$	DCI'
5	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	26	40	1.5385	26	40	1.5385	123	4.7308
		2	27	40	1.4815	27	40	1.4815	131	4.8519
		3	28	40	1.4286	28	40	1.4286	137	4.8929
		4	29	40	1.3793	29	40	1.3793	144	4.9655
		5	28	40	1.4286	29	39	1.3448	139	4.7931
	50	1	25	45	1.8000	26	50	1.9231	122	4.6923
		2	27	50	1.8519	27	50	1.8519	130	4.8148
		3	28	50	1.7857	28	50	1.7857	134	4.7857
		4	29	50	1.7241	30	50	1.6667	145	4.8333
		5	28	50	1.7857	29	50	1.7241	139	4.7931
10	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	33	40	1.2121	35	40	1.1429	165	4.7143
		2	31	40	1.2903	32	40	1.2500	152	4.7500
		3	33	40	1.2121	33	40	1.2121	164	4.9697
		4	34	40	1.1765	35	40	1.1429	172	4.9143
		5	32	40	1.2500	33	40	1.2121	161	4.8788
	50	1	33	50	1.5152	35	50	1.4286	162	4.6286
		2	31	50	1.6129	32	50	1.5625	150	4.6875
		3	33	50	1.5152	34	50	1.4706	166	4.8824
		4	34	50	1.4706	35	50	1.4286	171	4.8857
		5	32	50	1.5625	33	50	1.5152	160	4.8485

Table C.3: **DCI analysis of the models with fixed cost, τ=0.5.**

v_0	κ	Sample	Common Assortment: Number of Unique Products	Common Assortment: $\sum_{i\in M j\in N} O_{ij}$	Common Assortment: DCI	Customized Assortments: Number of Unique Products	Customized Assortments: $\sum_{i\in M j\in N} O_{ij}$	Customized Assortments: DCI	Customized Assortments: $\sum_{i\in M j\in N} m_{ij}$	Customized Assortments: DCI'
5	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	24	40	1.6667	26	40	1.5385	119	4.5769
		2	23	40	1.7391	25	40	1.6000	112	4.4800
		3	22	40	1.8182	22	40	1.8182	110	5.0000
		4	27	40	1.4815	27	40	1.4815	133	4.9259
		5	26	40	1.5385	27	40	1.4815	127	4.7037
	50	1	24	50	2.0833	26	50	1.9231	118	4.5385
		2	22	50	2.2727	29	50	1.7241	109	4.5417
		3	30	50	1.6667	24	50	2.0833	112	4.6667
		4	27	50	1.8519	28	50	1.7857	133	4.7500
		5	26	50	1.9231	26	50	1.9231	122	4.6923
10	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	28	40	1.4286	30	40	1.3333	139	4.6333
		2	28	40	1.4286	29	40	1.3793	139	4.7931
		3	30	40	1.3333	31	40	1.2903	153	4.9355
		4	32	40	1.2500	32	40	1.2500	160	5.0000
		5	30	40	1.3333	31	40	1.2903	153	4.9355
	50	1	29	50	1.7241	31	50	1.6129	139	4.4839
		2	28	50	1.7857	29	50	1.7241	138	4.7586
		3	30	50	1.6667	31	50	1.6129	150	4.8387
		4	32	50	1.5625	33	50	1.5152	160	4.8485
		5	31	50	1.6129	31	50	1.6129	150	4.8387

Table C.4: **DCI analysis of the models with fixed cost, τ=1.**

v_0	κ	Sample	Common Assortment: Number of Unique Products	Common Assortment: $\sum_{i\in M j\in N} O_{ij}$	Common Assortment: DCI	Customized Assortments: Number of Unique Products	Customized Assortments: $\sum_{i\in M j\in N} O_{ij}$	Customized Assortments: DCI	Customized Assortments: $\sum_{i\in M j\in N} m_{ij}$	Customized Assortments: DCI'
5	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	16	20	1.2500	17	20	1.1765	79	4.6471
		2	18	20	1.1111	18	20	1.1111	87	4.8333
		3	18	20	1.1111	19	20	1.0526	90	4.7368
		4	20	20	1	20	20	1	99	4.9500
		5	19	20	1.0526	19	20	1.0526	91	4.7895
	40	1	17	40	2.3529	22	40	1.8182	80	3.6364
		2	15	40	2.6667	17	40	2.3529	67	3.9412
		3	18	40	2.2222	20	40	2.0000	82	4.1000
		4	22	40	1.8182	26	40	1.5385	86	3.3077
		5	18	40	2.2222	21	40	1.9048	75	3.5714
	50	1	19	50	2.6316	24	50	2.0833	80	3.3333
		2	17	50	2.9412	19	50	2.6316	66	3.4737
		3	18	50	2.7778	20	50	2.5000	84	4.2000
		4	23	50	2.1739	26	50	1.9231	83	3.1923
		5	19	50	2.6316	23	50	2.1739	73	3.1739
10	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	25	40	1.6000	26	40	1.5385	119	4.5769
		2	24	40	1.6667	25	40	1.6000	111	4.4400
		3	22	40	1.8182	26	40	1.5385	114	4.3846
		4	27	40	1.4815	28	40	1.4286	134	4.7857
		5	26	40	1.5385	28	40	1.4286	128	4.5714
	50	1	25	50	2.0000	27	50	1.8519	117	4.3333
		2	23	50	2.1739	25	50	2.0000	110	4.4000
		3	23	50	2.1739	26	50	1.9231	115	4.4231
		4	28	50	1.7857	29	50	1.7241	136	4.6897
		5	26	50	1.9231	28	50	1.7857	119	4.2500

Appendix D

Performance Comparison of the Models Under Different Shipping Costs Setting

D.1 Common assortment under different costs

D.1.1 Results of problems with 50 products and 5 customer classes

Table D.1: **Common assortment under different costs- Sample 1 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	36.38%	720948	–	6.50%	64629	89.24	10.85%	204565	–	1.76%	6255	21.52	1.3163	1.6387
				6.62%			–			2.68%			–		
	20	14.23%	106871	–	3.69%	1602882	–	18.17%	129779	–	3.14%	151112	–	1.5716	1.3834
				9.32%			1.88%			8.62%			0.30%		
	40	7.54%	68767	–	3.10%	164001	–	19.48%	49307	–	3.45%	164376	–	1.6693	1.2857
				6.93%			1.88%			7.87%			1.83%		
	50	5.74%	67760	–	2.42%	275951	–	18.14%	100182	–	2.75%	177169	–	1.6977	1.2573
				5.23%			1.44%			–			1.29%		
10	10	46.91%	808094	–	5.45%	13820	19.42	6.40%	159776	–	0.66%	2195	5.78	0.8913	2.0637
				6.31%			–			1.06%			–		
	20	22.94%	124920	–	3.68%	1618932	–	11.40%	136088	–	1.46%	443304	–	1.1758	1.7792
				9.50%			1.07%			4.61%			0.33%		
	40	11.08%	178138	–	2.74%	1401223	–	14.75%	100234	–	1.70%	136555	–	1.3013	1.6537
				10.32%			1.89%			3.91%			0.75%		
	50	8.99%	48950	–	2.23%	1824200	–	13.82%	65011	–	1.40%	184823	–	1.3262	1.6288
				7.76%			1.57%			4.41%			0.56%		

Table D.2: **Common assortment under different costs- Sample 2 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	36.57%	626515	– 8.73%	6.07%	173485	264.35 –	12.75%	83049	– 3.95%	1.79%	8754	22.05 –	1.3340	1.6640
	20	14.88%	103180	– 10.14%	3.62%	1095959	– 1.72%	19.77%	103338	– 10.18%	3.00%	178732	– 0.43%	1.5859	1.4121
	40	8.64%	81390	– 8.18%	3.57%	1188328	– 2.64%	19.81%	60398	– 22.63%	3.74%	114489	– 2.03%	1.6769	1.3211
	50	6.69%	113318	– 6.22%	2.82%	1753697	– 2.09%	18.38%	47311	– 13.44%	3.05%	157626	– 1.42%	1.7076	1.2904
10	10	49.72%	846633	– 8.19%	5.37%	38347	57.71 –	7.36%	157258	– 1.66%	0.72%	12461	11.05 –	0.9188	2.0792
	20	22.08%	121146	– 9.75%	3.76%	1761962	– 1.11%	12.23%	190575	– 4.94%	1.44%	771745	– 0.10%	1.2083	1.7897
	40	10.32%	31914	– 8.39%	2.73%	1666982	– 2.09%	14.31%	73008	– 3.71%	1.95%	285623	– 0.98%	1.3370	1.6610
	50	8.32%	61569	– 7.78%	2.29%	1766684	– 1.75%	13.39%	67298	– 5.02%	1.64%	176917	– 0.80%	1.3617	1.6363

Table D.3: **Common assortment under different costs- Sample 3 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	39.12%	518158	– 9.93%	6.75%	181897	294.46 –	10.39%	132313	– 3.24%	1.88%	14003	47.24 –	1.3589	1.6071
	20	14.26%	90319	– 9.12%	3.06%	1642294	– 1.49%	16.43%	105127	– 6.84%	3.09%	147114	– 0.04%	1.6545	1.3115
	40	8.01%	63963	– 7.52%	3.13%	887700	– 2.28%	18.84%	46750	– 7.63%	3.44%	101155	– 1.76%	1.7502	1.2158
	50	6.29%	22791	– 5.77%	2.56%	1110955	– 1.70%	17.45%	60004	– 8.76%	2.81%	113111	– 1.21%	1.7786	1.1874
10	10	50.45%	302925	– 9.47%	5.55%	18795	29.89 –	5.66%	142520	– 1.28%	0.70%	6858	6.21 –	0.9397	2.0263
	20	21.51%	89046	– 9.19%	3.13%	1382284	– 0.98%	9.67%	155014	– 3.65%	1.42%	549795	– 0.07%	1.2596	1.7064
	40	10.23%	22690	– 7.71%	2.44%	1711059	– 1.78%	14.00%	91292	– 4.47%	1.89%	122536	– 0.64%	1.3885	1.5775
	50	8.24%	22788	– 6.93%	2.02%	1785501	– 1.50%	13.00%	57017	– 3.83%	1.57%	153032	– 0.51%	1.4140	1.5520

Table D.4: **Common assortment under different costs- Sample 4 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	42.97%	319249	–	8.48%	290341	558.10	11.84%	75165	–	1.97%	11453	53.29	1.2927	1.6883
				12.23%			–			4.13%			–		
	20	18.10%	151615	–	8.10%	1571910	–	18.40%	63912	–	5.73%	87720	–	1.5650	1.4160
				10.78%			5.05%			10.50%			1.46%		
	40	8.51%	61261	–	2.98%	1260150	–	18.33%	60239	–	3.24%	234264	–	1.7033	1.2777
				7.77%			2.13%			17.35%			1.31%		
	50	9.33%	82026	–	4.90%	1361811	–	19.66%	78861	–	5.77%	170049	–	1.6904	1.2906
				6.39%			1.78%			9.27%			1.05%		
10	10	52.11%	826750	–	8.10%	118641	142.10	6.96%	168389	–	0.82%	16090	20.92	0.8766	2.1044
				10.02%			–			1.65%			–		
	20	27.37%	89638	–	9.23%	1234827	–	11.39%	125542	–	2.92%	141655	–	1.1779	1.8031
				11.71%			4.74%			4.44%			0.62%		
	40	10.74%	54933	–	2.32%	1413549	–	13.60%	106194	–	1.70%	170768	–	1.3549	1.6261
				8.43%			1.74%			2.92%			0.52%		
	50	8.63%	40206	–	1.87%	1452509	–	12.60%	67668	–	1.38%	218137	–	1.3811	1.5999
				7.13%			1.44%			3.50%			0.41%		

Table D.5: **Common assortment under different costs- Sample 5 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	39.85%	460895	– 9.83%	7.50%	214893	324.65 –	11.49%	78207	– 3.92%	1.91%	9042	38.57 –	1.3231	1.6549
	20	15.82%	123565	– 9.81%	3.94%	1697521	– 1.85%	18.41%	96681	– 11.58%	3.18%	156229	– 0.18%	1.5976	1.3804
	40	8.53%	96915	– 7.88%	3.20%	1431149	– 2.51%	19.41%	52200	– 7.74%	3.67%	130480	– 1.68%	1.7049	1.2731
	50	6.56%	23201	– 6.09%	2.49%	1824569	– 2.00%	17.90%	43921	– 7.51%	2.92%	236221	– 1.17%	1.7364	1.2416
10	10	50.47%	751653	– 8.00%	6.70%	19154	30.65 –	6.49%	166149	– 1.41%	0.67%	3700	4.91 –	0.9031	2.0749
	20	23.45%	106254	– 10.44%	3.79%	1623744	– 0.98%	11.09%	136080	– 4.59%	1.50%	140534	409.83 –	1.2084	1.7696
	40	10.19%	322668	– 10.05%	2.45%	1416419	– 1.87%	14.35%	81617	– 4.09%	1.82%	164382	– 0.56%	1.3539	1.6241
	50	8.30%	50786	– 8.03%	2.15%	1506939	– 1.64%	13.50%	81283	– 3.68%	1.62%	219714	– 0.57%	1.3775	1.6005

D.1.2 Results of problems with 100 products and 10 customer classes

Table D.6: **Common assortment under different costs- Sample 1 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	61.41%	7128	–	13.80%	10418	–	14.41%	4247	–	3.18%	7097	–	1.3388	1.6542
				45.44%			11.18%			26.42%			2.38%		
	20	28.86%	87522	–	8.52%	31860	–	23.28%	1585	–	6.49%	8613	–	1.6770	1.3160
				28.20%			7.08%			40.30%			4.73%		
	40	16.42%	110071	–	7.36%	25829	–	34.38%	2584	–	10.14%	1920	–	1.8561	1.1369
				16.46%			6.11%			46.61%			8.25%		
	50	14.75%	112404	–	7.60%	26342	–	34.97%	2938	–	11.21%	1911	–	1.8832	1.1098
				14.55%			6.29%			34.77%			9.29%		
	100	9.42%	173958	–	5.77%	23902	–	29.99%	1500	–	9.49%	4846	–	1.9749	1.0181
				10.56%			5.00%			–			8.23%		
10	10	67.44%	9653	–	13.64%	14548	–	8.17%	6029	–	1.27%	23039	–	0.9147	2.0783
				40.88%			9.06%			9.83%			0.93%		
	20	46.29%	18447	–	9.79%	25854	–	14.25%	4572	–	3.08%	14021	–	1.2721	1.7209
				39.05%			8.19%			17.04%			2.34%		
	40	20.74%	21614	–	5.91%	50688	–	23.46%	3055	–	4.79%	4230	–	1.5413	1.4517
				19.33%			5.01%			30.53%			3.82%		
	50	18.00%	21423	–	6.20%	34870	–	26.56%	2939	–	5.60%	1673	–	1.5770	1.4160
				17.88%			5.33%			29.69%			4.31%		
	100	11.93%	21008	–	5.46%	40394	–	23.81%	2157	–	6.08%	6641	–	1.6625	1.3305
				13.31%			4.58%			29.44%			4.94%		

Table D.7: **Common assortment under different costs- Sample 2 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	61.07%	10054	– 46.11%	13.34%	11952	– 10.36%	14.53%	2031	– 34.96%	3.17%	7584	– 2.21%	1.2952	1.6758
	20	29.40%	72017	– 28.83%	9.00%	21087	– 7.68%	24.54%	3723	– 34.73%	5.98%	8993	– 4.35%	1.6121	1.3589
	40	17.62%	98537	– 17.52%	8.10%	25146	– 6.84%	34.92%	2234	– 39.65%	9.47%	1349	– 7.58%	1.7736	1.1974
	50	15.42%	101299	– 15.50%	7.89%	20977	– 6.84%	34.73%	2918	– 37.16%	10.15%	1887	– 8.18%	1.8075	1.1635
	100	10.02%	125619	– 10.78%	6.09%	22904	– 5.05%	30.16%	2749	– –	9.27%	3541	– 7.79%	1.8961	1.0749
10	10	65.44%	9119	– 40.58%	12.69%	19759	– 7.54%	8.32%	2741	– 19.13%	1.17%	14181	– 0.78%	0.8823	2.0887
	20	47.05%	13216	– 35.73%	10.01%	21185	– 8.14%	14.81%	6725	– 24.38%	2.86%	12403	– 1.90%	1.2242	1.7468
	40	22.33%	16410	– 21.29%	6.45%	43677	– 5.36%	24.04%	2325	– 25.41%	5.03%	2408	– 3.67%	1.4717	1.4993
	50	19.10%	22168	– 18.37%	6.54%	27507	– 5.50%	26.39%	2619	– 23.26%	5.74%	1586	– 4.28%	1.5115	1.4595
	100	12.36%	22275	– 14.94%	5.48%	28880	– 4.83%	23.76%	1357	– –	5.59%	5938	– 4.54%	1.6022	1.3688

Table D.8: **Common assortment under different costs- Sample 3 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	60.72%	7375	–	12.97%	7504	–	14.64%	1807	–	3.20%	7132	–	1.2980	1.6780
				48.12%			10.26%			30.17%			2.24%		
	20	29.13%	64200	–	8.67%	20772	–	24.62%	2966	–	5.95%	7627	–	1.6154	1.3606
				29.33%			7.67%			35.48%			4.27%		
	40	17.32%	110932	–	7.75%	21139	–	34.93%	2495	–	9.36%	1478	–	1.7781	1.1979
				17.66%			7.01%			40.77%			7.43%		
	50	15.26%	97118	–	7.68%	21757	–	34.85%	2979	–	10.21%	1984	–	1.8099	1.1661
				15.85%			6.90%			35.12%			8.25%		
	100	9.89%	135804	–	5.92%	21245	–	30.32%	2918	–	9.38%	4679	–	1.8984	1.0776
				11.00%			5.21%			–			7.95%		
10	10	64.68%	15730	–	10.93%	20682	–	8.24%	3103	–	1.17%	43478	–	0.9132	2.0628
				37.03%			6.32%			10.89%			0.62%		
	20	46.59%	13994	–	9.54%	21395	–	14.85%	4255	–	2.84%	14032	–	1.2281	1.7479
				37.58%			8.32%			24.69%			1.93%		
	40	22.20%	21903	–	6.22%	43823	–	24.21%	2364	–	5.13%	2909	–	1.4732	1.5028
				20.91%			5.43%			35.33%			3.75%		
	50	18.92%	21348	–	6.27%	24696	–	26.51%	2985	–	5.79%	1772	–	1.5138	1.4622
				18.49%			5.48%			24.35%			4.32%		
	100	12.06%	21925	–	5.13%	26730	–	23.78%	1733	–	5.55%	5857	–	1.6065	1.3695
				14.78%			4.94%			–			4.46%		

Table D.9: **Common assortment under different costs- Sample 4 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	63.57%	3537	–	13.90%	6574	–	14.74%	2213	–	3.02%	7446	–	1.2848	1.7052
				59.48%			10.97%			22.77%			2.25%		
	20	29.34%	62169	–	8.58%	25478	–	23.09%	2966	–	6.24%	8213	–	1.6249	1.3651
				28.69%			6.95%			44.91%			4.40%		
	40	17.12%	84003	–	7.75%	22109	–	34.06%	2696	–	9.74%	1674	–	1.7944	1.1956
				17.17%			6.39%			42.26%			7.70%		
	50	15.25%	70865	–	7.82%	20959	–	34.75%	2925	–	10.50%	1923	–	1.8235	1.1665
				15.64%			6.36%			32.87%			8.59%		
	100	9.45%	120790	–	5.56%	21015	–	29.75%	1755	–	8.36%	3971	–	1.9202	1.0698
				10.66%			4.86%			–			7.08%		
10	10	69.15%	13166	–	12.99%	14470	–	8.33%	3073	–	1.16%	22503	–	0.8770	2.1130
				42.63%			9.03%			8.44%			0.78%		
	20	47.64%	13457	–	9.45%	28615	–	14.02%	5956	–	2.94%	15366	–	1.2303	1.7597
				35.44%			7.57%			20.35%			1.96%		
	40	22.61%	22517	–	6.38%	38281	–	23.32%	2356	–	4.76%	3726	–	1.4814	1.5086
				19.89%			5.44%			28.85%			3.68%		
	50	19.90%	22990	–	6.83%	27111	–	26.35%	2636	–	5.55%	1980	–	1.5149	1.4751
				19.19%			5.71%			22.92%			4.40%		
	100	12.81%	22569	–	5.37%	26608	–	23.61%	4079	–	5.16%	3398	–	1.6102	1.3798
				15.63%			4.82%			–			4.06%		

Table D.10: **Common assortment under different costs- Sample 5 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	61.34%	9965	– 44.32%	13.41%	10547	– 10.29%	14.92%	1446	– 28.99%	3.03%	8622	– 1.95%	1.2344	1.6806
	20	30.67%	66693	– 30.08%	9.81%	20958	– 8.16%	25.24%	1540	– 34.79%	6.24%	6723	– 4.44%	1.5241	1.3909
	40	17.72%	105235	– 17.78%	8.10%	22133	– 7.05%	34.69%	2221	– 38.75%	9.22%	1757	– 7.32%	1.6917	1.2233
	50	15.58%	83042	– 15.82%	7.96%	21268	– 6.83%	8.51%	2734	– 25.96%	9.89%	1959	– 7.92%	1.7230	1.1920
	100	9.68%	147595	– 10.66%	5.76%	21598	– 4.93%	29.92%	3032	– –	8.06%	4883	– 6.81%	1.8158	1.0992
10	10	62.59%	12698	– 33.90%	12.02%	20830	– 6.38%	8.29%	3426	– 16.38%	1.21%	75093	– 0.65%	0.8406	2.0744
	20	49.58%	6158	– 47.02%	10.72%	20881	– 8.90%	15.40%	4804	– 25.90%	2.98%	12069	– 1.97%	1.1503	1.7647
	40	23.66%	4951	– 22.86%	7.07%	31753	– 5.86%	24.15%	2331	– 31.79%	5.07%	4381	– 3.86%	1.3913	1.5237
	50	20.71%	24388	– 19.66%	7.45%	24557	– 6.01%	26.79%	2735	– 25.96%	5.76%	1590	– 4.46%	1.4253	1.4897
	100	13.07%	21273	–	5.70%	29225	– 5.17%	23.78%	3356	– –	5.13%	3532	– 3.99%	1.5217	1.3933

D.2 Customized assortments under different costs

D.2.1 Results of problems with 50 products and 5 customer classes

Table D.11: **Customized assortments under different costs- Sample 1 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC				
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	36.38%	21261	– 11.30%	6.52%	110397	327.94 –	10.85%	48605	– 4.35%	1.76%	5288	17.38 –	1.3163	1.6387
	20	14.24%	42629	– 10.25%	3.79%	285039	– 2.69%	18.18%	48140	– 13.36%	3.14%	73951	– 0.37%	1.5714	1.3836
	40	7.40%	870137	– 7.22%	3.05%	163015	– 2.24%	19.37%	16344	– 9.49%	3.52%	86700	– 2.08%	1.6715	1.2835
	50	5.40%	731591	– 5.38%	2.20%	242718	– 1.69%	17.81%	11119	– 10.19%	2.56%	122614	– 1.34%	1.7032	1.2518
10	10	46.91%	129834	– 11.13%	5.49%	15590	40.58 –	6.40%	70089	– 1.52%	0.66%	2120	6.19 –	0.8913	2.0637
	20	22.94%	12406	– 5.84%	3.70%	687686	– 1.33%	11.40%	54877	– 10.59%	1.47%	318494	– 0.16%	1.1758	1.7792
	40	10.95%	22081	– 8.73%	2.75%	494777	– 2.36%	14.70%	28757	– 14.40%	1.69%	54997	– 0.85%	1.3028	1.6522
	50	8.89%	21522	– 8.24%	2.31%	460520	– 1.91%	13.78%	15829	– 6.89%	1.42%	67302	– 0.64%	1.3274	1.6276

Table D.12: **Customized assortments under different costs- Sample 2 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	36.57%	25287	– 11.68%	6.11%	319234	855.54 –	12.75%	69885	– 6.87%	1.79%	20546	47.42 –	1.3240	1.6640
	20	14.89%	47859	– 10.75%	3.68%	285668	– 2.66%	19.78%	49722	– 21.99%	3.01%	73973	– 0.54%	1.5757	1.4123
	40	8.60%	533342	– 8.52%	3.59%	436412	– 2.95%	19.79%	32145	– 18.46%	3.80%	87414	– 2.46%	1.6676	1.3204
	50	6.58%	882330	– 6.44%	2.80%	439198	– 2.25%	18.30%	29570	– 18.58%	3.04%	94837	– 1.92%	1.6994	1.2886
10	10	49.72%	76634	– 13.15%	5.39%	50941	101.91 –	7.36%	57878	– 2.70%	0.72%	9210	16.06 –	0.9088	2.0792
	20	22.08%	20740	– 7.71%	3.79%	753764	– 1.34%	12.23%	57270	– 11.11%	1.44%	252459	– 0.26%	1.1983	1.7897
	40	10.23%	362826	– 10.16%	2.69%	490321	– 2.28%	14.26%	33510	– 15.84%	1.92%	85036	– 0.94%	1.3282	1.6598
	50	8.21%	346914	– 8.24%	2.25%	455110	– 1.86%	13.32%	23812	– 11.20%	1.63%	123655	– 0.81%	1.3531	1.6349

Table D.13: **Customized assortments under different costs- Sample 3 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	39.12%	26106	– 12.79%	6.80%	401938	947.87 –	10.40%	71568	– 4.41%	1.88%	22931	70.93 –	1.3589	1.6071
	20	14.26%	41950	– 9.94%	3.13%	322613	– 2.23%	16.43%	52692	– 14.12%	3.09%	141124	– 0.10%	1.6545	1.3115
	40	8.09%	556201	– 7.94%	3.25%	375215	– 2.54%	18.94%	19484	– 10.21%	3.65%	53551	– 2.17%	1.7490	1.2170
	50	6.17%	361232	– 6.22%	2.51%	381432	– 2.00%	17.33%	29622	– 26.90%	2.79%	86508	– 1.50%	1.7807	1.1853
10	10	50.45%	89692	– 12.96%	5.60%	28081	59.92 –	5.66%	83817	– 1.49%	0.70%	6307	7.10 –	0.9397	2.0263
	20	21.51%	15264	– 5.19%	3.15%	804319	– 1.01%	9.67%	43216	– 5.75%	1.43%	51748	121.99 –	1.2596	1.7064
	40	10.09%	371159	– 10.08%	2.35%	428963	– 1.93%	13.91%	37711	– 14.75%	1.82%	69343	– 0.63%	1.3902	1.5758
	50	8.15%	354809	– 8.24%	1.99%	651698	– 1.67%	12.94%	35897	– 11.74%	1.54%	90168	– 0.60%	1.4151	1.5509

Table D.14: **Customized assortments under different costs- Sample 4 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	42.97%	17887	–	8.51%	369237	–	11.84%	70987	–	1.97%	12885	63.43	1.2927	1.6883
				12.32%			0.85%			6.06%			–		
	20	18.33%	42107	–	8.40%	324239	–	18.59%	34397	–	5.94%	70832	–	1.5619	1.4191
				11.09%			6.64%			13.47%			1.71%		
	40	8.52%	604720	–	3.08%	442851	–	18.36%	19868	–	3.34%	76300	–	1.7031	1.2779
				8.41%			2.50%			8.11%			1.71%		
	50	6.77%	636493	–	2.55%	543469	–	17.06%	19019	–	2.83%	102050	–	1.7311	1.2499
				6.61%			2.05%			9.83%			1.40%		
10	10	52.11%	83396	–	8.14%	144916	312.01	6.96%	65424	–	0.82%	32300	78.83	0.8766	2.1044
				12.88%			–			2.61%			–		
	20	27.37%	24264	–	9.36%	515904	–	11.39%	50292	–	2.92%	103384	–	1.1779	1.8031
				9.64%			6.43%			10.76%			0.72%		
	40	10.70%	22305	–	2.37%	777789	–	13.58%	46280	–	1.69%	84809	–	1.3553	1.6257
				8.64%			2.00%			13.86%			0.61%		
	50	8.61%	180005	–	1.92%	696528	–	12.59%	31853	–	1.41%	88763	–	1.3814	1.5996
				8.62%			1.64%			10.37%			0.50%		

Table D.15: **Customized assortments under different costs- Sample 5 results.**

v_0	κ	MILP			MILP+MC			CONIC			CONIC+MC			Max Obj	Min Obj
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	10	39.85%	19473	–	7.56%	365662	1220.20	11.49%	59244	–	1.91%	10044	50.09	1.3231	1.6549
				10.87%			–			5.81%			–		
	20	15.82%	41662	–	3.98%	217015	–	18.41%	27842	–	3.18%	104741	–	1.5976	1.3804
				10.53%			2.64%			10.64%			0.36%		
	40	8.40%	547912	–	3.23%	367932	–	19.30%	27795	–	3.55%	61440	–	1.7070	1.2710
				8.43%			2.76%			19.92%			1.90%		
	50	6.50%	672716	–	2.60%	423816	–	17.85%	23586	–	2.90%	76859	–	1.7375	1.2405
				6.49%			2.17%			22.18%			1.53%		
10	10	50.47%	90856	–	6.76%	28127	78.41	6.49%	104343	–	0.67%	6949	11.32	0.9031	2.0749
				12.73%			–			1.96%			–		
	20	23.45%	16657	–	3.83%	811591	–	11.09%	59652	–	1.50%	102597	597.67	1.2084	1.7696
				6.40%			1.22%			12.12%			–		
	40	10.13%	327059	–	2.46%	506147	–	14.33%	28901	–	1.80%	76992	–	1.3545	1.6235
				10.13%			2.10%			12.98%			0.59%		
	50	8.22%	325654	–	2.15%	624863	–	13.46%	26689	–	1.59%	89828	–	1.3785	1.5995
				8.23%			1.85%			12.17%			0.62%		

D.2.2 Results of problems with 100 products and 10 customer classes

Table D.16: **Customized assortments under different costs- Sample 1 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	61.41%	3758	– 56.50%	13.86%	7343	– 13.26%	14.41%	2453	– 29.24%	3.17%	11504	Gap 2.30%	1.3388	1.6542
	20	28.81%	2992	– 31.93%	8.59%	7819	– 8.30%	23.25%	1682	– 39.78%	6.44%	4553	4.51%	1.6777	1.3153
	40	16.50%	42068	– 16.79%	7.53%	9317	– 7.02%	34.46%	1534	– 49.50%	10.30%	1611	8.38%	1.8549	1.1381
	50	14.60%	64548	– 15.25%	7.56%	10080	– 6.95%	34.83%	367	– 39.63%	11.05%	2523	9.20%	1.8857	1.1073
	100	9.64%	90349	– 11.62%	6.06%	18183	– 5.14%	30.27%	28	– 35.99%	10.06%	2853	9.02%	1.9710	1.0220
10	10	67.44%	20967	– 47.30%	13.66%	13576	– 10.62%	8.17%	3128	– 15.93%	1.27%	52540	0.82%	0.9147	2.0783
	20	46.31%	3894	– 34.96%	9.85%	14933	– 9.55%	14.27%	1826	– 25.62%	3.09%	9894	2.31%	1.2719	1.7211
	40	20.97%	20308	– 21.44%	6.19%	14106	– 5.88%	23.61%	1726	– 23.67%	5.04%	1204	4.06%	1.5383	1.4547
	50	17.90%	27917	– 18.54%	6.17%	9806	– 6.07%	26.49%	1006	– 30.58%	5.57%	2018	4.27%	1.5784	1.4146
	100	11.70%	61592	– 13.71%	5.31%	12101	– 5.13%	23.63%	9	– 28.83%	5.90%	3372	4.85%	1.6660	1.3270

Table D.17: **Customized assortments under different costs- Sample 2 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	61.07%	3358	– 51.86%	13.44%	6540	– 12.72%	14.52%	1773	– 30.50%	3.17%	11935	Gap 2.02%	1.2952	1.6758
	20	29.48%	2992	– 31.90%	9.24%	5490	– 9.22%	24.60%	2135	– 40.01%	6.04%	2981	4.26%	1.6111	1.3599
	40	17.68%	28441	– 17.73%	8.26%	8858	– 8.03%	34.98%	3130	– 43.09%	9.65%	1370	7.74%	1.7726	1.1984
	50	15.34%	61893	– 16.18%	7.91%	8831	– 7.79%	34.68%	216	– 39.77%	10.15%	2581	8.47%	1.8086	1.1624
	100	10.21%	87648	– 11.85%	6.36%	14265	– 5.58%	30.40%	16	– 41.96%	9.63%	3193	8.63%	1.8928	1.0782
10	10	65.44%	19185	– 42.83%	12.72%	19438	– 11.93%	8.33%	1493	– 14.53%	1.16%	23908	0.71%	0.8823	2.0887
	20	46.89%	3819	– 36.61%	9.95%	9179	– 9.64%	14.73%	1817	– 24.00%	2.79%	8682	1.83%	1.2256	1.7454
	40	22.12%	5057	– 21.32%	6.35%	11761	– 6.12%	23.92%	2034	– 35.76%	4.87%	1155	3.50%	1.4741	1.4969
	50	19.11%	7959	– 18.15%	6.62%	7280	– 6.15%	26.39%	2183	– 29.29%	5.76%	1889	4.32%	1.5115	1.4595
	100	12.25%	56093	– 14.98%	5.45%	9917	– 5.25%	23.68%	15	– 27.54%	5.53%	4218	4.58%	1.6037	1.3673

Table D.18: **Customized assortments under different costs- Sample 3 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	60.72%	3081	– 60.39%	13.07%	6540	– 12.83%	14.64%	1661	– 30.97%	3.20%	11078	– 2.05%	1.2980	1.6780
	20	29.28%	3681	– 31.59%	8.97%	5490	– 9.51%	24.72%	2155	– 45.30%	6.08%	5526	– 4.36%	1.6136	1.3624
	40	17.53%	37038	– 17.96%	8.05%	8858	– 7.89%	35.10%	2203	– 42.23%	9.71%	1617	– 7.74%	1.7750	1.2010
	50	15.13%	51002	– 16.25%	7.65%	8831	– 8.06%	34.75%	46	– 40.03%	10.14%	2215	– 8.41%	1.8120	1.1640
	100	10.05%	76896	– 12.24%	6.17%	14265	– 5.70%	3.41%	8	– 28.54%	9.72%	2982	– 8.64%	1.8955	1.0805
10	10	64.68%	17685	– 41.28%	10.95%	19438	– 9.70%	8.24%	2881	– 18.08%	1.16%	38298	– 0.69%	0.9132	2.0628
	20	46.56%	2586	– 37.13%	9.58%	9179	– 9.71%	14.84%	1874	– 27.46%	2.81%	8082	– 1.78%	1.2283	1.7477
	40	22.00%	5934	– 22.22%	6.12%	11761	– 6.20%	24.08%	2045	– 34.91%	4.94%	1212	– 3.65%	1.4756	1.5004
	50	18.88%	7838	– 18.35%	6.31%	7280	– 6.29%	26.48%	3437	– 27.15%	5.78%	1732	– 4.33%	1.5144	1.4616
	100	12.06%	62672	– 14.69%	5.20%	9917	– 5.39%	23.79%	8	– 28.54%	5.59%	3980	– 4.62%	1.6065	1.3695

Table D.19: **Customized assortments under different costs- Sample 4 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	60.19%	2930	– 68.35%	11.58%	8169	– 13.57%	13.36%	1742	– 16.40%	1.45%	11211	– 2.05%	1.3120	1.6780
	20	29.12%	3823	– 35.39%	8.46%	6624	– 8.46%	22.94%	2189	– 36.72%	6.06%	3193	– 4.36%	1.6276	1.3624
	40	17.48%	38339	– 17.40%	8.15%	7199	– 7.19%	34.35%	3567	– 39.47%	10.16%	1719	– 7.74%	1.7890	1.2010
	50	15.10%	51744	– 16.39%	7.73%	8098	– 7.03%	34.61%	973	– 38.22%	10.35%	2201	– 8.41%	1.8260	1.1640
	100	10.06%	76548	– 12.65%	6.23%	14669	– 5.10%	30.46%	15	– 36.94%	9.43%	2729	– 8.64%	1.9095	1.0805
10	10	59.98%	17718	– 46.78%	6.89%	20875	– 10.54%	6.08%	2619	– 14.54%	1.24%	38934	– 0.69%	0.9272	2.0628
	20	46.21%	3958	– 35.85%	8.42%	13957	– 9.20%	13.42%	2190	– 27.37%	2.27%	8897	– 1.78%	1.2423	1.7477
	40	21.94%	4608	– 20.59%	5.87%	8105	– 6.33%	22.90%	1771	– 30.73%	4.25%	1200	– 3.65%	1.4896	1.5004
	50	18.85%	4840	– 19.45%	5.96%	5188	– 6.67%	25.67%	2852	– 34.28%	4.73%	1941	– 4.33%	1.5284	1.4616
	100	12.09%	50764	– 16.08%	4.76%	9904	– 5.54%	23.05%	14	– 30.08%	4.49%	3211	– 4.62%	1.6205	1.3695

Table D.20: **Customized assortments under different costs- Sample 5 results.**

		MILP			MILP+MC			CONIC			CONIC+MC				
v_0	κ	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Max Obj	Min Obj
5	10	61.34%	2573	– 57.47%	13.54%	6190	– 12.33%	14.92%	3909	– 25.25%	3.03%	8455	– 1.87%	1.2344	1.6806
	20	30.48%	3622	– 36.16%	9.82%	6236	– 9.61%	25.12%	1800	– 39.75%	6.09%	2633	– 4.24%	1.5263	1.3887
	40	17.85%	25277	– 18.00%	8.33%	7629	– 7.89%	34.78%	2823	– 38.87%	9.42%	1492	– 7.41%	1.6899	1.2251
	50	15.44%	50678	– 16.62%	7.95%	7188	– 7.68%	34.56%	45	– 38.34%	9.79%	2366	– 8.04%	1.7251	1.1899
	100	9.96%	87489	– 11.96%	6.15%	14713	– 5.31%	30.25%	25	– 46.63%	8.56%	2976	– 7.53%	1.8112	1.1038
10	10	62.59%	17117	– 39.67%	12.05%	13769	– 8.21%	8.30%	2773	– 16.86%	1.20%	34676	– 0.69%	0.8406	2.0744
	20	49.57%	4239	– 36.88%	10.78%	7165	– 10.53%	15.40%	1981	– 25.02%	2.97%	7731	– 1.95%	1.1503	1.7647
	40	23.66%	4397	– 21.62%	7.19%	7165	– 6.93%	24.15%	2467	– 31.63%	5.09%	1140	– 3.83%	1.3913	1.5237
	50	20.34%	5660	– 19.66%	7.20%	5071	– 7.04%	26.57%	4051	– 27.86%	5.52%	1773	– 4.28%	1.4297	1.4853
	100	13.00%	59857	– 15.84%	5.71%	7120	– 5.73%	23.75%	15	– 28.05%	5.13%	3771	– 4.19%	1.5226	1.3924

Appendix E

DCI Analysis of Different Costs Setting

Table E.1: **DCI analysis of the models with different costs.**

v_0	κ	Sample	Common Assortment: Number of Unique Products	Common Assortment: $\sum_{i\in M j\in N} O_{ij}$	Common Assortment: DCI	Customized Assortments: Number of Unique Products	Customized Assortments: $\sum_{i\in M j\in N} O_{ij}$	Customized Assortments: DCI	Customized Assortments: $\sum_{i\in M j\in N} m_{ij}$	Customized Assortments: DCI'
5	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	24	40	1.6667	26	40	1.5385	122	4.6923
		2	25	40	1.6000	27	40	1.4815	125	4.6296
		3	23	40	1.7391	26	40	1.5385	119	4.5769
		4	27	40	1.4815	29	40	1.3793	139	4.7931
		5	27	40	1.4815	28	40	1.4286	136	4.8571
	50	1	25	50	2.0000	26	50	1.9231	122	4.6923
		2	26	50	1.9231	27	50	1.8519	124	4.5926
		3	25	50	2.0000	27	50	1.8519	121	4.4815
		4	28	50	1.7857	29	50	1.7241	140	4.8276
		5	27	50	1.8519	28	50	1.7857	135	4.8214
10	10	1	10	10	1	10	10	1	50	5
		2	10	10	1	10	10	1	50	5
		3	10	10	1	10	10	1	50	5
		4	10	10	1	10	10	1	50	5
		5	10	10	1	10	10	1	50	5
	20	1	20	20	1	20	20	1	100	5
		2	20	20	1	20	20	1	100	5
		3	20	20	1	20	20	1	100	5
		4	20	20	1	20	20	1	100	5
		5	20	20	1	20	20	1	100	5
	40	1	30	40	1.3333	32	40	1.2500	152	4.7500
		2	28	40	1.4286	29	40	1.3793	141	4.8621
		3	31	40	1.2903	31	40	1.2903	153	4.9355
		4	33	40	1.2121	33	40	1.2121	163	4.9394
		5	31	40	1.2903	31	40	1.2903	154	4.9677
	50	1	30	50	1.6667	32	50	1.5625	151	4.7188
		2	28	50	1.7857	31	50	1.6129	143	4.6129
		3	31	50	1.6129	32	50	1.5625	156	4.8750
		4	33	50	1.5152	33	50	1.5152	162	4.9091
		5	31	50	1.6129	32	50	1.5625	156	4.8750