Decentralized Control: A Stable Proper Fractional Approach

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Abstract—Three decentralized control problems are considered in a fractional setup for two-channel multivariable systems. The problems are: i) making the system stabilizable and detectable through the first channel via dynamic output feedback around the second channel, ii) the first problem with the constraint of internal stability, and iii) the decentralized stabilization problem. It turns out that all three problems are equivalent as far as the solvability conditions are concerned. A characterization of all solutions in each case is given. The results in this paper apply to a class of systems having fractional representations over an arbitrary principal ideal domain.

I. INTRODUCTION

THIS paper considers three *local dynamic output feedback* problems posed on the following feedback configuration:

The system Σ is a linear, finite-dimensional, time invariant system of two input channels (vector-inputs) and two output channels. The systems Σ_{c1} , Σ_{c2} (*compensators*) are in the same class of systems as Σ and serve to modify the behavior of Σ . The first two problems concern the case $\Sigma_{c2} = 0$, i.e., the second loop is open. In the third problem, both loops may be closed. The following problem definitions are intended to serve us temporarily until they are replaced in Section III by more precise definitions. **P1:** Determine a dynamic compensator Σ_{c1} such that the closed-loop system is stabilizable from input channel u_2 and detectable from the output channel y_2 or, in short, the second channel is stabilizable and detectable. **P2:** Determine a Σ_{c1} which solves P1 and which internally stabilizes a canonical part of the subsystem from u_1 to y_1 . **P3:** Determine two dynamic compensators Σ_{c1} and Σ_{c2} such that the overall closed-loop system is internally stable. We call **P1**, the second channel canonicity problem (SCCP), **P2**, the second channel canonicity problem with internal stability (SCCPIS), and **P3**, the decentralized stabilization problem (DSP). All three problems are instances of *decentralized control* or *local output feedback* problems.

The decentralized control of linear multivariable systems is among the easily motivated problems since the need to use controls of special pattern arise in many different applications. The interest in decentralized control was at its peak around the 1970's when a solution to decentralized stabilization problem was given by Wang and Davison [15]. The solution introduced the idea of "fixed modes" under decentralized feedback, and

this has been the most important contribution to the algebraic theory of decentralized control so far. The paper by Corfmat and Morse [4] approached the decentralized stabilization problem by viewing it as a problem of making an N -channel system stabilizable and detectable through one channel via static feedbacks around the remaining $N - 1$ channels. They have shown that this problem has a solution in case the N -channel system is "strongly connected" and "complete." It has later been shown in Anderson and Clements [1] that the notion of completeness is closely related to fixed modes, as the case should be. The graph-theoretic notion of strong connectedness has to do with nonzeroness of the interacting subplants. This has no direct relevance to the decentralized stabilization problem, as the result of Wang and Davison [15] shows. The natural conclusion is that there should be a synthesis procedure for the decentralized stabilization problem starting with the notion of completeness and serving as an alternative to the synthesis procedure of Corfmat and Morse [4] which hinges upon identifying the "strongly connected components" in the composite plant.

In Özgüler [8], it was claimed and illustrated to a certain extent that the crucial idea which leads to such a direct synthesis procedure is to use "dynamic" feedbacks rather than "static" in $N - 1$ channels in making the system stabilizable and detectable through the remaining channel. The problem considered in [8] was that of making a two-channel system stabilizable in one channel via dynamic feedback around the other channel and was a one-sided version of SCCP above. An advantage of using dynamic compensators in one channel (even in the cases where DSP can be solved by a constant feedback at that channel) has been illustrated in [2] and [3].

This paper has the following objectives.

- To rigorously establish the equivalence of the decentralized stabilization problem and the problem of making a multichannel system stabilizable and detectable through one channel via dynamic feedbacks around the rest of the channels.
- To develop (what is hoped to be) a sound basis for a fractional approach to decentralized control problems.
- To obtain an explicit characterization of all solutions to the decentralized stabilization problem.

Initial results concerning a) and b) can be found in [8] as mentioned above. The fractional approach there was "polynomial" in contrast to the "proper stable" fractional approach of this paper. Although the polynomial fractional approach has certain advantages (such as rendering the consideration of pole-placement problems, being more suitable in keeping track of the McMillan degree of the compensators, etc.), the proper stable fractional approach has notational simplicity and it is much more suitable for characterizing all solutions. There have been other attempts to use the stable proper fractional approach in local output feedback problems, such as that of Tan and Ikeda [12]. However, the extent to which the approach is used has not gone further than applying the standard result for the characterization of internally stabilizing compensators for single-channel systems. In [5], the DSP has recently been considered also in a stable proper fractional setup, and an alternative characterization of all solutions to DSP has also been (independently) obtained. The central representation of the two-channel plant in [5] is a left (or a right) coprime fractional representation.

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The fractional representation chosen below for the two-channel plant Σ is a bicoprime fractional representation rather than a left or a right coprime fractional representation. There are at least two reasons for this choice. The first reason is that as the bicoprime fractions are more directly related to the standard state-space representations of the type $H(zI - F)^{-1}G + J$, the translation of the results (solvability conditions, synthesis procedures, and the characterization of solutions) to state-space terminology and their system theoretic interpretations are much more straightforward. The fact that the fractional representational approach streamlines and leads to conceptually simpler proofs should not blur the need to translate the eventual results to state-space context for computational convenience (at least at present). The second reason is more technical and is forced by the nature of the particular problem at hand. Note that *the unstabilizable and the undetectable modes at the second channel* are the crucial quantities for all three problems. Hence, a solution procedure which is able to keep track of these modes in all computations *separately* is bound to yield results in which the problem data appear with least distortion.

In Section III, we attain all our aims a)–c). Theorem 3.2 shows the equivalence of SCCP and DSP and shows that the internal stability constraint in SCCPIS can be satisfied without any extra effort if SCCP is solvable. This has applications in reliable decentralized stabilization. Finally, in Theorem 3.3 and Theorem 3.4, we give a characterization of all solutions to SCCP, SCCPIS, and DSP.

II. PRELIMINARIES

In this section, we set up the terminology and notation and state various results which are used in obtaining the main results in the next sections. The concept of “completeness” is defined in a fractional context, and consequences of this definition are also examined in this section.

$R(z)$ denotes the field of rational functions of z with real coefficients. Its subring, the ring of stable proper rational functions of z with real coefficients, will be denoted by S . For basic properties of S and matrices over S , the reader is referred to Vidyasagar [14]. Both S and $R[z]$, the ring of polynomials in z , are subrings of the ring of stable rational functions $R_s(z)$.

Let P be a principal ideal domain. The usual notation $P^{k \times l}$ denotes the set of $k \times l$ matrices with entries in P , or matrices over P . When $A \in P^{k \times k}$, by $|A|$, we denote the determinant of A . Let $A \in P^{k \times a}$ and $B \in P^{k \times b}$ be such that $[A:B]$ is of full row rank, i.e., one $k \times k$ minor of $[A:B]$ is nonzero. Then, a greatest common left factor of A and B , denoted by $\text{gclf}(A, B)$, is a greatest left factor of $[A:B]$, i.e., a matrix L such that $[A:B] = L[A:B]$ with $[A:B] \in P^{k \times (a+b)}$ having a right inverse over P . Thus, $[A:B]$ is *left unimodular*, or equivalently, the pair (A, B) is *left coprime*. The set of all greatest common left factors of $[A:B]$ are given by LU , where $U \in P^{k \times k}$ is an arbitrary *unimodular* matrix over P . Observe that (A, B) is left coprime iff (UAV, UBW) is left coprime for unimodular matrices U, V, W of appropriate sizes. A triple of matrices (Q, R_1, R_2) is said to be left coprime iff the matrix $[Q:R_1:R_2]$ is left unimodular.

All these definitions and facts apply to matrices of the form $\begin{bmatrix} C \\ D \end{bmatrix}$, where $C \in P^{c \times l}$ and $D \in P^{d \times l}$ on replacing the words “row, left” by “column, right,” respectively. For example, a triple of matrices (P_1, P_2, Q) is right coprime iff the matrix $[Q:P_1:P_2]^T$ is right unimodular, where “prime” denotes transpose.

A $p \times m$ transfer matrix is an element of $R(z)^{p \times m}$ with each of its entries proper. It is well known that any transfer matrix Z admits fractional representations of the form

$$Z = PQ^{-1}R + W = P_r Q_r^{-1} = Q_l^{-1} R_l \quad (1)$$

for matrices of suitable sizes over S . The first representation is called *bicoprime* if the pair (P, Q) is right coprime and the pair

(Q, R) is left coprime, equivalently, if the matrices

$$\begin{bmatrix} Q \\ P \end{bmatrix}, \quad [Q \ R]$$

are *right* and *left unimodular*, respectively. If the pair (P_r, Q_r) is right coprime, then the second representation above is called a *right coprime fractional representation* of Z . Similarly, if (Q_l, R_l) is left coprime, then the third representation in (1) is called *left coprime*. Coprime representations of all types also exist for Z . Given a right and a left coprime fractional representation of Z as above, there exist matrices K, L, M, N of appropriate sizes such that the following (which is sometimes called a “doubly coprime representation of Z ”) hold:

$$\begin{bmatrix} Q_l & R_l \\ -L & K \end{bmatrix} \begin{bmatrix} M & -P_r \\ N & Q_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (2)$$

It is clear that if only a right (or left) coprime representation of Z is given, then one can determine Q_l, R_l, K, L, M, N (or P_r, Q_r, K, L, M, N) such that (2) holds.

The relation between fractional representations of the type (1) and the state-space realizations of Z has been examined in detail in Khargonekar and Özgüler [7]. The following fact is particularly illuminating.

Proposition 1.1: Given a realization $\Sigma = (F, G, H, J)$ of Z , there exists a fractional representation of the type $Z = PQ^{-1}R + W$ over S which is equivalent (in the sense of Fuhrmann) over $R_s(z)$ to the representation $H(zI - F)^{-1}G + J$ of Z , i.e., there exist matrices A, B, C, D over $R_s(z)$ such that (C, Q) is right coprime, $(A, zI - F)$ is left coprime over $R_s(z)$, and the equality

$$\begin{bmatrix} A & 0 \\ B & I \end{bmatrix} \begin{bmatrix} Q & R \\ -P & W \end{bmatrix} = \begin{bmatrix} zI - F & G \\ -H & J \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}$$

holds. Conversely, given a representation over S of the type $Z = PQ^{-1}R + W$ there exists a realization (F, G, H, J) such that the two representations $Z = PQ^{-1}R + W = H(zI - F)^{-1}G + J$ are strictly system equivalent over $R_s(z)$.

Proof: This follows easily from the results in [7, Section 5]. We omit the details of this proof. $\square$

The significance of such an equivalence over $R_s(z)$ is that the equivalent representations have the same “unstable modes.” For example, if $Z = PQ^{-1}R + W$ and $Z = H(zI - F)^{-1}G + J$ are the two corresponding equivalent realizations, then the first representation is bicoprime over $R_s(z)$ if and only if the state-space system (F, G, H, J) is stabilizable and detectable. This fact allows one to transform various problems concerning (F, G, H, J) to an equivalent problem concerning the fractional representation $Z = PQ^{-1}R + W$.

An example of this transformation is seen in the problem of internally stabilizing a canonical realization of Z . Let Z be strictly proper and let all representations in (1) be coprime. It is well known that (see [14]) the set of transfer matrices Z_{cs} , such that the pair (Z, Z_{cs}) is internally stable (equivalently, a canonical realization of Z is internally stabilized by a canonical realization of Z_{cs}), is given by

$$\{Z_{cs} = (N + Q_r X)(M - P_r X)^{-1} : X \text{ is over } S\}$$

or, equivalently by,

$$\{Z_{cs} = (K + YR_l)^{-1}(L - YQ_l) : Y \text{ is over } S\}$$

where the matrices M, N, K, L satisfy (2). It is also well known that the pair of transfer matrices (Z, Z_c) is internally stable iff the matrix

$$\begin{bmatrix} Q & RP_c \\ -R_c P & Q_c + R_c W_p \end{bmatrix}$$

is unimodular for some fractional representation $Z_c = P_c Q_c^{-1} R_c$ of Z_c over S .

In the next sections, we shall have to view the ring S as a normed algebra and consider the extended topology to matrices over S . The details of this viewpoint have been worked out in [14, Sect. 2.2 and Appendix C]. We are here mainly interested in results concerning "genericity" of sets, and adapt the following terminology. A subset $\mathcal{S}$ of S is called *generic* in S iff it is an open and dense subset of S viewed as a topological space. Similarly, a subset of $S^{k \times l}$ is said to be *generic* in $S^{k \times l}$ iff it is open and dense in $S^{k \times l}$. Alternatively, we sometimes say that a property holds true for *almost all elements* of $S^{k \times l}$ provided the set of elements with this property is open and dense in $S^{k \times l}$.

We now state three results concerning genericity of certain sets. The first of these states roughly (when B is nonsingular) in system theoretic words that "the closed-loop characteristic polynomial of a canonical system is coprime with a given fixed polynomial under *almost all* stable dynamic compensators." This is reminiscent of the well-known result for constant output feedbacks. The second lemma can be given a similar interpretation concerning "stable" systems, and is used to prove Lemma 2.3 which is a key result used in the proof of Theorem 3.2.

Lemma 2.1: Let $A \in S^{k \times k}$ be nonsingular and let $B \in S^{k \times k}$, $C \in S^{k \times c}$ be such that the pair (B, C) is left coprime. The set of X such that $(A, B + CX)$ is left coprime is generic in $S^{c \times k}$.

Proof: See the Appendix. $\square$

Lemma 2.2: Let $A \in S^{k \times b}$, $B \in S^{b \times b}$, and $C \in S^{b \times c}$, with B nonsingular. The set of X such that $(AB, I + ACX)$ is left coprime is generic in $S^{c \times k}$.

Proof: See the Appendix. $\square$

A consequence of Lemma 2.2 above is the following.

Lemma 2.3: Consider the matrices $\Theta \in S^{p \times s}$, $\Theta_1 \in S^{s \times p}$, $\Theta_2 \in S^{s \times m}$, $\Theta_3 \in S^{s \times s}$, and $\Psi \in S^{q \times m}$, $\Psi_1 \in S^{m \times q}$, $\Psi_2 \in S^{p \times q}$, $\Psi_3 \in S^{q \times q}$, where Θ_3 and Ψ_3 are nonsingular. Let $\text{gclf}(\Theta_2, \Theta_3)$ be a left factor of Θ , and $\text{grcf}(\Psi_2, \Psi_3)$ be a right factor of Ψ_1 . The set of X such that

$$\begin{aligned} (\Theta\Theta_3, I + \Theta\Theta_1 + \Theta\Theta_2 X) & \text{ is left coprime and} \\ (\Psi_3\Psi, I + \Psi_1\Psi + X\Psi_2\Psi) & \text{ is right coprime} \end{aligned}$$

is generic in $S^{m \times p}$.

Proof: See the Appendix. $\square$

A central concept to all local feedback problems is the concept of "completeness" first introduced in [4]. We now define completeness for fractional representations over S as follows.

Definition: Let $Z = PQ^{-1}R + W$ be a given fractional representation. The quadruple (P, Q, R, W) is *complete* (over S) if the Smith normal form over S of the system matrix

$$\Pi := \begin{bmatrix} Q & R \\ -P & W \end{bmatrix}$$

has at least $r := \text{size}(Q)$ identity elements, i.e., iff at least r invariant factors of Π over S are identity. (Note: Although a more descriptive term is perhaps "complete modulo stable modes," we shall use "complete" throughout the paper for the sake of brevity.)

This definition has the following consequences.

i) (P, Q, R, W) is complete iff there exist unimodular matrices $U \in S^{r+p \times r+p}$ and $V \in S^{r+m \times r+m}$ such that

$$U \begin{bmatrix} Q & R \\ -P & W \end{bmatrix} V = \begin{bmatrix} I_r & 0 \\ 0 & \Omega \end{bmatrix}$$

for some $\Omega \in S^{p \times m}$, where I_r denotes the identity matrix of size r . (Note that for any such Ω , we must have

$$\text{rank}(Z) = \text{rank}(\Omega)$$

since $\text{rank}(\Pi) = r + \text{rank}(Z)$.)

ii) It is possible that $Z = 0$ and (P, Q, R, W) is complete. (Example: $(0, I, 0, 0)$.)

iii) Let $\Sigma = (H, F, G, J)$ be a realization of Z such that $(H, zI - F, G, J)$ and (P, Q, R, W) are system equivalent over $R_s(z)$. Then, (P, Q, R, W) is complete iff

$$\text{rank} \begin{bmatrix} zI - F & G \\ -H & J \end{bmatrix} \geq \text{size}(F)$$

for all unstable eigenvalues of F .

iv) The quadruple (P, Q, R, W) is complete iff there exist matrices $K \in S^{r \times r}$, $L \in S^{r \times p}$, $M \in S^{r \times r}$, $N \in S^{m \times r}$ such that

$$\begin{bmatrix} K & L \end{bmatrix} \begin{bmatrix} Q & R \\ -P & W \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = I.$$

v) The quadruples (I, Q, R, W) and (P, Q, I, W) are complete for all P, Q, R, W .

This last fact can be interpreted to mean that a *detectable and/or stabilizable system is complete*.

The proofs of facts i)–v) are either easy or can be carried out similarly to results in [8]; hence they are omitted. A more involved consequence is stated in Lemma 2.5, the proof of which requires the following result.

Lemma 2.4: Let

$$U \begin{bmatrix} C & \Theta \\ 0 & \Gamma \end{bmatrix} V = \begin{bmatrix} I & 0 \\ 0 & \Lambda \end{bmatrix}$$

for some matrices $C \in S^{r \times r}$, $\Theta \in S^{r \times m}$, $\Gamma \in S^{p \times m}$, $\Lambda \in S^{p \times m}$ and for some unimodular matrices U and V , where I is the identity matrix of size r . Then, the pair (C, Θ) is left coprime if and only if the pair (U_{22}, Λ) is left coprime, where U_{22} is the lower right $p \times p$ submatrix of U .

Proof: Partitioning U and its inverse $\hat{U}$, we have

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix} = I \quad (3)$$

and

$$\begin{bmatrix} C & \Theta \\ 0 & \Gamma \end{bmatrix} V = \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12}\Lambda \\ \hat{U}_{21} & \hat{U}_{22}\Lambda \end{bmatrix}.$$

By unimodularity of V , (C, Θ) is left coprime iff $(\hat{U}_{11}, \hat{U}_{12}\Lambda)$ is left coprime. Now, let $(\hat{U}_{11}, \hat{U}_{12}\Lambda)$ be coprime so that $\hat{U}_{11}K + \hat{U}_{12}\Lambda L = I$ for some K and L . Then, using (3), it is easy to verify that

$$U_{22}(\hat{U}_{22} - \hat{U}_{22}\Lambda L\hat{U}_{12} - \hat{U}_{21}K\hat{U}_{12}) + \Lambda L\hat{U}_{12} = I.$$

Conversely, if (U_{22}, Λ) is left coprime, then for some K, L we have $U_{22}K + \Lambda L = I$ which implies by (3) that

$$\hat{U}_{11}(U_{11} - U_{12}KU_{21}) + \hat{U}_{12}\Lambda LU_{21} = I,$$

proving the result. $\square$

Lemma 2.5: Given a quadruple (P, Q, R, W) , let D be a left factor of Q such that (D, R) is left coprime. Write $\bar{Q} = DQ$ and let $D^{-1}R = \bar{R}\bar{D}^{-1}$ for a right coprime pair $(\bar{R}, \bar{D})$ of matrices over S . Also let $C := \text{grcf}(P, \bar{Q})$ and write $P = \hat{P}C$, $\bar{Q} = \hat{Q}C$. Then, the following statements are equivalent:

a) $(P, \bar{Q}, R, W)$ is complete;

b) $(P, \bar{Q}, \bar{R}, W, \bar{D})$ is complete;

c) There exist matrices $A \in S^{r \times r}$ and $B \in S^{r \times p}$ such that $A\hat{Q} + B\hat{P} = I$ and the pair $(C, A\bar{R} - B\bar{W}\bar{D})$ is left coprime.

Proof: [a] $\Rightarrow$ b): If a) holds, then, by iv) above, there exist K, L, M, N of appropriate sizes such that $K(QM + RN) + L(WN - PM) = I$. Let $\bar{D}^{-1}N = N_0\bar{D}_0^{-1}$ for a right coprime pair of matrices over S . Thus, $KD(\bar{Q}M\bar{D}_0 + \bar{R}N_0) +$

$L(W\bar{D}N_0 - PMD_0) = D_0$, where $(\bar{Q}MD_0 + \bar{R}N_0, D_0)$ is right coprime by right coprimeness of (N_0, D_0) and $(\bar{R}, \bar{D})$. Let $\bar{K}_0, L_0$ be such that $\bar{K}_0(\bar{Q}MD_0 + \bar{R}N_0) + L_0D_0 = I$ and note that $(K_0 + L_0KD)(\bar{Q}MD_0 + \bar{R}N_0)$

$$+ L_0L(W\bar{D}N_0 - PMD_0) = I.$$

This implies, by iv), that $(P, \bar{Q}, \bar{R}, W\bar{D})$ is complete.

[b) $\Rightarrow$ c)]: Let $\hat{P}\hat{Q}^{-1} = Q_1^{-1}R_1$ for a left coprime pair of matrices over S and note that there exist matrices $\hat{A}, \hat{B}, A_1, B_1$ satisfying

$$\begin{bmatrix} \hat{A} & \hat{B} \\ -R_1 & Q_1 \end{bmatrix} \begin{bmatrix} \hat{Q} & -B_1 \\ \hat{P} & A_1 \end{bmatrix} = I. \quad (4)$$

If b) holds, then there exist unimodular matrices $\bar{U}, \bar{V}$ such that the following holds with the matrix in Smith normal form:

$$\bar{U} \begin{bmatrix} \hat{Q}C & \bar{R} \\ -\hat{P}C & W\bar{D} \end{bmatrix} \bar{V} = \begin{bmatrix} I & 0 \\ 0 & \Lambda \end{bmatrix}. \quad (5)$$

Here, $\Lambda = \begin{bmatrix} \bar{\Lambda} \\ 0 \end{bmatrix}$ with $\bar{\Lambda}$ of full row rank equal to $\text{rank}(Z)$. Note that (4) and (5) together imply that

$$\bar{U} \begin{bmatrix} C & \hat{A}\bar{R} - \hat{B}W\bar{D} \\ 0 & Q_1W\bar{D} + R_1\bar{R} \end{bmatrix} \bar{V} = \begin{bmatrix} I & 0 \\ 0 & \Lambda \end{bmatrix}. \quad (6)$$

Now, $Q_1W\bar{D} + R_1\bar{R} = Q_1Z\bar{D}$, and hence it has rank equal to $\text{rank}(Z) = \text{rank}(\Lambda)$. Let $\hat{U}$ be a unimodular matrix such that

$$\hat{U}(Q_1W\bar{D} + R_1\bar{R}) = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}$$

for a full row rank matrix Γ . Note that Γ and $\bar{\Lambda}$ have the same number of rows. It follows, by (6), eliminating some zero entries, that

$$\hat{U} \begin{bmatrix} C & \Theta \\ 0 & \Gamma \end{bmatrix} \bar{V} = \begin{bmatrix} I & 0 \\ 0 & \bar{\Lambda} \end{bmatrix}$$

where $\Theta := \hat{A}\bar{R} - \hat{B}W\bar{D}$, for some matrix $\hat{U}$. It is not difficult to see that $\hat{U}$ is nonsingular and actually unimodular using various rank conditions and noting that the matrix on the right-hand side is the Smith normal form of the second matrix on the left. Partitioning $\hat{U}$ conformably, we get

$$\begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ \tilde{U}_{21} & \tilde{U}_{22} \end{bmatrix} \begin{bmatrix} C & \Theta \\ 0 & \Gamma \end{bmatrix} \bar{V} = \begin{bmatrix} I & 0 \\ 0 & \bar{\Lambda} \end{bmatrix}$$

which implies

$$\begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} + \tilde{U}_{11}\tilde{X} \\ \tilde{U}_{21} & \tilde{U}_{22} + \tilde{U}_{21}\tilde{X} \end{bmatrix} \begin{bmatrix} C & \Theta - \tilde{X}\Gamma \\ 0 & \Gamma \end{bmatrix} \bar{V} = \begin{bmatrix} I & 0 \\ 0 & \bar{\Lambda} \end{bmatrix}$$

for any $\tilde{X}$ over S . By Lemma 2.1, there exists $\tilde{X}$ such that $(\tilde{U}_{22} + \tilde{U}_{21}\tilde{X}, \bar{\Lambda})$ is left coprime. If $\tilde{X}$ is chosen as such, then, by Lemma 2.4, the pair $(C, \Theta - \tilde{X}\Gamma)$ is left coprime. Define

$$X := [\tilde{X} \quad \hat{X}] \hat{U},$$

where $\hat{X}$ is any matrix over S . Then, it is easy to see that the pair

$$[C, \hat{A}\bar{R} - \hat{B}W\bar{D} - X(Q_1W\bar{D} + R_1\bar{R})]$$

is left coprime. Letting

$$A := \hat{A} - XR_1, \quad B := \hat{B} + XQ_1$$

we see that $A\hat{Q} + B\hat{P} = I$ and $(C, A\bar{R} - BW\bar{D})$ is left coprime yielding c).

[c) $\Rightarrow$ a)]: If c) holds, then there exist matrices M and N such that

$$(A\hat{Q}C + B\hat{P}C)M + (A\bar{R} - BW\bar{D})N = I.$$

Let $AD^{-1} = \tilde{D}^{-1}\tilde{A}$ for some left coprime matrices over S . Then,

$$(\tilde{A}Q + \tilde{D}BP)M + (\tilde{A}R - \tilde{D}BW\bar{D})N = \tilde{D}$$

where the pair $(\tilde{A}R - \tilde{D}BW\bar{D}, \tilde{D})$ is left coprime by left coprimeness of $(\tilde{A}, \tilde{D})$ and (D, R) . Therefore, a) holds by iv). $\square$

A closer inspection of our proof of [a) $\Rightarrow$ c)] reveals a stronger fact.

Lemma 2.6: The quadruple (P, Q, R, W) is complete if and only if for almost all matrices A, B satisfying

$$A\hat{Q} + B\hat{P} = I,$$

we have that

$$(C, A\bar{R} - BW\bar{D}) \quad \text{is left coprime,} \quad (7)$$

where $\hat{Q}, \hat{P}, \bar{R}, \bar{D}, C$ are as in Lemma 2.5.

Proof: Considering the proof in Lemma 2.5 of "a) implies c)," the set of $\tilde{X}$ such that $(C, \Theta - \tilde{X}\Gamma)$ is left coprime is generic in $S^{r \times \text{rank}(Z)}$. Now $X := [\tilde{X} \quad \hat{X}]\hat{U}$, where $\hat{U}$ is unimodular and $\hat{X}$ is arbitrary in $S^{r \times p}$. Consequently, the set of X such that

$$[C, \hat{A}\bar{R} - \hat{B}W\bar{D} - X(Q_1W\bar{D} + R_1\bar{R})]$$

is left coprime is generic in $S^{r \times p}$. In other words, for almost any $A := \hat{A} - XR_1$ and $B := \hat{B} + XQ_1$, we have (7) holding. $\square$

The significance of Lemmas 2.5 and 2.6 is that *completeness is reduced to a (generic) coprimeness condition*.

III. PROBLEM DEFINITIONS AND MAIN RESULTS

This section contains the definitions of problems **P1-P3** of Section I in matrix fractional terminology, and the main results of the paper.

Consider a two-channel plant in matrix fractional representation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ = \left(\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} Q_{11}^{-1} \begin{bmatrix} R_1 & R_2 \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $P_1 \in S^{p \times r}$, $P_2 \in S^{q \times r}$, $Q_{11} \in S^{r \times r}$, $R_1 \in S^{r \times m}$, $R_2 \in S^{r \times s}$, $W_{11} \in S^{p \times m}$, $W_{12} \in S^{p \times s}$, $W_{21} \in S^{q \times m}$, $W_{22} \in S^{q \times s}$ with Q_{11} nonsingular.

We assume that the representation is bicoprime, i.e.,

$$(P_1, P_2, Q_{11}) \quad \text{is right coprime,}$$

$$(Q_{11}, R_1, R_2) \quad \text{is left coprime,}$$

and assume that

$$Z_{11} \quad \text{is strictly proper.}$$

Let a compensator $Z_c \in R(z)^{m \times p}$ be applied to the first channel by

$$u_1 = -Z_c y_1$$

resulting in the closed-loop plant

$$y_2 = \left(\begin{bmatrix} P_2 - W_{21}P_c \\ -P_1 \end{bmatrix} \begin{bmatrix} Q_{11} & R_1P_c \\ Q_c + W_{11}P_c \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} R_2 \\ W_{12} \end{bmatrix} + W_{22} u_2 \quad (8)$$

where $P_cQ_c^{-1}$ is a right coprime fractional representation of the compensator Z_c .

The problems to be considered are now defined.

Second Channel Canonicity Problem (SCCP): Given the two-channel plant in bicoprime fractional representation, determine a compensator in a right coprime fractional representation $Z_c = P_cQ_c^{-1}$ such that the resulting fractional representation (8) of the second channel subplant is bicoprime.

Second Channel Canonicity Problem with Internal Stability (SCCPIS): Given the two-channel plant, determine a solution Z_c to SCCP such that the pair of plants (Z_{11}, Z_c) is internally stable.

Decentralized Stabilization Problem (DSP): Given the two-channel plant, determine compensators $Z_{c1} \in \mathcal{R}(z)^{m \times p}$ and $Z_{c2} \in \mathcal{R}(z)^{s \times q}$ such that the pair of plants

$$\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \begin{bmatrix} Z_{c1} & 0 \\ 0 & Z_{c2} \end{bmatrix} \right)$$

is internally stable.

In view of our discussion in Section II, these rather technical definitions are equivalent to the problems posed in Section I in state-space terminology.

In [8], the one-sided versions of SCCP have been solved via a polynomial fractional approach and can be easily adapted to our present setup. The following results on SCCP can hence be viewed as an extension of the results in [8].

Consider the fractional representation

$$Z_{11} = P_1Q_{11}^{-1}R_1 + W_{11} \quad (9)$$

of the subplant Z_{11} . Let

$$C_1 := \text{grcf}(P_1, Q_{11}), \quad D_1 := \text{gclf}(Q_{11}, R_1)$$

so that

$$Q_{11} = Q_1C_1 = D_1Q_2, \quad P_1 = PC_1, \quad R_1 = D_1R_0$$

for a right coprime pair (P, Q_1) and a left coprime pair (Q_2, R_0) of matrices. Further, let

$$C := \text{grcf}(P_1, Q_2), \quad D := \text{gclf}(Q_1, R_1)$$

so that

$$P_1 = P_0C, \quad Q_2 = Q_0C, \quad Q_1 = DQ, \quad R_1 = DR$$

for a right coprime pair (P_0, Q_0) and a left coprime pair (Q, R) of matrices. By definitions of grcf and gclf , it follows that

$$C_1 = C_0C, \quad D_1 = DD_0$$

for some matrices C_0 and D_0 over S . It holds that

$$Q_0C_0 = D_0Q_0$$

where (Q, D_0) is left coprime and (Q_0, C_0) is right coprime by coprimeness of the pairs (Q, R) and (P_0, Q_0) , respectively. Moreover, both of the fractional representations

$$Z_{11} = P_0Q_0^{-1}R_0 + W_{11} = PQ^{-1}R + W_{11}$$

are bicoprime. Various coprimeness conditions above yield further that

$$(P, DQ) \text{ is right coprime, and} \quad (10)$$

$$(Q_0C, R_0) \text{ is left coprime.} \quad (11)$$

The assumptions of strict properness of $P_1Q_{11}^{-1}$ and $Q_{11}^{-1}R_1$ imply that PQ^{-1} , $Q^{-1}R$, $P_0Q_0^{-1}$, $Q_0^{-1}R_0$ are all strictly proper. This, combined with coprimeness of these four fractions, yields that Q , Q_0 are biproper and that P , R , P_0 , R_0 are strictly proper. (These properness conditions assure that all the matrix inverses used below exist, and that various properness requirements are met.)

Following Rosenbrock [10], we call the unstable zeros of $|C_1|$ the *unstable output decoupling zeros*; the unstable zeros of $|D_1|$ are the *unstable input decoupling zeros*; and the unstable zeros of $|C_0| = |D_0|$ are the *unstable input-output decoupling zeros* of the fractional representation (9) of Z_{11} . The reader will notice that it is these zeros which make SCCP and SCCPIS nontrivial problems; since if D_1 and C_1 are unimodular matrices, then any internally stabilizing compensator for $P_1Q_{11}^{-1}R_1 + W_{11}$ is a solution to both problems.

We now give a description of all compensators Z_c and of all internally stabilizing compensators Z_{cs} for (9). Let

$$PQ^{-1} = Q_l^{-1}R_l, \quad Q^{-1}R = P_rQ_r^{-1}$$

for a left coprime pair (Q_l, R_l) and a right coprime pair (P_r, Q_r) of matrices and note that there exist matrices $E, F, E_r, F_r, G, H, G_l, H_l$ satisfying

$$\begin{bmatrix} E & F \\ -R_l & Q_l \end{bmatrix} \begin{bmatrix} Q & -F_r \\ P & E_r \end{bmatrix} = I, \quad \begin{bmatrix} Q & R \\ -H_l & G_l \end{bmatrix} \begin{bmatrix} G & -P_r \\ H & Q_r \end{bmatrix} = I. \quad (12)$$

Now,

$$Z_{11} = Q_l^{-1}(R_lR + Q_lW_{11}) = (PP_r + W_{11}Q_r)Q_r^{-1} \quad (13)$$

are left and right coprime fractions for Z_{11} and, by (12), we have

$$\begin{bmatrix} Q_l & R_lR + Q_lW_{11} \\ -H_lF & G_l + H_l(ER - FW_{11}) \end{bmatrix} \cdot \begin{bmatrix} E_r + (PG - W_{11}H)F_r & -(PP_r + W_{11}Q_r) \\ HF_r & Q_r \end{bmatrix} = I. \quad (14)$$

It immediately follows that given any transfer matrix $Z_c = P_cQ_c^{-1}$ in right fractional representation, there exists a transfer matrix $X = X_2X_1^{-1}$ such that

$$\begin{bmatrix} Q_c \\ P_c \end{bmatrix} = \begin{bmatrix} E_r + (PG - W_{11}H)F_r & -(PP_r + W_{11}Q_r) \\ HF_r & Q_r \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad (15)$$

and any pair of matrices Q_c, P_c defined as above for some X_1, X_2 such that X_1 is nonsingular and $X_2X_1^{-1}$ is proper yields a transfer matrix $Z_c = P_cQ_c^{-1}$ in right fractional representation. Moreover, the pair (P_c, Q_c) is right coprime iff (X_2, X_1) is right coprime. Similarly, given any transfer matrix $\bar{Z}_c = \bar{Q}_c^{-1}\bar{P}_c$ in left fractional representation, there exist $Y = Y_1^{-1}Y_2$ such that

$$[\bar{Q}_c \quad -\bar{P}_c] = [Y_1 \quad -Y_2] \begin{bmatrix} G_l + H_l(ER - FW_{11}) & -H_lF \\ + (R_lR + Q_lW_{11}) & Q_l \end{bmatrix} \quad (16)$$

and conversely. The pair $(\bar{Q}_c, \bar{R}_c)$ is left coprime iff (Y_1, Y_2) is left coprime. It is easy to see from (14)–(16) that $Z_c = \bar{Z}_c$ iff $X_2 X_1^{-1} = Y_1^{-1} Y_2$.

Lemma 3.1: A transfer matrix Z_{cs} is such that the pair (Z_{11}, Z_{cs}) is internally stable if and only if

$$\begin{aligned} Z_{cs} &= (HF_r + Q_r X) [E_r + (PG - W_{11}H)F_r \\ &\quad - (PP_r + W_{11}Q_r)X]^{-1} \\ &= [G_l + H_l(ER - FW_{11}) - X(R_l R + Q_l W_{11})]^{-1} \\ &\quad \cdot (H_l F + XQ_l) \end{aligned}$$

for some $X \in S^{m \times p}$.

Proof: The equality (13) yields coprime fractions for Z_{11} . The result now follows directly by the matrix equality (14) and the standard result on the characterization of all internally stabilizing compensators. See, e.g., [14, Sect. 5]. $\square$

Considering now our two-channel plant, we note that the central issue pertaining to SCCP and SCCPIS is how the unstable input and output decoupling zeros of the subplant (9) interact with the second channel quantities P_2 and R_2 . We now examine this more closely and define some matrices that will be helpful in clarifying this interaction.

By right coprimeness of (P_1, P_2, Q_{11}) , the pair (P_2, C) as well as the pair (P_2, C_1) are right coprime. Similarly, by left coprimeness of (Q_{11}, R_1, R_2) , the pairs (D, R_2) and (D_1, R_2) are both left coprime. Let

$$P_2 C^{-1} = \bar{C}^{-1} \bar{T}, \quad D^{-1} R_2 = \bar{S} \bar{D}^{-1}$$

for some left coprime $(\bar{C}, \bar{T})$ and for some right coprime $(\bar{S}, \bar{D})$. Also let

$$\bar{T} \bar{C}_0^{-1} = \bar{C}_0^{-1} T, \quad \bar{D}_0^{-1} \bar{S} = \bar{S} \bar{D}_0^{-1}$$

for some left coprime $(\bar{C}_0, T)$ and for some right coprime $(S, \bar{D}_0)$. If we now define

$$\bar{C}_1 := \bar{C}_0 \bar{C}, \quad \bar{D}_1 := \bar{D}_0 \bar{D}$$

then

$$P_2 C_1^{-1} = \bar{C}_1^{-1} T, \quad D_1^{-1} R_2 = \bar{S} \bar{D}_1^{-1}$$

where $(\bar{C}_1, T)$ is left coprime and $(S, \bar{D}_1)$ is right coprime.

We can now state and prove our main result.

Theorem 3.2: The following statements are equivalent:

- i) SCCP is solvable;
- ii) SCCPIS is solvable;
- iii) DSP is solvable;
- iv) the quadruples $(P_1, Q_{11}, R_2, W_{12})$ and $(P_2, Q_{11}, R_1, W_{21})$ are both complete.

Proof: First observe by problem definitions that SCCP is solvable iff there exists $Z_c = P_c Q_c^{-1}$ such that the matrices below are right and left unimodular, respectively:

$$\begin{bmatrix} Q_{11} & R_1 P_c \\ -P_1 & Q_c + W_{11} P_c \\ -P_2 & W_{21} P_c \end{bmatrix}, \quad \begin{bmatrix} Q_{11} & R_1 P_c & R_2 \\ -P_1 & Q_c + W_{11} P_c & W_{12} \\ -P_2 & W_{21} P_c & W_{22} \end{bmatrix}. \quad (17)$$

Similarly, DSP is solvable iff there exists $Z_{c1} = P_{c1} Q_{c1}^{-1}$ and $Z_{c2} = P_{c2} Q_{c2}^{-1}$ such that the matrix below is unimodular:

$$\begin{bmatrix} Q_{11} & R_1 P_{c1} & R_2 P_{c2} \\ -P_1 & Q_{c1} + W_{11} P_{c1} & W_{12} P_{c2} \\ -P_2 & W_{21} P_{c1} & Q_{c2} + W_{22} P_{c2} \end{bmatrix}. \quad (18)$$

Comparing the matrix in (18) to those in (17), it is immediate that iii) implies i). By problem definitions again, it is also clear that ii) implies i). The fact that i) implies iv) already follows by [5, Theorem (3.3)] and by the dual of that result. Since the fractional setup here is different than that of [8] in spite of the obvious analogy, we now repeat the main steps of the proof. Considering (17) let $(Q_c + W_{11} P_c)^{-1} P_1 = LK^{-1}$ for some right coprime pair (L, K) of matrices. It follows that

$$\begin{bmatrix} Q_{11} K + R_1 P_c L \\ 0 \\ -P_2 K + W_{21} P_c L \end{bmatrix} = \begin{bmatrix} Q_{11} & R_1 P_c \\ -P_1 & Q_c + W_{11} P_c \\ -P_2 & W_{21} P_c \end{bmatrix} \begin{bmatrix} K \\ L \end{bmatrix}$$

is also right unimodular, i.e., there exist matrices M and N such that

$$M(Q_{11} K + R_1 P_c L) + N(-P_2 K + W_{21} P_c L) = I,$$

which implies by iv) of Section II that $(P_2, Q_{11}, R_1, W_{21})$ is complete. The completeness of $(P_1, Q_{11}, R_2, W_{12})$ follows by dual manipulations. This establishes that i) implies iv).

To complete the proof, we only need to show that iv) implies iii). Below, we establish this by showing that iv) implies i) and i) implies iii). The redundant step that iv) implies ii) is also shown below, since this has some significance on its own.

[iv) $\Rightarrow$ i) and ii): Let iv) hold. By completeness of $(P_1, Q_{11}, R_2, W_{12})$ and Lemma 2.5, there exist matrices E_0, F_0, M_0, N_0 satisfying

$$E_0 Q_0 + F_0 P_0 = I, \quad (19)$$

$$CM_0 + (E_0 S - F_0 W_{12} \bar{D}_1) N_0 = I. \quad (20)$$

Since $P_0 = PC_0$, equality (19) implies $C_0 E_0 D_0^{-1} Q + C_0 F_0 P = I$, where $E := C_0 E_0 D_0^{-1}$ is over S by left coprimeness of (D_0, Q_0) . Also letting $F := C_0 F_0$, we have

$$EQ + FP = I.$$

There also exist matrices F_r, E_r satisfying

$$\begin{bmatrix} E & F \\ -R_l & Q_l \end{bmatrix} \begin{bmatrix} Q & -F_r \\ P & E_r \end{bmatrix} = I. \quad (21)$$

It is easy to see, by $EF_r = C_0 F_0 E_r$, that

$$F_{r0} := D_0^{-1} F_r \quad (22)$$

is a matrix over S . Since $\bar{D}_1 = \bar{D} \bar{D}_0$ and $D_0 S = \bar{S} \bar{D}_0$, equality (20) implies

$$C_1 M_0 + (E \bar{S} - F W_{12} \bar{D}) \bar{D}_0 N_0 = C_0. \quad (23)$$

Observe that $C_0^{-1}(E \bar{S} - F W_{12} \bar{D}) = (E_0 S - F_0 W_{12} \bar{D}_1) \bar{D}_0^{-1}$, where both fractions are coprime by left coprimeness of (C_0, E) and $(D_0, \bar{S})$ and by right coprimeness of (E_0, D_0) and $(S, \bar{D}_0)$. It follows by (23) that there exist matrices M, N over S satisfying

$$C_1 M + (E \bar{S} - F W_{12} \bar{D}) N = I.$$

There also exist matrices N_l, M_l, S_r, C_r such that

$$\begin{bmatrix} C_1 & E \bar{S} - F W_{12} \bar{D} \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M & -S_r \\ N & C_r \end{bmatrix} = I. \quad (24)$$

By completeness of $(P_2, Q_{11}, R_1, W_{21})$ and by Lemma 2.5, there exist G, H, K, L such that

$$\begin{aligned} QG + RH &= I, \\ KD + L(TG - \bar{C}_1 W_{21} H) &= I. \end{aligned}$$

There also exist $H_l, G_l, D_l, T_l, L_r, K_r$ satisfying

$$\begin{bmatrix} Q & R \\ -H_l & G_l \end{bmatrix} \begin{bmatrix} G & -P_r \\ H & Q_r \end{bmatrix} = I. \quad (25)$$

$$\begin{bmatrix} K & L \\ -T_l & D_l \end{bmatrix} \begin{bmatrix} D & -L_r \\ TG - \bar{C}_1 W_{21} H & K_r \end{bmatrix} = I. \quad (26)$$

Note that, since $R = D_0 R_0$, the equality (25) implies that

$$C_0^{-1} G = G_0 D_0^{-1} \quad (27)$$

for some matrix G_0 over $\mathcal{S}$, where both fractions are coprime. Let us define

$$\Omega_{12} := R_l \bar{S} + Q_l W_{12} \bar{D}, \quad \Omega_{21} := TP_r + \bar{C}_1 W_{21} Q_r$$

and

$$A_1 := \Omega_{12} NGF_r, \quad A_2 := \Omega_{12} NP_r, \quad A_3 := \Omega_{12} C_r,$$

$$B_1 := H_l EL \Omega_{21}, \quad B_2 := R_l L \Omega_{21}, \quad B_3 := D_l \Omega_{21}.$$

Note by (25) that $QP_r = D_0 R_0 Q_r$, implying that $P_r = C_0 P_{r0}$ for some matrix P_{r0} over $\mathcal{S}$. Hence, $Q_0 P_{r0} = R_0 Q_r$, where $|Q_0| = |Q_r| = |\bar{Q}|$ and the pair (Q_0, R_0) is left coprime. It follows that (P_{r0}, Q_r) is right coprime. By left coprimeness of $(Q_0 C, R_0)$, it now follows that (C, P_{r0}) is left coprime. Let matrices J_1, J_2 be such that $CJ_1 + P_{r0} J_2 = I$. Then, using (22), (24), and (27), we have

$$C_r(N_l J_1 G_0 F_{r0}) + NP_r(J_2 G_0 F_{r0}) = NGF_r$$

which imply that $\text{gclf}(NP_r, C_r)$ is a left factor of NGF_r . On the other hand, by right coprimeness of (P, QD) , it follows that (R_l, D) is right coprime, i.e., $K_1 D + K_2 R_l = I$ for some K_1 and K_2 . Hence,

$$(H_l EK_1 L_r) D_r + (H_l EK_2) R_l L = H_l EL$$

which implies that $\text{grcf}(D_l, R_l L)$ is a right factor of $H_l EL$. Therefore, by Lemma 2.3, there exists $X = X_2 X_1^{-1} = Y_1^{-1} Y_2$ such that

$$(A_3, X_1 + A_1 X_1 + A_2 X_2) \text{ is left coprime, and} \quad (28)$$

$$(B_3, Y_1 + Y_1 B_1 + Y_2 B_2) \text{ is right coprime.} \quad (29)$$

(In particular, one can choose $Y_1 = I$ and $X_1 = I$.) Let us now consider the compensator $Z_c = P_c Q_c^{-1} = \bar{Q}_c^{-1} \bar{R}_c$, with $P_c, Q_c, \bar{Q}_c, \bar{R}_c$ defined by (15) and (16), where the matrices X_1, X_2, Y_1, Y_2 are chosen as above. We show that this solves SCCP.

Since

$$\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} R_2 \\ W_{12} \end{bmatrix} = \begin{bmatrix} \bar{S} \\ W_{12} \bar{D} \end{bmatrix} \bar{D}^{-1}$$

where both fractions are coprime, the second matrix in (17) is left unimodular if and only if the matrix

$$\begin{bmatrix} QC_1 & RP_c & \bar{S} \\ -PC_1 & Q_c + W_{11} P_c & W_{12} \bar{D} \end{bmatrix} \quad (30)$$

is left unimodular. Multiplying (30) on the left and right, respectively, by

$$\begin{bmatrix} E & -F \\ R_l & Q_l \end{bmatrix}, \quad \begin{bmatrix} M & 0 & -S_r \\ 0 & I & 0 \\ N & 0 & C_r \end{bmatrix}$$

both of which are unimodular, we obtain

$$\begin{bmatrix} I & P_r X_2 - GF_r X_1 & 0 \\ \Omega_{12} N & X_1 & (Q_l W_{12} \bar{D} + R_l \bar{S}) C_r \end{bmatrix}$$

which is clearly left unimodular iff the pair

$$[\Omega_{12} C_r, X_1 - \Omega_{12} N(P_r X_2 - GF_r X_1)]$$

is left coprime. This holds by our choice of X_1 and X_2 . On the other hand, the first matrix in (17) is right unimodular iff the matrix

$$\begin{bmatrix} Q_{11} & R_1 \\ -\bar{R}_c P_1 & \bar{Q}_c + \bar{R}_c W_{11} \\ -P_2 & W_{21} \end{bmatrix} \quad (31)$$

is right unimodular, by the fact that $P_c Q_c^{-1} = \bar{Q}_c^{-1} \bar{R}_c$, where both fractions are coprime. Since

$$\begin{bmatrix} -P_2 & W_{21} \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix}^{-1} = \bar{C}_1^{-1} [-T \quad \bar{C}_1 W_{21}]$$

where both fractions are coprime, it follows that (31) is right unimodular iff the matrix

$$\begin{bmatrix} DQ & DR \\ -\bar{R}_c P & \bar{Q}_c + \bar{R}_c W_{11} \\ -T & \bar{C}_1 W_{21} \end{bmatrix} \quad (32)$$

is right unimodular. Multiplying (32) on the left and right, respectively, by the unimodular matrices

$$\begin{bmatrix} K & 0 & I \\ 0 & I & 0 \\ -T_l & 0 & D_l \end{bmatrix}, \quad \begin{bmatrix} G & -P_r \\ H & Q_r \end{bmatrix}$$

we obtain

$$\begin{bmatrix} I & -L \Omega_{21} \\ Y_1 H_l E - Y_2 R_l & Y_1 \\ 0 & D_l (TP_r + \bar{C}_1 W_{21} Q_r) \end{bmatrix}$$

which is right unimodular iff the pair

$$[D_l \Omega_{21}, Y_1 + (Y_1 H_l E - Y_2 R_l) L \Omega_{21}]$$

is right coprime. This holds by our choice of Y_1 and Y_2 . Therefore, the matrices in (17) are right and left unimodular, respectively, proving that Z_c above is a solution to SCCP provided $X = X_2 X_1^{-1} = Y_1^{-1} Y_2$ is chosen to satisfy (28) and (29). On the other hand, the special choice $X_1 = I, Y_1 = I$, and $X_2 = Y_2 = X$, where X is such that

$$(A_3, I + A_1 + A_2 X) \text{ is left coprime,} \quad (33)$$

$$(B_3, I + B_1 + XB_2) \text{ is right coprime} \quad (34)$$

is possible by Lemma 2.3. The compensator $Z_c = P_c Q_c^{-1} = \bar{Q}_c^{-1} \bar{R}_c$ defined through the equalities (15) and (16) is shown above to solve SCCP. Moreover, by Lemma 3.1, the pair of transfer matrices (Z_{c1}, Z_c) is internally stable. $\square$

[i] $\Rightarrow$ [iii]: Let $Z_{c1} = P_c Q_c^{-1}$ be a solution to SCCP so that the representation (8) is bicoprime. Let Z_{c2} be any internally stabilizing compensator for the transfer matrix of (8). It follows that the matrix in (18) is unimodular, where $P_{c1} := P_c, Q_{c1} := Q_c$. Consequently, the compensators Z_c and Z_{c2} solve DSP. $\square$

Remark: Observe that the left or right fractional representations where $[P'_1 : P'_2]' = I$ or $[R_1 : R_2] = I$ are special cases of

our two-sided representation. Although it is possible to formulate the notion of completeness (see [1]) and specialize our results to such representations, in the final results (especially in the matrices A_i , B_i of Theorem 3.2) the problem data will appear in a much more alienated form. (The reader may compare our results and those in [5] where the two-channel plant is in fact represented in a one-sided coprime fraction.) $\triangle$

The proof of Theorem 3.2 suggests that one can give a characterization of all solutions to SCCP, SCCPIS, and DSP. For this purpose, let $\mathcal{X}_c$ denote the set of all solutions to SCCP and let $\mathcal{X}_{cs}$ denote the set of all solutions to SCCPIS. Let us also denote

$$\mathcal{X} := \left\{ \begin{array}{l} X = X_2 X_1^{-1} = Y_1^{-1} Y_2 : X_2, \\ Y_2 \in S^{m \times p}, X_1 \in S^{m \times m}, Y_1 \in S^{p \times p}, \\ (A_3, X_1 + A_1 X_1 + A_2 X_2) \quad \text{is left coprime,} \\ (B_3, Y_1 + Y_1 B_1 + Y_2 B_2) \quad \text{is right coprime} \end{array} \right\}$$

and

$$\mathcal{X}_s := \left\{ X \in S^{m \times p} : \begin{array}{l} (A_3, I + A_1 + A_2 X) \\ (B_3, I + B_1 + X B_2) \end{array} \begin{array}{l} \text{is left coprime and} \\ \text{is right coprime} \end{array} \right\}.$$

Theorem 3.3: Let $(P_1, Q_{11}, R_2, W_{12})$ and $(P_2, Q_{11}, R_1, W_{21})$ be complete. Then,

$$\begin{aligned} \mathcal{X}_c &= \{ (HF_r X_1 + Q_r X_2) [E_r X_1 + (PG - W_{11}H)F_r X_1 \\ &\quad - (PP_r + W_{11}Q_r)X_2]^{-1} : X := X_2 X_1^{-1} \in \mathcal{X} \}, \\ \mathcal{X}_{cs} &= \{ (HF_r + Q_r X) [E_r + (PG - W_{11}H)F_r \\ &\quad - (PP_r + W_{11}Q_r)X]^{-1} : X \in \mathcal{X}_s \}. \end{aligned}$$

Proof: It is clear by the proof of Theorem 3.2 that any $Z_c \in \mathcal{X}_c$ is a solution to SCCP, and similarly, any $Z_{cs} \in \mathcal{X}_{cs}$ is a solution to SCCPIS. The fact that any solution Z_c of SCCP is in $\mathcal{X}_c$ follows by tracing backwards: any solution $Z_c = P_c Q_c^{-1} = \bar{Q}_c^{-1} \bar{R}_c$ to SCCP can be expressed by (15) and (16) for some $X = X_2 X_1^{-1} = Y_1^{-1} Y_2$ which are such that (28) and (29) hold. Similarly, any solution to SCCPIS can be expressed as in Lemma 3.1 for some X which satisfies (33) and (34) with $X_1 = I$, $Y_1 = I$ and with $X := X_2 = Y_2$. $\square$

Let us denote

$$\hat{Z}_{22} := [P_2 \quad -W_{21}P_c] \begin{bmatrix} Q_{11} & R_1 P_c \\ -P_1 & Q_c + W_{11}P_c \end{bmatrix}^{-1} \begin{bmatrix} R_2 \\ W_{12} \end{bmatrix} + W_{22}$$

which is the transfer matrix of the closed-loop system around the second channel when $Z_c = P_c Q_c^{-1}$ is applied to the first channel [see (8)]. The following result yields a characterization of all solutions to DSP via the characterization of all solutions to SCCP.

Theorem 3.4: The sets of all solutions $\mathcal{X}_c$ to DSP are given by

$$\mathcal{X}_c = \left\{ \begin{bmatrix} Z_{c1} & 0 \\ 0 & Z_{c2} \end{bmatrix} : Z_{c1} \in \mathcal{X}_c, (Z_{c2}, \hat{Z}_{22}) \right. \\ \left. \text{is internally stable} \right\}.$$

Proof: If Z_{c1} , Z_{c2} solve DSP, then the matrix in (18) is unimodular. Clearly, Z_{c1} is a solution to SCCP and, hence, by Theorem 3.3, we have $Z_{c1} \in \mathcal{X}_c$. On the other hand, given any element of $\mathcal{X}_c$, it is easy to see that the matrix in (18) is unimodular. $\square$

Remarks:

i) The significance of this result is that for any fixed solution Z_{c1} of SCCP, one can obtain an explicit characterization of all Z_{c2} such that Z_{c1} , Z_{c2} together solve DSP easily via the standard characterization for all internally stabilizing compensators—Lemma 3.1. This allows one to reflect with ease any further constraints imposed on Z_{c2} through the free parameter that enters. The weakness of the result, however, lies in the fact that the characterization of Theorem 3.3 does not involve truly free parameters X_1 , X_2 . These are “almost free” by the result on genericity. This type of characterization result seems to be inevitable in local feedback problems (compare [5]). We also note that the roles played by channels 1 and 2 can be changed leading to a slightly different characterization.

ii) One obvious use of Theorems 3.3 and 3.4 is in pinpointing the source of trouble in the nongeneric cases. In fact, by these results, the “bad” choices for the compensators result from the choices for X that do not satisfy the coprimeness conditions, i.e., that are not members of the sets $\mathcal{X}$ or $\mathcal{X}_s$. A closer look at these sets in a given application will yield a more detailed classification of “bad” compensators.

iii) Observe that the subclass of $\mathcal{X}_c$ in which Z_{c1} is chosen from $\mathcal{X}_{cs}$ leads to a *reliable synthesis for the loop around the first channel*. In fact, if for some reason the loop around channel 2 is opened, then the remaining loop being internally stable is protected against component damages. There has been some interest in reliable decentralized control. See, e.g., Siljak [11], Ozgüler and Hiraoglu [9], Tan and Ikeda [12]. For N -channel systems, the above procedure is generalized to yield a “hierarchically reliable” synthesis procedure for decentralized controllers [13]. $\triangle$

IV. CONCLUSIONS

The results of Section III apply with minor changes to systems having fractional representations over a principal ideal domain P and to some more general rings although the exposition was restricted to the more familiar ring S . Examples of various rings of system theoretic significance have been listed in Desoer *et al.* [6]. Thus, for instance, the results of Section III can also be applied in decentralized control of *lumped-distributed multi-variable plants* by letting P be the ring of matrices whose entries are H_∞ functions on the imaginary axis.

We have examined the two-channel case which can be considered as the “core problem” of the N -channel case. The generalization to N -channel systems is possible, and all our results extend to N -channel systems.

There are many directions for further research even in the two-channel case considered here: i) now that an explicit description of all solutions to DSP is available, it is worthwhile to examine the effect of various H_∞ constraints on the outputs of the resulting system; ii) interpretation of the completeness conditions for various domains P ; iii) stating “weak” sufficient conditions on the interaction subplants Z_{12} and Z_{21} for the completeness conditions to be automatically satisfied; and iv) giving upper bounds on the number of unstable poles of the decentralized compensators via a closer inspection of the set $\mathcal{X}_c$.

APPENDIX

In this Appendix, we give proofs of Lemmas 2.1, 2.2, and 2.3 of Section II.

Proof of Lemma 2.1: We need to show that the set

$$\mathcal{X} := \{ X \in S^{c \times k} : (A, B + CX) \text{ is left coprime} \}$$

is open and dense in $S^{c \times k}$. If $C = 0$, then by left coprimeness

of (B, C) , B is unimodular and thus $\mathcal{X} = S^{c \times k}$. We hence assume that $C \neq 0$. Let $X_0 \in \mathcal{X}$ so that $AK + (B + CX_0)L = I$ for some $K, L \in S^{k \times k}$. If $L = 0$, then A is unimodular and $\mathcal{X}$ is trivially open in $S^{c \times k}$. If $L \neq 0$, then $\|C\| \|L\| > 0$. For all matrices X over S satisfying $\|X - X_0\| < (\|C\| \|L\|)^{-1}$, we have

$$AK + (B + CX)L = I + C(X - X_0)L$$

where the matrix on the right-hand side is unimodular since $\|C(X - X_0)L\| < 1$. Consequently, $(A, B + CX)$ is left coprime for all such X proving that $\mathcal{X}$ is open in $S^{c \times k}$. On the other hand, consider an element $X_1 \in S^{c \times k}$ not in $\mathcal{X}$ so that for some $G \in S^{k \times k}$ such that $|G|$ is nontrivial, we have

$$A = GA_1, \quad B + CX_1 = GB_1,$$

$$(A_1, B_1) \quad \text{is left coprime,}$$

for some A_1, B_1 , over S . By left coprimeness of (B, C) , the pair (C, GB_1) is also left coprime. Let $Y_1 \in S^{c \times k}$ be such that $\|Y_1\| < \epsilon/2$ and $GB_1 + CY_1$ is nonsingular. Such a matrix exists by [14, Lemma (11), Sect. 5.2]. Now, if $(A, GB_1 + CY_1)$ is left coprime, then $X := X_1 + Y_1$ is such that $\|X - X_1\| < \epsilon$ and $(A, B + CX)$ is left coprime. Otherwise, there exists $G_1 \in S^{k \times k}$ such that $|G_1|$ is nontrivial and

$$A = G_1 A_2, \quad B + C(X_1 + Y_1) = G_1 B_2,$$

$$(A_2, B_2) \quad \text{is left coprime,}$$

for some A_2, B_2 over S . By left coprimeness of (B, C) , the pair $(G_1 B_2, C)$ is also left coprime. Working through the Smith-McMillan form of $(G_1 B_2)^{-1}C$ over S [14], it is straightforward to construct a matrix $Y_2 \in S^{c \times k}$ such that $\|Y_2\| < \epsilon/2$ and $(|A|, |G_1 B_2 + CY_2|)$ is coprime. This yields that $(A, G_1 B_2 + CY_2)$ is left coprime. Letting $X := X_1 + Y_1 + Y_2$, it follows that $\|X - X_1\| < \epsilon$ and $(A, B + CX)$ is left coprime; proving that $\mathcal{X}$ is dense in $S^{c \times k}$. $\square$

Proof of Lemma 2.2: Let U be a unimodular matrix such that $UA = \begin{bmatrix} \bar{A} \\ 0 \end{bmatrix}$ and $\bar{A}$ is of full row rank. Set $\bar{X} := XU^{-1} = [\bar{X}_1; \bar{X}_2]$ with $\bar{X}_1 \in S^{c \times \text{rank}(A)}$. It is easy to see that $(AB, I + ACX)$ is left coprime iff $(\bar{A}B, I + \bar{A}C\bar{X}_1)$ is left coprime. Let V be a unimodular matrix such that $\bar{A}BV = [\bar{A}; 0]$ for some full column rank $\bar{A}$. By nonsingularity of B and by the fact that $\bar{A}$ is of full row rank, it follows that $\bar{A}$ is square (and nonsingular). Further, $(AB, I + ACX)$ is left coprime iff $(\bar{A}, I + \bar{A}C\bar{X}_1)$ is left coprime. By Lemma 2.1, the set of $\bar{X}_1$ for which the latter holds is generic in $S^{c \times \text{rank}(A)}$. Since $\bar{X}_2$ is completely arbitrary, it easily follows that the set of $X = \bar{X}U$ for which the former holds is generic in $S^{c \times k}$. $\square$

Proof of Lemma 2.3: By the fact that $\text{gclf}(\Theta_2, \Theta_3)$ divides Θ_1 , there exist matrices J_2, J_3 such that

$$\Theta_2 J_2 + \Theta_3 J_3 = \Theta_1.$$

Similarly, there exist matrices K_2, K_3 such that

$$K_2 \Psi_2 + K_3 \Psi_3 = \Psi_1.$$

Clearly, the assertion of the lemma holds if and only if

$$[\Theta \Theta_3, I + \Theta \Theta_2(X + J_2)]$$

is left coprime and

$$[\Psi_3 \Psi, I + (X + K_2) \Psi_2 \Psi]$$

is right coprime. In each case, the set of such X is open and dense in $S^{m \times p}$, by Lemma 2.2 and by the translation invariance [10] of generic sets. Therefore, the set of X simultaneously satisfying both conditions is also open and dense in $S^{m \times p}$. $\square$

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