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Maximum entanglement and its proper measure

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Abstract
We discuss a definition of maximally entangled states in terms of maximum uncertainty of corresponding measurements. We describe a method of construction of bases of maximally entangled states. The entangled states that can be obtained from the maximally entangled states by means of SLOCC (stochastic local operations assisted by classical communications) we consider as semistable vectors. We discuss a measure of entanglement expressed in terms of a geometric invariant.

Keywords: dynamic symmetry, entanglement, quantum fluctuations, quantum information

1. Introduction

For a long time, entanglement was considered as an academic problem related to the foundation of quantum mechanics (Einstein *et al* 1935, Schrödinger 1935, Bohm 1952, Bell 1966, Wigner 1970). The realization that quantum entanglement is an unexpectedly efficient alternative to classical information (Bennett and Shor 1998)—namely the discovery of quantum cryptography (Bennett and Brassard 1984, Ekert 1991) and the transmission of intact quantum states (quantum teleportation Bennett *et al* 1993)—has led to development of quantum information science as an emerging field with the potential to cause revolutionary advances in science and technology. The notion of entanglement lies at the very heart of this new science.

It is possible to say that now everything is known about the *bipartite* entanglement of pure states. The reason is that the bipartite entanglement has a simple mathematical structure described by the Schmidt decomposition (Ekert and Knight 1995, Eberly *et al* 2003).

The situation is far more complicated in the case of bipartite *mixed* entangled states and *multipart* entanglement. For example, some bipartite entangled mixed states exhibit bound entanglement—though entanglement is necessary to create these states, none of this entanglement can be distilled (Horodecki *et al* 1998). Moreover, the quantum states with no entanglement can exhibit a peculiar kind of quantum nonlocality. An important example is provided by the coherent states, which, in a sense, represent an exact antithesis to entangled states (Klyachko 2002).

It is appropriate to mention here that entanglement is usually defined to be a feature of *composite* quantum systems that cannot be created through the use of local operations, acting on the different parties separately, or by means of classical communications between the parties.

Such a definition assigns primary importance to the *nonlocality* of the system and hence sets aside a *single-particle* entanglement with respect to internal degrees of freedom (Kim 2003) as well as entanglement of atoms in a Bose–Einstein condensate, when nonlocality is meaningless because of the strong overlap of the wavefunctions of different atoms (Leggett 2001). Other examples are provided by an ensemble of fermions (a cloud of electrons with an overlapping wavefunction) and photons in a field.

The difficulties in the description of multipart entanglement and extension of the notion of entanglement to local systems require a certain revision and sharpening of the very definition of quantum entanglement and its proper measure. This apparently needs the revealing of the mathematical structure hidden behind the entanglement and the development of novel methods for studying and implementation of entanglement.

It was shown that the mathematical structure of entanglement is adequately reflected by the *dynamic symmetry method* and *geometric invariant theory* (Klyachko 2002). Further development of the approach have led to a number of new results (Klyachko and Shumovsky 2003a, 2003b).

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There are three main objectives of this paper. First, we discuss a new definition of *maximum entanglement*, which has a simple physical meaning and is appropriate for the composite and local systems. Second, we consider a general method of construction of bases of maximally entangled states. Third, we discuss a new measure of entangled states.

The choice of maximum entanglement as a key notion is motivated by observation of the fact that entanglement can be either increased or decreased by means of a certain operations such as the Lorentz transformation (Peres *et al* 2002, Peres and Terno 2002, Bergou *et al* 2003) and SLOCC (stochastic operations assisted by classical communications) (Bennett *et al* 1995, Dür *et al* 2000, Bennett *et al* 2001, Verstraete *et al* 2002). The point is that all entangled states of a given system can be constructed from the maximally entangled states by means of a certain operation.

The paper is organized as follows. In section 2, we introduce a new definition of maximum entanglement based on the rate of quantum fluctuations. That is, the amount of quantum fluctuations in a given state is considered as the measure of *remoteness* of this state from 'classical realism'. Then, the maximally entangled states are specified by the maximum remoteness. Physically this means that the maximally entangled states represent a manifestation of quantum fluctuations at their extreme. In section 3, we discuss the choice of *generic* maximally entangled states such that the basis of maximally entangled states can be constructed from the generic one through the use of a local cyclic permutation operator. In section 4, we discuss a measure of entanglement represented by the length of a minimal vector in the complex orbit of the entangled state. The last section 5 contains a brief summary and concluding remarks. The discussion is accompanied by a number of examples.

2. Definition of maximum entanglement

2.1. The criterion of maximum entanglement

All one can assume is that the entangled states are maximally remote from the classical states. The main difference between the classical and quantum descriptions of a system is that the observables in the latter case are represented by operators and manifest *quantum uncertainties* (quantum fluctuations). The range of quantum fluctuations depends on the specification of the quantum state. For example, *coherent states* manifest minimal uncertainties (Delbargo and Fox 1970, Perelomov 1986), and therefore they are usually interpreted as almost classical states.

Following this ideology, we can state that the maximally entangled states provide the maximum range of quantum fluctuations (Can *et al* 2002a, Klyachko 2002, Klyachko and Shumovsky 2003a, 2003b).

To formulate a rigorous mathematical definition from this intuitive statement, consider a system S defined in the Hilbert space $\mathbb{H}(S)$ (not necessary a nonlocal system). We choose to specify the system by the Lie algebra $\mathcal{L}$, generated by *all essential observables*, and by the corresponding dynamic symmetry group $G = \exp\mathcal{L}$. The choice of the essential observables depends on the set of measurements that we are going to perform over the system, or, what is the same, on the Hamiltonians, which are accessible for the manipulation with quantum states. The possible choice of essential measurements is discussed below in this section.

Let us denote by M_i the measurements, forming a basis of $\mathcal{L}$. The uncertainties of measurements M_i are specified by the *variances*

$$\mathbb{V}_i(\psi) \equiv \langle\psi|(\Delta M_i)^2|\psi\rangle = \langle\psi|M_i^2|\psi\rangle - \langle\psi|M_i|\psi\rangle^2, \quad (1)$$

where ψ is a pure state in $\mathbb{H}(S)$. In the case of mixed states ρ, instead of equation (1) we get

$$\mathbb{V}_i(\rho) = \mathrm{Tr}(\rho M_i^2) - [\mathrm{Tr}(\rho M_i)]^2. \quad (2)$$

We choose to specify the range of quantum fluctuations (the 'remoteness' of the quantum state in $\mathbb{H}(S)$ from the classical one) by the *total variance*

$$\mathbb{V}(\psi) = \sum_i \mathbb{V}_i(\psi). \quad (3)$$

Definition. *A vector* $\psi_{\mathrm{ME}} \in \mathbb{H}(S)$ *is called the maximally entangled state of S iff it provides the maximum of the total variance*

$$\mathbb{V}(\psi_{\mathrm{ME}}) = \max_{\{\psi\}} \mathbb{V}(\psi). \quad (4)$$

In the case of mixed states, equation (4) should be replaced by the following:

$$\mathbb{V}(\rho_{\mathrm{ME}}) = \max_{\{\rho\}} \mathbb{V}(\rho). \quad (5)$$

This definition seems to be quite general. For example, it is independent of whether S is a composite system or not. Then, it deals with a simple physical quantity and has clear physical meaning. At the same time, the magnitude of the total variance (3) cannot be used as a measure of entanglement. This is discussed in detail in section 4.

The definition (4) represents a kind of *variational principle* in quantum mechanics, defining the maximally entangled states. In a sense, this is similar to the principle of maximum entropy in statistical mechanics.

In special cases of interest, certain properties of the dynamic symmetry group G enable us to somewhat simplify the analysis of maximum entanglement. That is, if the enveloping algebra of essential observables contains the uniquely determined Casimir operator

$$\sum_i M_i^2 = \mathbf{C} \times \mathbb{I},$$

where $\mathbb{I}$ is the unit operator in $\mathbb{H}(S)$, then it is clear that

$$\mathbb{V}(\psi_{\mathrm{ME}}) = \max_{\psi} \mathbb{V}(\psi) = \mathbf{C}, \quad (6)$$

provided by the condition

$$\forall i, \qquad \langle\psi_{\mathrm{ME}}|M_i|\psi_{\mathrm{ME}}\rangle = 0, \quad (7)$$

or

$$\forall i, \qquad \mathrm{Tr}(\rho_{\mathrm{ME}} M_i) = 0, \quad (8)$$

in the case of mixed maximally entangled states. Conditions (7) and (8) are simpler than (4) and (5) from the operational point of view because they deal with the averages, whose measurement is easier than that of variances. In fact, equations (7) and (8) can be used as an operational definition of maximum entanglement (Can *et al* 2002a), that is the definition in terms of what can be measured. Consider now a few examples.

2.2. A system of N qubits

As an illustrative example of some considerable interest, we examine a system of N qubits defined in the Hilbert space

$$\mathbb{H}_{2,N} = \bigotimes_{j=1}^{N} \mathbb{H}_2 = \mathbb{H}_2^N, \tag{9}$$

where the two-dimensional space of each party is spanned by the basis vectors

$$\mathbf{e}_\ell = |\ell\rangle, \qquad \ell = 0, 1. \tag{10}$$

The dynamic symmetry group in (9) is

$$G = SU(2) \times \cdots \times SU(2) = \prod_{j=1}^{N} SU(2).$$

To choose the set of essential observables, assume that the total access to the *local measurements* is allowed. In the case of qubits (spin-$\frac{1}{2}$ particles), the local measurements are provided by the Pauli operators (e.g., see Nielsen and Chuang 2000)

$$\begin{aligned} M_1 &= \sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|, \\ M_2 &= \sigma_2 = -\mathrm{i}|0\rangle\langle 1| + \mathrm{i}|1\rangle\langle 0|, \\ M_3 &= \sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|, \end{aligned} \tag{11}$$

forming a representation of the infinitesimal generators of the $SL(2, \mathbb{C})$ algebra, which corresponds to the *complexification* of the $SU(2)$ algebra. Let us stress that the complexification of Lie algebra plays an important role here (Klyachko 2002). Physically, the complexification is associated with SLOCC (Dür *et al* 2000, Verstraete *et al* 2002). Since $\sigma_i^2 = \mathbb{I}_2$, where $\mathbb{I}_2$ is the unit operator in $\mathbb{H}_2$, the maximum of the total variance (3) is

$$\max \mathbb{V}_{2,N} = 3N. \tag{12}$$

The so-called Bell states of two qubits manifest the maximum value in (12) at $N = 2$ and hence fit the above definition of maximum entanglement. It should be emphasized that condition (7) defines infinitely many maximally entangled states in addition to the Bell states. In more detail, this is discussed in section 3 (also see Klyachko and Shumovsky 2003a). In turn, the Greenberger–Horne–Zeilinger (GHZ) states of three qubits also give the maximum value of the total variance (12) at $N = 3$ and obey the definition of maximum entanglement.

At the same time, the simple separable states such as $|00\rangle$ in the case of two qubits and $|000\rangle$ in the case of three qubits give the minimum value of the total variance (12) ($\mathbb{V}_{2,2} = 4$ and $\mathbb{V}_{2,3} = 6$, respectively). Hence, they can be interpreted as the (generalized) coherent states.

It is easily seen that a single qubit does not obey the conditions (7) and therefore cannot manifest entanglement.

2.3. A system of qutrits

Consider now the case of N qutrits (three degrees of freedom per party), when the system S is defined in the Hilbert space $\mathbb{H}_{3,N}$. We again assume that total access to the local measurements is allowed. We can choose the set of local measurements as the generators of the $SL(2, \mathbb{C})$ algebra that have the representation

$$\begin{aligned} J_x &= (|0\rangle\langle 1| + |1\rangle\langle 2| + \text{H.c.}), \\ J_y &= (-|0\rangle\langle 1| - |1\rangle\langle 2| - \text{H.c.}), \\ J_z &= |0\rangle\langle 0| - |2\rangle\langle 2|, \end{aligned} \tag{13}$$

in the case of three dimensions (spin-1 particles). At the same time, the dynamic symmetry of the Hilbert space $\mathbb{H}_{3,N}$ is

$$G = SU(3) \times \cdots \times SU(3) = \prod_{j=1}^{N} SU(3)$$

which allows another choice of the local measurements provided by the generators

$$\begin{gathered} |0\rangle\langle 0| - |1\rangle\langle 1|, \qquad |1\rangle\langle 1| - |2\rangle\langle 2|, \qquad |2\rangle\langle 2| - |0\rangle\langle 0|, \\ \tfrac{1}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|), \qquad \tfrac{1}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|), \\ \tfrac{1}{2}(|2\rangle\langle 0| + |0\rangle\langle 2|), \\ \frac{1}{2\mathrm{i}}(|0\rangle\langle 1| - |1\rangle\langle 0|), \qquad \frac{1}{2\mathrm{i}}(|1\rangle\langle 2| - |2\rangle\langle 1|), \\ \frac{1}{2\mathrm{i}}(|2\rangle\langle 0| - |0\rangle\langle 2|). \end{gathered} \tag{14}$$

Only eight out of nine operators in (14) are independent because the sum of the operators in the first row in (14) is equal to zero. To clarify the physical difference between (13) and (14), we note that the former corresponds to the measurement of states of spin 1 in a uniform field, while the latter can be associated with the Stern–Gerlach apparatus.

Consider the states of a *single* qutrit, that can be written as follows:

$$|\psi\rangle = \sum_{\ell=0}^{2} \psi_\ell |\ell\rangle, \qquad \sum_{\ell=0}^{2} |\psi_\ell|^2 = 1, \qquad \ell = 0, 1, 2. \tag{15}$$

Let us choose (13) as the essential observables. Then, from the condition (7) we get

$$\begin{aligned} (\psi_0^* + \psi_2^*)\psi_1 + \text{c.c.} &= 0, \\ (\psi_2^* - \psi_0^*)\psi_1 - \text{c.c.} &= 0, \\ |\psi_0|^2 - |\psi_2|^2 &= 0. \end{aligned} \tag{16}$$

These equations (16) together with the normalization condition in (15) have the following solutions:

$$(1) \quad |\psi_\ell| = \begin{cases} 1/\sqrt{2}, & \ell = 0, 2, \\ 0, & \ell = 1, \end{cases} \tag{17}$$

$$(2) \quad |\psi_\ell| = \begin{cases} 0, & \ell = 0, 2, \\ 1, & \ell = 1, \end{cases} \tag{18}$$

$$\begin{gathered} (3) \quad 2|\psi_0|^2 + |\psi_1|^2 = 1, \\ |\psi_0| = |\psi_2|, \\ 2 \arg \psi_1 - \arg \psi_0 - \arg \psi_2 = \pm\pi + 2n\pi. \end{gathered} \tag{19}$$

The corresponding maximally entangled states are

$$(1) \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |2\rangle), \tag{20}$$

$$(2) \quad |\psi\rangle = |1\rangle, \tag{21}$$

$$(3) \quad |\psi\rangle = |\psi_0|(\mathrm{e}^{\mathrm{i}\varphi_0}|0\rangle + \mathrm{e}^{\mathrm{i}\varphi_2}|2\rangle) \pm \mathrm{i}|\psi_1|\mathrm{e}^{\mathrm{i}(\varphi_0+\varphi_2)/2}|1\rangle, \tag{22}$$

respectively. In the case of spin-1 particle, we can interpret ℓ in terms of the projection of spin $m = 0, \pm 1$ as $\ell = 1 - m$. Thus, a single spin-1 particle can be observed in the maximally entangled states with respect to the intrinsic degrees of freedom. The interpretation of this result lies outside of quantum information theory and could be of interest for particle physics (Klyachko 2002, Klyachko and Shumovsky 2003a, 2003b). In the case of a photon emitted by an electric dipole transition, $|\ell\rangle$ can be interpreted as a state with a given projection of the angular momentum of the photon (Shumovsky 1999).

It is also seen that a single qutrit cannot manifest maximum entanglement with respect to the measurements (14). In general, entanglement of an N-qutrit system with respect to the local measurements (14) also gives entanglement with respect to (13) but not vice versa.

2.4. *Entanglement of a photon field*

Consider a monochromatic photon field and assume that the observables are provided by the quadrature operators

$$q = \tfrac{1}{2}(a + a^+), \qquad p = -\mathrm{i}\tfrac{1}{2}(a - a^+), \qquad [a, a^+] = 1. \tag{23}$$

It is well known that, in the coherent state, $\langle(\Delta q)^2\rangle = \langle(\Delta p)^2\rangle = 1/4$, so the total variance (3) is $\mathbb{V}_{\text{coherent}} = 1/2$. At the same time, the condition (7) gives

$$\langle q\rangle = \mathrm{Re}(\alpha), \qquad \langle p\rangle = \mathrm{Im}(\alpha),$$

where α is the parameter of the coherent state. Thus, the coherent state does not obey the condition of maximum entanglement (7).

In turn, in the case of the Fock number state $|n\rangle$ we get $\langle n|q|n\rangle = \langle n|p|n\rangle = 0$, so this state obeys the condition (7). The total variance (3) in this case has the form

$$\mathbb{V}(|n\rangle) = \frac{2n+1}{2}, \tag{24}$$

which is always greater than $\mathbb{V}_{\text{coherent}} = 1/2$ at $n \geqslant 1$. Thus, the Fock number state of photons $|n\rangle$ manifests the maximum entanglement with respect to the measurement of the field quadratures (23). At the same time, equation (24) corresponds to the definition of a *parametric* maximum entanglement, because it is seen that the remoteness from the almost classical coherent state increases with increase of the number of photons n.

Another example of maximum entanglement of a photon field is provided by the squeezed vacuum state when $\langle q\rangle = \langle p\rangle = 0$ and

$$\mathbb{V}_{\text{squeezed}} = \frac{2\cosh r - 1}{2}, \tag{25}$$

where r is the parameter of squeezing. This is again the case of a parametric maximally entangled state. A certain difference between the total variances (24) and (25) is caused by the fact that the Weyl–Heisenberg algebra of photon operators (23) has no uniquely defined Casimir operator.

Probably the entanglement with respect to measurements of field quadratures (23) has only academic interest and is useless for the purposes of quantum information processing. The above examples show that the definition of maximum entanglement (4) also works in the case of photon fields. A more detailed consideration of different states of two and more photons requires taking into consideration the intrinsic degrees of freedom of photons as well.

3. The basis of maximally entangled states

3.1. *Generic states*

There are usually infinitely many maximally entangled states of a given system S, determined by the condition (7). In fact, we should be interested in the sets of maximally entangled states forming a basis in $\mathbb{H}(S)$. Exactly these basis states are important for quantum information processes such as quantum teleportation (Bennett *et al* 1993). Construction of such a basis consists of the two steps. First, in the space $\mathbb{H}_{n,N}$ of N parties with n degrees of freedom per party we should determine the set of n *generic* maximally entangled states. Then, by the action of a special local cyclic permutation operator, we can complete this set within the basis in $\mathbb{H}_{n,N}$.

Denote the local degree of freedom by $\ell = 0, 1, \ldots, n-1$ in the system with n degrees of freedom per party. Consider the homogeneous state

$$|\ell\ell\cdots\ell\rangle = \bigotimes_{j=0}^{N} |\ell\rangle_j$$

in the Hilbert space $\mathbb{H}_{n,N}$ and the linear combination

$$|\psi_{nN}^{(\text{gen})}\rangle = \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1}|\ell\ell\cdots\ell\rangle. \tag{26}$$

We call (26) the *generic* maximally entangled state. In the case of $n = 2$ and $N = 2$, equation (26) gives the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. At $n = 2$ and $N = 3$, it coincides with the GHZ state. At $n = 3$ and $N = 2$, (26) takes the form

$$|\psi_{3,2}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{27}$$

Similar states were discussed in the context of the 'biphoton' (Burlakov *et al* 1999) and in connection with quantum cryptography (Bechman-Pasquinucci and Peres 2000).

Since the dynamic symmetry group in $\mathbb{H}_{n,N}$ is $G = \prod_{j=1}^{N} SU(n)$, it is a straightforward matter to check that the states (26) obey the condition of maximum entanglement (7) with respect to the Lie algebra of the complexified group G^{c}.

Beginning with the generic state (26), we can construct a set of n mutually orthogonal states, consisting of the same uniform vectors as (26). Consider the operator

$$\Xi(\xi, n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \xi & 0 & \cdots & 0 \\ 0 & 0 & \xi^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \xi^{n-1} \end{pmatrix} \tag{28}$$

where $\xi^n = 1$, so $\xi = \exp(2ik\pi/n)$, $k = 0, 1, \ldots, n-1$. Acting with (28) on the generic state (26), we get the set of n states

$$|\psi_{n,N}^{(\mathrm{gen})}(k)\rangle = \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1} \mathrm{e}^{\mathrm{i}\ell\phi_k(n)}|\ell\ell\cdots\ell\rangle, \qquad \phi_k(n) = \frac{2k\pi}{n}, \tag{29}$$

which are mutually orthogonal (Kostrikin *et al* 1983). It is again possible to check that all n states in (29) obey the condition of maximum entanglement (7).

3.2. The local cyclic permutation operator

Equation (29) represents n orthonormal states, while the basis in $\mathbb{H}_{n,N}$ should consist of the $\dim \mathbb{H}_{n,N} = n^N$ vectors. To complete the set of generic states (29) with respect to the basis, we introduce the following local cyclic permutation operator (LCPO):

$$\mathcal{C}(n) = |0\rangle\langle 1| + |1\rangle\langle 2| + \cdots + |n-1\rangle\langle 0|, \tag{30}$$

acting in the single-party subspace $\mathbb{H}_n \subset \mathbb{H}_{n,N}$. In the matrix representation, it has the form of the $(n \times n)$ matrix

$$\mathcal{C}(n) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is seen that LCPO (30) obeys the condition $\mathcal{C}^n = \mathbb{I}_n$ and that the action of different powers of the LCPO on the single-party states produces cyclic permutations:

$$\mathcal{C}(n)|0\rangle = |n-1\rangle, [\mathcal{C}(n)]^2|0\rangle = |n-2\rangle, \ldots, [\mathcal{C}(n)]^{n-1} = |1\rangle.$$

Taking into account this property, we can act with (30) on the generic states (29) $(n^N - n)$ times to build the basis of maximally entangled states in $\mathbb{H}_{n,N}$.

To illustrate the process, consider a three-qubit system ($n = 2, N = 3$). Then, the generic states (29) take the form

$$|\psi_{2,3}^{(\mathrm{gen})}(k)\rangle = \frac{1}{\sqrt{2}}(|000\rangle + \mathrm{e}^{\mathrm{i}k\pi}|111\rangle), \qquad k = 0, 1, \tag{31}$$

which coincide with the GHZ states. In turn, LCPO (30) takes the form

$$\mathcal{C}(2) = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{32}$$

which coincides with the Pauli operator σ_1 in (11). Acting successively with (22) on the states of the first, second and third parties in the generic state (31), we get the set of six states

$$\frac{1}{\sqrt{2}}(|100\rangle + \mathrm{e}^{\mathrm{i}k\pi}|011\rangle),$$
$$\frac{1}{\sqrt{2}}(|010\rangle + \mathrm{e}^{\mathrm{i}k\pi}|101\rangle), \tag{33}$$
$$\frac{1}{\sqrt{2}}(|001\rangle + \mathrm{e}^{\mathrm{i}k\pi}|110\rangle),$$

that, together with the two states (31), completes the basis of eight maximally entangled states in $\mathbb{H}_{2,3}$. The maximum entanglement of states (33) immediately follows from the condition (7).

In the case of four qubits ($n = 2, N = 4$), the same procedure with the generic states

$$|\psi_{2,4}^{(\mathrm{GHZ})}(k = 0, 1)\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + \mathrm{e}^{\mathrm{i}k\pi}|1111\rangle), \tag{34}$$

and LCPO (32), gives the following fourteen states:

$$\frac{1}{\sqrt{2}}(|1000\rangle + \mathrm{e}^{\mathrm{i}k\pi}|0111\rangle), \qquad \frac{1}{\sqrt{2}}(|0100\rangle + \mathrm{e}^{\mathrm{i}k\pi}|1011\rangle),$$
$$\frac{1}{\sqrt{2}}(|0010\rangle + \mathrm{e}^{\mathrm{i}k\pi}|1101\rangle), \qquad \frac{1}{\sqrt{2}}(|0001\rangle + \mathrm{e}^{\mathrm{i}k\pi}|1110\rangle),$$
$$\frac{1}{\sqrt{2}}(|1100\rangle + \mathrm{e}^{\mathrm{i}k\pi}|0011\rangle), \qquad \frac{1}{\sqrt{2}}(|1010\rangle + \mathrm{e}^{\mathrm{i}k\pi}|0101\rangle),$$
$$\frac{1}{\sqrt{2}}(|0110\rangle + \mathrm{e}^{\mathrm{i}k\pi}|0110\rangle),$$

completing (34) as the basis of maximally entangled states in $\mathbb{H}_{2,4}$.

In the case of two qutrits ($n = 3, N = 2$), the generic states (29) have the form

$$|\psi_{3,2}^{(\mathrm{GHZ})}(k)\rangle = \frac{1}{\sqrt{3}}(|00\rangle + \mathrm{e}^{2\mathrm{i}k\pi/3}|11\rangle + \mathrm{e}^{4\mathrm{i}k\pi/3}|22\rangle),$$
$$k = 0, 1, 2, \tag{35}$$

and LCPO (30) is represented as follows:

$$\mathcal{C}_3 = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 0| = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{36}$$

Then the states

$$\frac{1}{\sqrt{3}}(|10\rangle + \mathrm{e}^{2\mathrm{i}k\pi/3}|21\rangle + \mathrm{e}^{4k\pi/3}|02\rangle),$$
$$\frac{1}{\sqrt{3}}(|20\rangle + \mathrm{e}^{2\mathrm{i}k\pi/3}|01\rangle + \mathrm{e}^{4\mathrm{i}k\pi/3}|12\rangle),$$

obtained from (35) by means of (36), complete (35) as the basis of maximally entangled states in $\mathbb{H}_{3,2}$. In this case, the maximum entanglement is provided by the condition (7) with the measurements (14).

Generally, the basis of maximally entangled states of a bipartite system in $\mathbb{H}_{n,2}$ is represented by the states

$$|\psi_n(k,m)\rangle = \frac{1}{\sqrt{2}}\sum_{\ell,m=0}^{n-1} \mathrm{e}^{\mathrm{i}\ell k\pi/n}|(\ell - m)\mathrm{mod}(n)\rangle \otimes |\ell\rangle,$$
$$k = 0, 1, \ldots, n-1.$$

(see Bennett *et al* 1993).

4. Entangled states

So far, we have considered maximally entangled states. In addition, there is an important class of entangled states that do not obey the condition of maximum entanglement (7). By definition, the entangled states can be constructed from the maximally entangled states by means of SLOCC (Verstraete *et al* 2002). There is a problem of how to distinguish between the maximally entangled, entangled and unentangled states and how to classify and quantify the entanglement (e.g., see Acin *et al* 2002, Dür *et al* 2000, Luque and Thibon 2003, Pleisch and Bužek 2003, Řeháček and Hradil 2003, Verstraete *et al* 2002). Solution of the problem requires a certain measure of entanglement.

It should be stressed that the total variance (3) cannot be used as a measure of entanglement. Below in this section we show that unentangled states may have quite strong quantum fluctuations that even exceed the level corresponding to certain entangled but not maximally entangled states.

To choose a proper measure of entanglement, we appeal to the geometric invariant theory. It has been proved (Klyachko 2002) that the definition of entanglement based on the notion of maximum entanglement and SLOCC is equivalent to the following one.

Definition. *Any state $\psi \in \mathbb{H}(S)$ is entangled (not necessarily maximally entangled) iff ψ is a semistable vector.*

Within the geometric invariant theory, the notion of *semistability* means that the state can be separated from zero by a G^c-invariant function $I(g\psi) = I(\psi) \neq 0$ (Vinberg and Popov 1992), where $g \in G^c$ and G^c is the complexified dynamic symmetry group in $\mathbb{H}(S)$, corresponding to the Lie algebra of all essential measurements. From the physical point of view, the invariants $I(\psi)$ correspond to the integrals of motion of the quantum system S.

According to the above definition, the *unentangled* states are represented by *unstable* vectors that can fall into zero ($g\psi \to 0$). In other words, they cannot be separated from zero by invariants. This means that the geometric invariants can be used to measure the entanglement. As the simplest measure, the length of the minimal vector in the complex orbit

$$\mu(\psi) = \min_{g \in G^c} |g\psi|^2 \tag{37}$$

can be used (Klyachko 2002). First of all, (37) represents an entanglement *monotone*. The notion of a 'monotone' was introduced by Vidal (Vidal 2000), and it is known that any proper measure of entanglement should obey this property (Eisert *et al* 2003). In the case of bipartite systems, (37) coincides with the *concurrence* (Hill and Wooters 1997) to within an inessential factor. For a three-part system, (37) amounts to the square root of a 3-tangle (Coffman *et al* 2000) that can be expressed in terms of Cayley's hyperdeterminant (Miyake 2003).

To illustrate the use of the measure (37), consider the case of three qubits, when an arbitrary pure state in $\mathbb{H}_{2,3}$ is represented as follows:

$$|\psi\rangle = \sum_{p,q,r} \psi_{pqr} \mathbf{e}_p \otimes \mathbf{e}_q \otimes \mathbf{e}_r. \tag{38}$$

Here $\mathbf{e}_\ell$ are the basis vectors (10) in $\mathbb{H}_2$. The coefficients ψ_{pqr} are the elements of a three-dimensional matrix (see figure 1). Concerning the multidimensional matrices and determinants, see Gelfand *et al* 1994. The measure (37) coincides in this case with Cayley's hyperdeterminant

$$\begin{aligned}\mathrm{Det}[\psi] = {} & \psi_{+++}^2\psi_{---}^2 + \psi_{++-}^2\psi_{--+}^2 + \psi_{+-+}^2\psi_{-+-}^2 \\ & + \psi_{+--}^2\psi_{-++}^2 - 2[\psi_{+++}(\psi_{++-}\psi_{--+} + \psi_{+-+}\psi_{-+-} \\ & + \psi_{+--}\psi_{-++})\psi_{---} + \psi_{++-}\psi_{+-+}\psi_{-+-}\psi_{--+} \\ & + \psi_{++-}\psi_{+--}\psi_{--+}\psi_{-++} + \psi_{+-+}\psi_{+--}\psi_{-+-}\psi_{-++}] \\ & + 4(\psi_{+++}\psi_{+--}\psi_{-+-}\psi_{--+} + \psi_{++-}\psi_{+-+}\psi_{-++}\psi_{---}). \end{aligned} \tag{39}$$

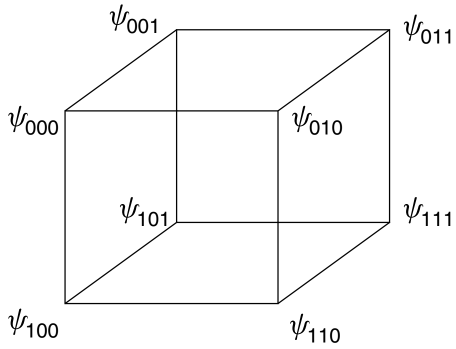

Figure 1. Structure of the three-dimensional matrix, corresponding to the state (38) of three qubits. Vertices of the cube correspond to the coefficients ψ_{pqr}.

Let us analyse some states of three qubits with the aid of (39). Consider first the maximally entangled states (31) and (33) with the total variance $\mathbb{V}_{\max} = 9$. In this case, $\mathrm{Det}[\psi] = 1/4$ is separated from zero. In the case of separable states of the type $|\ell\ell\ell\rangle$ ($\ell = 0, 1$), $\mathbb{V} = \mathbb{V}_{\min} = 6$ and $\mathrm{Det}[\psi] = 0$, so these states are coherent and do not manifest any entanglement. For a slightly more complicated separable state

$$\frac{1}{\sqrt{2}}(|011\rangle \pm |110\rangle), \tag{40}$$

we get $\mathbb{V} = 8 > \mathbb{V}_{\min}$ and $\mathrm{Det}[\psi] = 0$. Thus, we again have an unentangled state, which however is incoherent. At the same time, the relatively strong level of quantum fluctuations is caused by the quantum correlations between the first and third parties in (40).

Consider now the so-called W-states (Zeilinger *et al* 1997, Dür *et al* 2000). In view of the results of the previous section, the complete set of mutually orthogonal W-states of three qubits has the following form:

$$\begin{aligned} |W_k^{(0)}\rangle &= \frac{1}{\sqrt{3}}(|011\rangle + e^{i\phi_k}|101\rangle + e^{2i\phi_k}|110\rangle), \\ |W_k^{(1)}\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + e^{i\phi_k}|010\rangle + e^{2i\phi_k}|001\rangle), \end{aligned} \tag{41}$$

where $\phi_k = 2k\pi/3$. It is seen that the states (41) form a basis of a six-dimensional subspace in the eight-dimensional Hilbert space $\mathbb{H}_{2,3}$, corresponding to the discarding of the directions $|000\rangle$ and $|111\rangle$. In the case of W-states (41), the total variance (3) is very high: $\mathbb{V}(W) = 8 + 2/3$. At the same time, calculation of Cayley's hyperdeterminant (39) gives $\mathrm{Det}[W] = 0$ for all states in (41). Thus, equation (41) represents unstable vectors, corresponding to unentangled states.

Consider one more example of a state that can be maximally entangled, simply entangled and unentangled depending of the choice of parameters:

$$|\psi_x\rangle = x(|000\rangle + |111\rangle) + y(|001\rangle + |110\rangle), \qquad x^2 + y^2 = 1/2. \tag{42}$$

This state is a linear superposition of the two maximally entangled states. At $x = 0$ and $1/\sqrt{2}$, this state (42) is reduced to one of the maximally entangled states with $\mathbb{V}(\psi_x) = \mathbb{V}_{\max} = 9$ and $\mathrm{Det}[\psi_x] = 1/4$. At $x = 1/2$, (42) degenerates into a separable state with $\mathbb{V}(\psi_x) = 8$ and $\mathrm{Det}[\psi_x] = 0$

(no entanglement). When $x \in (0, 1/2)$ and $x \in (1/2, 1/\sqrt{2})$, we get

$$8 < \mathbb{V}(\psi_x) < 9, \qquad 0 < \mathrm{Det}[\psi_x] < 1/4, \qquad (43)$$

which corresponds to entanglement but not maximum entanglement. It is also seen that in the case of $0.122 < x < 0.5$ and $0.5 < x < 0.696$, the state (42) manifests entanglement, but has total variance less than that of the unentangled W-states.

The measure (37) can also be calculated explicitly in the case of four qubits and in many other cases. In fact all invariants and covariants in the case of four qubits have been calculated recently (Luque and Thibon 2003, Briand *et al* 2003).

5. Summary and conclusion

We have discussed a novel definition of maximum entanglement based on the choice of essential measurements and estimation of the range of quantum fluctuations of these measurements. This definition has a clear physical meaning. It specifies the states that are maximally remote from the classical states. In a sense, this definition fits Zeilinger's interpretation of entanglement and quantum information in terms of correlations (Zeilinger 1998), because the maximum of the total variance also means the maximum of correlations.

At the same time, the existence of strong correlations can be observed in the case of separable states such as (40) and nonseparable but unentangled states such as (41). Hence, neither the existence of quantum correlations nor a high range of quantum fluctuations is evidence of entanglement. Only the maximum range of quantum fluctuations corresponds to the maximally entangled states.

Let us note that the possible relation between maximum entanglement and quantum fluctuations (uncertainties) of certain measurements is widely discussed (Can *et al* 2002a, Gühne *et al* 2002, Gühne 2003, Gühne *et al* 2003, Klyachko 2002, Klyachko and Shumovsky 2002, Klyachko and Shumovsky 2003a, Klyachko and Shumovsky 2003b and references therein). The new result discussed in the present paper is the definition of maximally entangled states in terms of variational principle (4). In other words, the maximum entanglement is defined to be the manifestation of quantum fluctuations of observables at their extreme. This assumes maximum remoteness of maximally entangled states from the classical states.

It should be stressed that the maximality property (4) of the total variance plays here an important heuristic role peculiar to the variational principles. In particular, it helps one to understand the stabilizing effect of the environment on entanglement. That is, to prepare a persistent maximally entangled state, we should bring the system into a state with maximum possible value of the total variance (maximal level of quantum fluctuations) and then decrease its energy up to a (local) minimum, conserving the range of quantum fluctuations. For implementations of this mechanism, see Can *et al* (2002b), Can *et al* (2003).

Besides that, the definition (4) is independent of whether the system can be decomposed into separated subsystems or not. This opens the way to consideration of entanglement of a single particle with respect to the intrinsic degrees of freedom (Klyachko and Shumovsky 2003a, 2003b).

We showed that the bases of maximally entangled states of N qudits can be constructed from the generic maximally entangled state through the successive use of the two operations. The first one acts on the whole state $\psi \in \mathbb{H}_{n,N}$ and extends the generic maximally entangled state to a set of n mutually orthogonal maximally entangled states. The rest of the basis in $\mathbb{H}_{n,N}$ is constructed through the use of the LCPO, producing cyclic permutations of states of individual parties.

We discussed the definition of entangled (not necessarily maximally entangled) states in terms of the semistable vectors and showed that the simplest measure is provided by the length of the minimal vector (37). In particular, we showed that the so-called W-states of three qubits are unentangled although they manifest quite strong quantum fluctuations, which evidence the high level of quantum correlations between the parties.

Most of the results in this paper were obtained for pure states. They can be generalized to the mixed states because a mixed state can be formally treated as a pure state of a doublet, consisting of the system S and its 'mirror image' (Takahashi and Umezawa 1996).

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