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THE FRACTIONAL FOURIER TRANSFORM

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Abstract

A brief introduction to the fractional Fourier transform and its properties is given. Its relation to phase-space representations (time- or space-frequency representations) and the concept of fractional Fourier domains are discussed. An overview of applications which have so far received interest are given and some potential application areas remaining to be explored are noted.

1 Introduction

The purpose of this paper is to provide a brief introduction to the fractional Fourier transform (FRT). Since the ordinary Fourier transform and related techniques are of importance in the control field, it is natural to expect the fractional Fourier transform to find many applications as well. This expectation is further supported by the fact that the fractional Fourier transform has already found many applications in the areas of signal processing and communications. This paper will provide general motivation and mention some of the more important properties of the transform. Those interested in learning more are referred to a recent book on the subject [1] or the chapter-length treatment [2].

The fractional Fourier transform is a generalization of the ordinary Fourier transform with an order (or power) parameter a. The ath order fractional Fourier transform operator is the ath power of the ordinary Fourier transform operator. (Readers not familiar with functions of operators may think of them in analogy with functions of matrices. In the discrete case, where the discrete ordinary and fractional Fourier transform operators are represented by matrices, this is actually the case.) If we denote the ordinary Fourier transform operator by $\mathcal{F}$, then the ath order fractional Fourier transform operator is denoted by $\mathcal{F}^a$. The zeroth-order fractional Fourier transform operator $\mathcal{F}^0$ is equal to the identity operator $\mathcal{I}$. The first-order fractional Fourier transform operator $\mathcal{F}^1$ is equal to the ordinary Fourier transform operator. Integer values of a correspond to repeated application of the Fourier transform; for instance, $\mathcal{F}^2$ corresponds to the Fourier transform of the Fourier transform. $\mathcal{F}^{-1}$ corresponds to the inverse Fourier transform operator. The a'th order transform of the ath order transform is equal to the $(a'+a)$th order transform; that is $\mathcal{F}^{a'}\mathcal{F}^a = \mathcal{F}^{a'+a}$, a property referred to as index additivity. For instance, the 0.5th fractional Fourier transform operator $\mathcal{F}^{0.5}$, when applied twice, amounts to ordinary Fourier transformation. Or, the 0.4th transform of the 0.3rd transform is the 0.7th transform. The order a may assume any real value, however the operator $\mathcal{F}^a$ is periodic in a with period 4; that is $\mathcal{F}^{a+4j} = \mathcal{F}^a$ where j is any integer. This is because $\mathcal{F}^2$ equals the parity operator $\mathcal{P}$ which maps $f(u)$ to $f(-u)$ and $\mathcal{F}^4$ equals the identity operator. Therefore, the range of a is usually restricted to $(-2,2]$ or $[0,4)$. Complex-ordered transforms have also been discussed by some authors, although there remains much to do in this area both in terms of theory and applications.

The same facts can also be thought of in terms of the functions which these operators act on. For instance, the 0th order fractional Fourier transform of the function $f(u)$ is merely the function itself, and the 1st order transform is its ordinary Fourier transform $F(\mu)$, where μ denotes the frequency domain variable. The ath fractional Fourier transform of $f(u)$ is denoted by $f_a(u)$ so that $f_0(u) = f(u)$ and $f_1(\mu) = F(\mu)$ (or $f_1(u) = F(u)$ since the functional equality does not depend on the dummy variable employed).

An example is given in figure 1, where we see the magnitude of the fractional Fourier transforms of the rectangle function for different values of the order $a \in [0,1]$. We observe that as a varies from 0 to 1, the rectangle function evolves into a sinc function, which is the ordinary Fourier transform of the rectangle function. Such two-dimensional functions $f_a(u)$ with variables a and u are known as rectangular time-order or space-order representations of the function $f(u)$, depending on whether the variable u is interpreted as time or space (or something else) [1].

The earliest known references dealing with the transform go back to 1920s and 1930s; since then the transform has been reinvented several times. It has received the attention of a few mathematicians during the eighties [3, 4, 5]. However, interest in the transform really grew with its reinvention/reintroduction by researchers in the fields of optics and signal processing, who noticed its relevance for a variety of application areas [6, 7, 8, 9]. A detailed account of the history of the transform may be found in [1].

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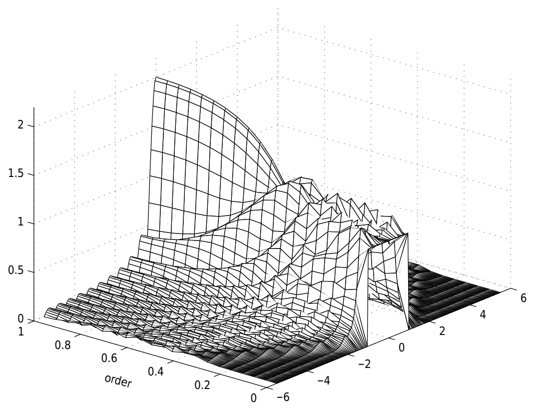

Figure 1:

2 Definition

The most straightforward way of defining the fractional Fourier transform is as an integral transform as follows:

$$f_a(u) = \int_{-\infty}^{\infty} K_a(u,u') f(u')\, du', \tag{1}$$

$$K_a(u,u') = A_\alpha \exp\left[i\pi(\cot\alpha\, u^2 - 2\csc\alpha\, uu' + \cot\alpha\, u'^2)\right],$$

$$A_\alpha = \sqrt{1 - i\cot\alpha} \quad \alpha = \frac{a\pi}{2}$$

when $a \neq 2j$. When $a = 4j$ the transform is defined as $K_a(u,u') = \delta(u - u')$ and when $a = 4j+2$ the transform is defined as $K_a(u,u') = \delta(u+u')$. It can be shown that the above kernel for $a \neq 2j$ indeed approaches these delta function kernels as a approaches even integers.

It is not easy to see from the above definition that the transform is indeed the operator power of the ordinary Fourier transform. In order to find the operator power of the ordinary Fourier transform, we first consider its eigenvalue equation:

$$\mathcal{F}\psi_n(u) = e^{-in\pi/2}\psi_n(u). \tag{2}$$

Here $\psi_n(u)$, $n = 0,1,2\ldots$ are the Hermite-Gaussian functions defined as $\psi_n(u) = (2^{1/4}/\sqrt{2^n n!})$ $H_n(\sqrt{2\pi}\,u)\,\exp(-\pi u^2)$, where $H_n(u)$ are the standard Hermite polynomials. $\exp(-in\pi/2)$ is the eigenvalue associated with the nth eigenfunction $\psi_n(u)$. Now, following a standard procedure also used to define functions of matrices, the fractional Fourier transform may be defined such that it has the same eigenfunctions but the eigenvalues raised to the ath power:

$$\mathcal{F}^a\psi_n(u) = (e^{-in\pi/2})^a\psi_n(u). \tag{3}$$

This definition is not unique for at least two reasons. First, it depends on the choice of the Hermite-Gaussian set as the set of eigenfunctions (which is not the only such possible set). Second, it depends on how we resolve the ambiguity in evaluating $[\exp(-in\pi/2)]^a$. The particular definition which has so far received the greatest attention, has the most elegant properties, and which has found the most applications follows from choosing $[\exp(-in\pi/2)]^a = \exp(-ian\pi/2)$. With this choice, the fractional Fourier transform of a square-integrable function $f(u)$ can be found by first expanding it in terms of the Hermite-Gaussian functions as

$$f(u) = \sum_{n=0}^{\infty} C_n\psi_n(u), \tag{4}$$

$$C_n = \int \psi_n(u) f(u)\, du, \tag{5}$$

and then applying $\mathcal{F}^a$ to both sides to obtain

$$\mathcal{F}^a f(u) = \sum_{n=0}^{\infty} C_n\mathcal{F}^a\psi_n(u) \tag{6}$$

$$f_a(u) = \sum_{n=0}^{\infty} C_n e^{-ian\pi/2}\psi_n(u), \tag{7}$$

$$f_a(u) = \int \left[\sum_{n=0}^{\infty} e^{-ian\pi/2}\psi_n(u)\psi_n(u')\right] f(u')\, du' \tag{8}$$

The final form can be shown to be equal to that given by equation 1 through a standard identity.

3 Fractional Fourier domains

One of the most important concepts in Fourier analysis is the concept of the Fourier (or frequency) domain. This "domain" is understood to be a space where the Fourier transform representation of the signal lives, with its own interpretation and qualities. This naturally leads one to inquire into the nature of the domain where the fractional Fourier transform representation of a function lives. This is best understood by referring to figure 2 which shows the phase space spanned by the axes u (usually time or space) and μ (temporal or spatial frequency). This phase space is also referred to as the time-frequency or space-frequency plane in the signal processing literature. The horizontal axis u is simply the time or space domain, where the original function lives. The vertical axis μ is simply the frequency (or Fourier) domain where the ordinary Fourier transform of the function lives. Oblique axes making angle α constitute domains where the ath order fractional Fourier transform lives, where a and α are related through $\alpha = a\pi/2$. Notice that this description is consistent with the fact that the second Fourier transform is equal to the parity operation (associated with the $-u$ axis), the fact that the -1st transform corresponds to the inverse Fourier transform (associated with the $-\mu$ axis), and the periodicity of $f_a(u)$ in a (adding a multiple of 4 to a corresponds to adding a multiple of 2π to α).

For those familiar with phase spaces from a mechanics—rather than signal analysis—perspective, we note that the correspondence between spatial frequency and momentum allows one to construct a correspondence between the familiar mechanical phase space of a single degree of freedom (defined by the

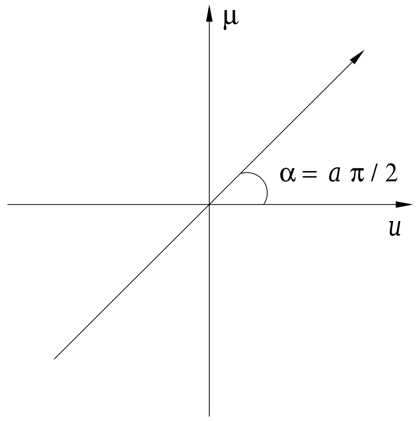

Figure 2:

space axis and the momentum axis), and the phase space of signal analysis (defined by the space axis and the spatial frequency axis). What is important to understand for the present purpose is that the phase space or time/space-frequency plane we are talking about is essentially the same physical construct as the classical phase space of mechanics.

Referring to axes making angle $\alpha = a\pi/2$ with the u axis as the "ath fractional Fourier domain" is supported by several of the properties of the fractional Fourier transform to be discussed further below. However, the most substantial justification is based on the fact that *fractional Fourier transformation corresponds to rotation in phase space*. This can be formulated in many ways, the most straightforward being to consider a phase-space distribution (or time/space-frequency representation) of the function $f(u)$, such as the Wigner distribution $W_f(u,\mu)$, which is defined as

$$W_f(u,\mu) = \int f(u+u'/2) f^*(u-u'/2) e^{-i2\pi\mu u'}\, du'. \quad (9)$$

The many properties of the Wigner distribution [10] support its interpretation as a function giving the distribution of signal energy in phase space (the time- or space-frequency plane). That is, the Wigner distribution answers the question "How much of the signal energy is located near this time and frequency?" (Naturally, the answer to this question can only be given within limitations imposed by the uncertainty principle.) Three of the important properties of the Wigner distribution are

$$\int W_f(u,\mu)\, d\mu = \mathcal{R}_0[W_f(u,\mu)] = |f(u)|^2, \quad (10)$$

$$\int W_f(u,\mu)\, du = \mathcal{R}_{\pi/2}[W_f(u,\mu)] = |F(\mu)|^2, \quad (11)$$

$$\iint W_f(u,\mu)\, du\, d\mu = \|f\|^2 = \text{Signal Energy}. \quad (12)$$

Here $\mathcal{R}_\alpha$ denotes the integral projection (or Radon transform) operator which takes an integral projection of the two-dimensional function $W_f(u,\mu)$ onto an axis making angle α with the u axis, to produce a one-dimensional function.

Now, it is possible to show that the Wigner distribution $W_{f_a}(u,\mu)$ of $f_a(u)$ is a clockwise rotated version of the Wigner distribution $W_f(u,\mu)$ of $f(u)$. Mathematically,

$$W_{f_a}(u,\mu) = W_f(u\cos\alpha - \mu\sin\alpha, u\sin\alpha + \mu\cos\alpha). \quad (13)$$

That is, the act of fractional Fourier transformation on the original function, corresponds to rotation of the Wigner distribution. An immediate corollary of this result, supported by figure 3, is

$$\mathcal{R}_\alpha[W_f(u,\mu)] = |f_a(u)|^2, \quad (14)$$

which is a generalization of equations 10 and 11. This equation means that the projection of the Wigner distribution of $f(u)$ onto the axis making angle α gives us $|f_a(u)|^2$, the squared magnitude of the ath fractional Fourier transform of the function. Since projection onto the u axis (the time or space domain) gives $|f(u)|^2$ and projection onto the $\mu = u_1$ axis (the frequency domain) gives $|F(\mu)|^2$, it is natural to refer to the axis making angle α as the ath order fractional Fourier domain.

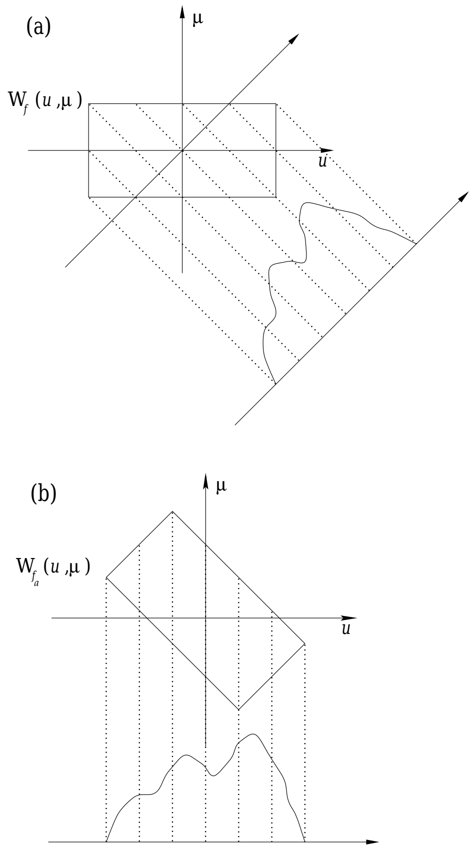

Figure 3:

4 Applications

We begin by highlighting some of the applications of the fractional Fourier transform which have received the greatest interest so far. A more comprehensive treatment and an extensive list of references may once again be found in [1] and [2].

The fractional Fourier transform has received a great deal of interest in the area of optics and especially optical signal processing (also known as Fourier optics or information optics) [11, 12, 13, 14]. Optical signal processing is an analog signal processing method which relies on the representation of signals by light fields and their manipulation with optical elements such as lenses, prisms, transparencies, holograms and so forth. Its key component is the optical Fourier transformer which can be realized using one or two lenses separated by certain distances from the input and output planes. It has been shown that the fractional Fourier transform can be optically implemented with equal ease as the ordinary Fourier transform, allowing a generalization of conventional approaches and results to their more flexible or general fractional analogs.

The fractional Fourier transform has also been shown to be intimately related to wave and beam propagation and diffraction. The process of diffraction of light, or any other disturbance satisfying a similar wave equation, has been shown to be nothing but a process of continual fractional Fourier transformation; the distribution of light becomes fractional Fourier transformed as it propagates, evolving through continuously increasing orders.

The transform has also found widespread use in signal and image processing, in areas ranging from time/space-variant filtering, perspective projections, phase retrieval, image restoration, pattern recognition, tomography, data compression, encryption, watermarking, and so forth (for instance, [8, 15, 16, 17, 18, 19, 20]). Concepts such as "fractional convolution" and "fractional correlation" have been studied. One of the most striking applications is that of filtering in fractional Fourier domains [15]. In traditional filtering, one takes the Fourier transform of a signal, multiplies it with a Fourier-domain transfer function, and inverse transforms the result. Here, we take the fractional Fourier transform, apply a filter function in the fractional Fourier domain, and inverse transform to the original domain. It has been shown that considerable improvement in performance is possible by exploiting the additional degree of freedom coming from the order parameter a. This improvement comes at no additional cost since computing the fractional Fourier transform is not more expensive than computing the ordinary Fourier transform [21]. The concept has been generalized to multi-stage and multi-channel filtering systems which employ several fractional Fourier domain filters of different orders [22]. These schemes provide flexible and cost-efficient means of designing time/space-variant filtering systems to meet desired objectives and may find use in control systems.

The fractional Fourier transform is intimately related to the harmonic oscillator in both its classical and quantum-mechanical forms. The kernel $K_a(u, u')$ given in equation 1 is precisely the Green's function (time-evolution operator kernel) of the quantum-mechanical harmonic oscillator differential equation. In other words, the time evolution of the wave function of a harmonic oscillator corresponds to continual fractional Fourier transformation. In classical mechanics, the relationship can be most easily seen by noting that—with properly normalized coordinates—the phase space point describing harmonic oscillation follows circular trajectories; that is, it rotates in phase space. Therefore, one can expect the fractional Fourier transform to play an important role in the study of vibrating systems, an application area which has so far not received attention.

Another potential application area is the solution of time-varying differential equations. Namias and McBride and Kerr [3, 4, 23] have shown how the fractional Fourier transform can be used to solve certain differential equations. Constant coefficient (time-invariant) equations can be solved with the ordinary Fourier or Laplace transforms. It has been shown that certain kinds of second-order differential equations with non-constant coefficients can be solved by exploiting the additional degree of freedom associated with the order parameter a. One proceeds by taking the fractional Fourier transform of the equation and then choosing a such that the second-order term disappears, leaving a first-order equation whose exact solution can always be written. Then, an inverse transform (of order $-a$) provides the solution of the original equation. It remains to be seen if this method can be generalized to higher-order equations by reducing the order from n to $n-1$ and proceeding recursively down to a first-order equation, by using a different-ordered transform at each step.

We believe that the fractional Fourier transform is of potential usefulness in every area in which the ordinary Fourier transform is used. The typical pattern of discovery of a new application is to concentrate on an application where the ordinary Fourier transform is used and ask if any improvement or generalization might be possible by using the fractional Fourier transform instead. The additional order parameter often allows better performance or greater generality because it provides an additional degree of freedom over which to optimize.

Typically, improvements are observed or are greater when dealing with time/space-variant signals or systems. Furthermore, very large degrees of improvement often becomes possible when signals of a chirped nature or with nearly-linearly increasing frequencies are in question, since chirp signals are the basis functions associated with the fractional Fourier transform (just as harmonic functions are the basis functions associated with the ordinary Fourier transform).

The fractional Fourier transform has spurred interest in many other fractional transforms; see [1] for further references. The fractional Laplace and z-transforms, however, have so far not received sufficient attention.

5 Transforms of some common functions

Below we list the fractional Fourier transforms of some common functions. Transforms of most other functions must usually be computed numerically. It has been shown that the transform of a continuous function whose time- or space-bandwidth product is N can be computed in the order of $N \log N$ time [21], just like the ordinary Fourier transform. Therefore any improvements that come with use of the fractional Fourier transform come at no additional cost. The discrete fractional Fourier transform has been defined and studied in [24].

Unit function: The fractional Fourier transform of $f(u) = 1$ is

$$\mathcal{F}^a[1] = \sqrt{1 + i\tan\alpha}\, e^{-i\pi u^2 \tan\alpha}. \tag{15}$$

This equation is valid when $a \neq 2j+1$ where j is an arbitrary integer. The transform is $\delta(u)$ when $a = 2j+1$.

Delta function: The fractional Fourier transform of a delta function $f(u) = \delta(u - u_0)$ is

$$\mathcal{F}^a[\delta(u-u_0)] = \sqrt{1 - i\cot\alpha}\, e^{i\pi(u^2\cot\alpha - 2uu_0\csc\alpha + u_0^2\cot\alpha)}. \tag{16}$$

This expression is valid when $a \neq 2j$. The transform of $\delta(u - u_0)$ is $\delta(u-u_0)$ when $a = 4j$ and $\delta(u+u_0)$ when $a = 4j+2$.

Harmonic Function: The fractional Fourier transform of a harmonic function $f(u) = \exp(i2\pi\mu_0 u)$ is

$$\mathcal{F}^a[e^{i2\pi\mu_0 u}] = \sqrt{1 + i\tan\alpha}\, e^{-i\pi(u^2\tan\alpha - 2u\mu_0\sec\alpha + \mu_0^2\tan\alpha)}. \tag{17}$$

This equation is valid when $a \neq 2j+1$. The transform of $\exp(i2\pi\mu_0 u)$ is $\delta(u-\mu_0)$ when $a = 4j+1$ and $\delta(u+\mu_0)$ when $a = 4j+3$.

General chirp function: The fractional Fourier transform of a general chirp function $f(u) = \exp[i\pi(\chi u^2 + 2\xi u)]$ is

$$\mathcal{F}^a[e^{i\pi(\chi u^2+2\xi u)}] = \sqrt{\frac{1+i\tan\alpha}{1+\chi\tan\alpha}} \times e^{i\pi[u^2(\chi-\tan\alpha)+2u\xi\sec\alpha-\xi^2\tan\alpha]/[1+\chi\tan\alpha]} \tag{18}$$

This equation is valid when $a - (2/\pi)\arctan\chi \neq 2j+1$. The transform of $\exp(i\pi\chi u^2)$ is $\sqrt{1/(1-i\chi)}\,\delta(u)$ when $[a - (2/\pi)\arctan\chi] = 2j+1$ and $\sqrt{1/(1-i\chi)}$ when $[a - (2/\pi)\arctan\chi] = 2j$.

Hermite-Gaussian functions: The fractional Fourier transform of a Hermite-Gaussian function $f(u) = \psi_n(u)$ is

$$\mathcal{F}^a[\psi_n(u)] = e^{-in\alpha}\,\psi_n(u). \tag{19}$$

General Gaussian function: The fractional Fourier transform of a general Gaussian function $f(u) = \exp[-\pi(\chi u^2 + 2\xi u)]$ is

$$\mathcal{F}^a[e^{-\pi(\chi u^2+2\xi u)}] = \sqrt{\frac{1-i\cot\alpha}{\chi - i\cot\alpha}} \times e^{i\pi\cot\alpha[u^2(\chi^2-1)+2u\chi\xi\sec\alpha+\xi^2]/[\chi^2+\cot\alpha]} \times e^{-\pi\csc^2\alpha(u^2\chi+2u\xi\cos\alpha-\chi\xi^2\sin^2\alpha)/(\chi^2+\cot\alpha)}. \tag{20}$$

Here $\chi > 0$ is required for convergence.

6 Properties

Linearity: Let $\mathcal{F}^a$ denote the ath order fractional Fourier transform operator. Then $\mathcal{F}^a[\sum_k b_k f_k(u)] = \sum_k b_k[\mathcal{F}^a f_k(u)]$.

Integer orders: $\mathcal{F}^k = (\mathcal{F})^k$ where $\mathcal{F}$ denotes the ordinary Fourier transform operator. This property states that when a is equal to an integer k, the ath order fractional Fourier transform is equivalent to the kth integer power of the ordinary Fourier transform, defined by repeated application. It also follows that $\mathcal{F}^2 = \mathcal{P}$ (the parity operator), $\mathcal{F}^3 = \mathcal{F}^{-1} = (\mathcal{F})^{-1}$ (the inverse transform operator), $\mathcal{F}^4 = \mathcal{F}^0 = \mathcal{I}$ (the identity operator), and $\mathcal{F}^j = \mathcal{F}^{j \bmod 4}$.

Inverse: $(\mathcal{F}^a)^{-1} = \mathcal{F}^{-a}$. In terms of the kernel, this property is stated as $K_a^{-1}(u,u') = K_{-a}(u,u')$.

Unitarity: $(\mathcal{F}^a)^{-1} = (\mathcal{F}^a)^{\mathrm{H}} = \mathcal{F}^{-a}$ where $()^{\mathrm{H}}$ denotes the conjugate transpose of the operator. In terms of the kernel, this property can stated as $K_a^{-1}(u,u') = K_a^*(u',u)$.

Index additivity: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_2+a_1}$. In terms of kernels this can be written as $K_{a_2+a_1}(u,u') = \int K_{a_2}(u,u'')K_{a_1}(u'',u')\,du''$.

Commutativity: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_1}\mathcal{F}^{a_2}$.

Associativity: $\mathcal{F}^{a_3}(\mathcal{F}^{a_2}\mathcal{F}^{a_1}) = (\mathcal{F}^{a_3}\mathcal{F}^{a_2})\mathcal{F}^{a_1}$.

Eigenfunctions: $\mathcal{F}^a[\psi_n(u)] = \exp(-ian\pi/2)\psi_n(u)$.

Parseval: $\int f^*(u)g(u)du = \int f_a^*(u)g_a(u)du$. This property is equivalent to unitarity. Energy or norm conservation ($\mathrm{En}[f] = \mathrm{En}[f_a]$ or $\|f\| = \|f_a\|$) is a special case.

Time reversal: Let $\mathcal{P}$ denote the parity operator: $\mathcal{P}[f(u)] = f(-u)$, then

$$\mathcal{F}^a\mathcal{P} = \mathcal{P}\mathcal{F}^a \tag{21}$$

$$\mathcal{F}^a[f(-u)] = f_a(-u) \tag{22}$$

Transform of a scaled function: Let $\mathcal{M}_M$ and $\mathcal{Q}_q$ denote the scaling $\mathcal{M}_M[f(u)] = |M|^{-1/2}f(u/M)$ and chirp multiplication $\mathcal{Q}_q[f(u)] = e^{-i\pi q u^2}f(u)$ operators respectively. Then

$$\mathcal{F}^a\,\mathcal{M}_M = \mathcal{Q}_{[-\cot\alpha(1-(\cos^2\alpha')/(\cos^2\alpha))]}\,\mathcal{M}_{[M\sin\alpha'/\sin\alpha]}\,\mathcal{F}^{a'}, \tag{23}$$

$$\mathcal{F}^a[|M|^{-1/2}f(u/M)] = \sqrt{\frac{1-i\cot\alpha}{1-iM^2\cot\alpha}} \times e^{i\pi u^2\cot\alpha\left(1-(\cos^2\alpha')/(\cos^2\alpha)\right)}\, f_{a'}\left(\frac{Mu\sin\alpha'}{\sin\alpha}\right). \tag{24}$$

Here $\alpha' = \arctan(M^{-2}\tan\alpha)$ and α' is taken to be in the same quadrant as α. This property is the generalization of the ordinary Fourier transform property stating that the Fourier transform of $f(u/M)$ is $|M|F(M\mu)$. Notice that the fractional Fourier transform of $f(u/M)$ cannot be expressed as a scaled version of $f_a(u)$ for the same order a. Rather, the fractional Fourier transform of $f(u/M)$ turns out to be a scaled and chirp modulated version of $f_{a'}(u)$ where $a' \neq a$ is a different order.

Transform of a shifted function: Let $\mathcal{SH}_{u_0}$ and $\mathcal{PH}_{\mu_0}$ denote the shift $\mathcal{SH}_{u_0}[f(u)] = f(u-u_0)$ and the phase shift $\mathcal{PH}_{\mu_0}[f(u)] = \exp(i2\pi\mu_0 u)f(u)$ operators respectively. Then

$$\mathcal{F}^a\,\mathcal{SH}_{u_0} = e^{i\pi u_0^2 \sin\alpha\cos\alpha}\mathcal{PH}_{-u_0\sin\alpha}\,\mathcal{SH}_{u_0\cos\alpha}, \tag{25}$$

$$\mathcal{F}^a[f(u-u_0)] = e^{i\pi\sin\alpha(u_0^2\cos\alpha - 2uu_0)}\,f_a(u-u_0\cos\alpha). \tag{26}$$

We see that the $\mathcal{SH}_{u_0}$ operator, which simply results in a translation in the u domain, corresponds to a translation followed by a phase shift in the ath fractional domain. The amount of translation and phase shift is given by cosine and sine multipliers which can be interpreted in terms of "projections" between the axes.

Transform of a phase-shifted function:

$$\mathcal{F}^a\,\mathcal{PH}_{\mu_0} = e^{-i\pi\mu_0^2\sin\alpha\cos\alpha}\mathcal{PH}_{\mu_0\cos\alpha}\,\mathcal{SH}_{\mu_0\sin\alpha} \tag{27}$$

$$\mathcal{F}^a[f(u-u_0)] = e^{-i\pi\cos\alpha(\mu_0^2\sin\alpha - 2u\mu_0)}\,f_a(u-\mu_0\sin\alpha) \tag{28}$$

Similar to the shift operator, the phase-shift operator which simply results in a phase shift in the u domain, corresponds to a translation followed by a phase shift in the ath fractional domain. Again the amount of translation and phase shift are given by cosine and sine multipliers.

Transform of a coordinate multiplied function: Let $\mathcal{U}$ and $\mathcal{D}$ denote the coordinate multiplication $\mathcal{U}[f(u)] = uf(u)$ and differentiation $\mathcal{D}[f(u)] = (i2\pi)^{-1}df(u)/du$ operators respectively. Then

$$\mathcal{F}^a\,\mathcal{U}^n = [\cos\alpha\,\mathcal{U} - \sin\alpha\,\mathcal{D}]^n\mathcal{F}^a \tag{29}$$

$$\mathcal{F}^a[u^n f(u)] = [\cos\alpha\,u - \sin\alpha\,(i2\pi)^{-1}d/du]^n f_a(u) \tag{30}$$

When $a=1$ the transform of a coordinate multiplied function $uf(u)$ is the derivative of the transform of the original function $f(u)$, a well-known property of the Fourier transform. For arbitrary values of a, we see that the transform of $uf(u)$ is a linear combination of the coordinate-multiplied transform of the original function and the derivative of the transform of the original function. The coefficients in the linear combination are $\cos\alpha$ and $-\sin\alpha$. As a approaches 0, there is more $uf(u)$ and less $df(u)/du$ in the linear combination. As a approaches 1, there is more $df(u)/du$ and less $uf(u)$.

Transform of the derivative of a function:

$$\mathcal{F}^a\,\mathcal{D}^n = [\sin\alpha\,\mathcal{U} + \cos\alpha\,\mathcal{D}]^n\mathcal{F}^a \tag{31}$$

$$\mathcal{F}^a[[(i2\pi)^{-1}d/du]^n f(u)] = [\sin\alpha\,u + \cos\alpha\,(i2\pi)^{-1}d/du]^n f_a(u) \tag{32}$$

When $a = 1$ the transform of the derivative of a function $df(u)/du$ is the coordinate-multiplied transform of the original function. For arbitrary values of a, we see that the transform is again a linear combination of the coordinate-multiplied transform of the original function and the derivative of the transform of the original function.

Transform of a coordinate divided function:

$$\mathcal{F}^a[f(u)/u] = -i\csc\alpha\,e^{i\pi u^2\cot\alpha}\int_{-\infty}^{2\pi u} f_a(u')e^{(-i\pi u'^2\cot\alpha)}\,du' \tag{33}$$

Transform of the integral of a function:

$$\mathcal{F}^a[\int_{u_0}^{u} f(u')\,du'] = \sec\alpha\,e^{-i\pi u^2\tan\alpha}\int_{u_0}^{u} f_a(u')e^{i\pi u'^2\tan\alpha}\,du' \tag{34}$$

A few additional properties are

$$\mathcal{F}^a[f^*(u)] = f^*_{-a}(u), \tag{35}$$

$$\mathcal{F}^a[(f(u)+f(-u))/2] = (f_a(u)+f_a(-u))/2, \tag{36}$$

$$\mathcal{F}^a[(f(u)-f(-u))/2] = (f_a(u)-f_a(-u))/2. \tag{37}$$

It is also possible to write convolution and multiplication properties for the fractional Fourier transform, though these are not of great simplicity [1].

We may finally note that the transform is continuous in the order a. That is, small changes in the order a correspond to small changes in the transform $f_a(u)$. Nevertheless, care is always required in dealing with cases where a approaches an even integer, since in this case the kernel approaches a delta function.

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