An Iterative Algorithm for Signal Reconstruction from Bispectrum

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Abstract—This paper presents a procedure to reconstruct a (minimum or nonminimum phase) discrete-time signal from its bispectrum. The algorithm is iterative and based on using the method of projections onto convex sets (POCS) in an inner product space in which the vector space consists of real sequences, and vector addition is defined in terms of the convolution operation. Prior information, such as an energy bound on the signal, can be incorporated into this algorithm. An algorithm that employs energy information was found to produce better results in simulation than one that ignores energy constraints.

I. INTRODUCTION

TECHNIQUES based on higher order correlations [1], [2] have recently been applied to many problems in a number of diverse fields (e.g., see [3]–[13]) such as optics [3], [4], image processing [13], and geophysical signal processing [7]–[9]. This paper presents an algorithm to reconstruct a (minimum or nonminimum phase) discrete-time signal from its bispectrum. This iterative algorithm is devised by using the method of projections onto convex sets (POCS) [14], [15] in an inner product space framework, l^* [16], [17], in which the vector space consists of real sequences, and vector addition is defined in terms of the convolution operation. Signals with a prescribed bispectrum constitute a closed and convex set in this framework, a condition which is not valid in the commonly used l_2 or (L_2) Hilbert space.

The method of POCS is an effective numerical technique for signal recovery (or reconstruction) with partial information in the form of some given constraints. Algorithms devised by using the method of POCS have been applied to some important signal reconstruction [18], [19] and synthesis problems [20], [21].

The signal reconstruction problem can be described as follows: Let $x = \{x(n)\}$ be a real sequence. The autotriplecorrelation of x is defined as follows:

$$r_x(n_1, n_2) = \sum_{n=-\infty}^{\infty} x(n)x(n+n_1)x(n+n_2). \quad (1)$$

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The bispectrum, $B_x(\omega_1, \omega_2)$, of the sequence $\{x(n)\}$ is defined as the Fourier transform of the two-dimensional autotriplecorrelation sequence, i.e.,

$$B_x(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} r_x(n_1, n_2) \cdot \exp[-j(\omega_1 n_1 + \omega_2 n_2)]. \quad (2)$$

It can be easily shown that [1]–[4]

$$B_x(\omega_1, \omega_2) = X(\omega_1)X(\omega_2)X^*(\omega_1 + \omega_2) \quad (3)$$

where $X(\omega)$ is the Fourier transform of x , i.e.,

$$F[x](\omega) \stackrel{\text{def}}{=} X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \exp(-j\omega n). \quad (4)$$

In this paper, the problem of reconstructing the sequence x from its bispectrum $B_x(\omega_1, \omega_2)$ is considered. This signal reconstruction problem arises in many practical situations including optics [3], [4] and geophysics [7]–[9].

Other prior information such as an energy bound on the desired signal x can also be used in our algorithm.

Section II reviews the method of POCS. In Section III the inner product space framework l^* and some closed and convex sets in l^* are described. In Section IV the iterative signal reconstruction algorithm and some simulation examples are presented.

II. THE METHOD OF PCOS

The method of POCS [14] is used to devise the iterative signal reconstruction algorithm. In this method the basic approach is to use a Hilbert space framework for signal representation and to associate closed and convex sets with available information and constraints. In this signal reconstruction problem the prescribed bispectrum and an upper bound on the signal energy constitute the available prior information. An iterative signal reconstruction algorithm can be obtained by making alternating orthogonal projections onto the convex sets corresponding to constraints and prior information.

Let the sets, C_m , $m = 1, 2, \dots, M$, which are associated with available constraints and prior information be closed and convex in a Hilbert space. If these constraints are compatible, then the intersection

$$C_o = \bigcap_{m=1}^M C_m \quad (5)$$

is a nonempty, closed, and convex set which contains the desired signal. Any vector in the set C_o is a feasible solution to the signal reconstruction problem. In order to find a vector in the set C_o a procedure based on the following theorem (Bregman's theorem) can be used.

Theorem (Bregman) [14]: Let $P_m, m = 1, 2, \dots, M$, denote the projection operators onto the sets, $C_m, m = 1, 2, \dots, M$, which are closed and convex in a Hilbert space. Let y_o be an arbitrary vector in the Hilbert space. Define a sequence of iterates $\{y_k\}$ as follows:

$$y_{k+1} = P_{k \bmod (M) + 1} y_k, \quad k \geq 0. \quad (6)$$

If $C_o = \bigcap_{m=1}^M C_m \neq \emptyset$, then the sequence $\{y_k\}$ converges weakly to some $y^* \in C_o$.

There is no restriction that the number of sets to be finite. Convexity of the sets, $C_m, m = 1, 2, \dots, M$, are essential for the convergence of the iterates y_k . Other versions of this theorem can be found in [14].

III. THE INNER PRODUCT SPACE l^* AND SOME CONVEX SETS IN l^*

This section describes the inner product space l^* which is related to a Banach algebra $l_1(\mathbf{Z})$ where the vector multiplication is defined as the convolution operation [16], [17]. The inner product space l^* is used in minimum-phase signal recovery from Fourier transform magnitude in [17].

The main idea of the method of POCS is to associate convex sets to the constraints and prior information about the desired signal. In this method corresponding to the prescribed bispectrum values $B_g(\omega_1, \omega_2)$ convex sets are associated. The signals whose bispectrums are equal to a prescribed value, $B_g(\omega_{1i}, \omega_{2i})$, at a frequency pair, $(\omega_{1i}, \omega_{2i})$, do not constitute a convex set in the commonly used l_2 Hilbert space. In other words, the sets

$$C_{i,l} \stackrel{\text{def}}{=} \{x: X(\omega_{1i})X(\omega_{2i})X^*(\omega_{1i}, \omega_{2i}) = B_g(\omega_{1i}, \omega_{2i}), \} \\ i = 1, 2, \dots, I; l = 1, 2, \dots, L \quad (7)$$

are not convex in the l_2 Hilbert space. However, the sets defined in (7) are convex in a recently proposed Hilbert space framework [16], [17] in which the main idea is to allow the use of convolution operation for vector addition with suitably defined scalar multiplication. The resulting vector space is referred as l^* . This vector space framework is now briefly described.

Let us consider the vector space $(V, \mathbf{R}, *, \odot)$ where V is the set of sequences $y = \{y(n), n = 0, \pm 1, \pm 2, \dots\}$ satisfying the following conditions:

$$\sum_{n=-\infty}^{\infty} |y(n)| < \infty, \quad y(n) \in \mathbf{R} \quad (8)$$

and

$$F[y](\omega) = Y(\omega) \neq 0 \quad \text{for all } \omega \in [-\pi, \pi]. \quad (9)$$

Here $\mathbf{R}$ denotes the set of real numbers and the set V consists of the regular elements of the Banach algebra $l_1(\mathbf{Z})$ [22], [23].

The Fourier transform of any sequence $y \in V$ can be expressed as

$$Y(\omega) = |Y(\omega)| \exp(j\Psi(\omega)) \quad (10)$$

where in view of (8) and (9) $\Psi(\omega)$ can be defined as a continuous function with $\Psi(0) = 0$. While $|Y(\omega)|$ is a periodic function of ω with period 2π , $\Psi(\omega)$ is not necessarily a periodic function. But note that for any $y \in V$ there exists an integer $K = \Psi(2\pi)/2\pi$ such that $\Psi(\omega) - K\omega$ is a periodic function of ω with period 2π [23], [24]. For any $y \in V$ let us define

$$\log(Y(\omega)) = \log(|Y(\omega)|) + j\Phi(\omega) \quad (11)$$

where $\Phi(\omega) = \Psi(\omega) - K\omega$. In (11) both $\log(|Y(\omega)|)$ and $\Phi(\omega)$ are periodic and continuous functions. $\Phi(\omega)$ is called the unwrapped phase of the sequence y . In this manner the set V has been partitioned into equivalence classes such that $z \equiv x$ if and only if $x = D_{n_o}z$ for some integer n_o where $(D_{n_o}z)(n) = z(n - n_o)$.

Note that for any $y \in V$ the following integral exists [17], [23]:

$$\int_{-\pi}^{\pi} |\log(Y(\omega))|^2 d\omega < \infty \quad (12)$$

where $\log(Y(\omega))$ is defined as above.

Vector "addition" $y_1 * y_2$ of any two vectors y_1 and $y_2 \in V$ is defined by

$$(y_1 * y_2)(n) = \sum_{k=-\infty}^{\infty} y_1(n-k)y_2(k), \\ n = 0, \pm 1, \pm 2, \dots \quad (13)$$

The additive identity in this vector space is the unit sample sequence, δ where

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Also y^{-1} , the additive inverse of y , is defined as

$$y^{-1}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{Y(\omega)} \exp(-j\omega n) d\omega \\ n = 0, \pm 1, \pm 2, \dots \quad (15)$$

Clearly, $y^{-1} * y = \delta$. Scalar multiplication is defined by assigning to each $c \in \mathbf{R}$ and $y \in V$ a vector $c \odot y \in V$ such that

$$c \odot y = F^{-1} \{\exp(c \log(Y(\omega)))\} = F^{-1}[Y^c(\omega)] \quad (16)$$

where $\log(Y(\omega))$ is defined as in (11).

The inner product (x, y) for any x and $y \in V$ are defined as follows:

$$(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log X(\omega) (\log(Y(\omega)))^* d\omega. \quad (17)$$

It is easy to verify that the axioms of the inner product are satisfied [17]. Let us define the norm on the vector space $(V, \mathbf{R}, *, \odot)$ as that induced by the inner product. The

norm $\|y\|_*$ of any y is given by

$$\|y\|_* = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\log(Y(\omega))|^2 d\omega \right)^{1/2}. \quad (18)$$

Note that since any inner product space can be completed [25], there is a Hilbert space which is the completed version of the inner product space l^* .

Some closed and convex sets which may be used in the signal reconstruction algorithms in l^* are defined as follows:

1) C_e : The set of sequences whose energy is bounded by ϵ_o , i.e.,

$$C_e \stackrel{\text{def}}{=} \{x: \|x\|_*^2 \leq \epsilon_o\}. \quad (19)$$

The proof of convexity and closure of the set C_e in l^* is given in Appendix A.

2) $C_{i,l}$, $i = 1, 2, \dots, I$; $l = 1, 2, \dots, L$: These sets are defined in (7). Their proof of convexity and closure is also given in Appendix A. These sets are not convex in l_2 .

3) $C_{i,l}^m$: The set of sequences whose bispectral magnitudes are equal to a prescribed value, $|B_g(\omega_{1i}, \omega_{2l})|$, at $(\omega_{1i}, \omega_{2l})$, i.e.,

$$C_{i,l}^m \stackrel{\text{def}}{=} \{x: |X(\omega_{1i})X(\omega_{2l})X(\omega_{1i} + \omega_{2l})| = |B_g(\omega_{1i}, \omega_{2l})|, x \in l^*\}. \quad (20)$$

The sets, $C_{i,l}^m$, are not convex in l_2 , either.

4) $C_{i,l}^p$: The set of sequences whose bispectral phases are equal to a prescribed value $\arg[B_g(\omega_{1i}, \omega_{2l})]$ at $(\omega_{1i}, \omega_{2l})$, i.e.,

$$C_{i,l}^p \stackrel{\text{def}}{=} \{x: \arg[X(\omega_{1i})X(\omega_{2l})X(\omega_{1i} + \omega_{2l})] = \arg[B_g(\omega_{1i}, \omega_{2l})]\}. \quad (21)$$

It should be noted that the bispectral phase values $\arg[B_g(\omega_{1i}, \omega_{2l})]$ are unwrapped phase values. Unwrapped bispectrum phase values can be obtained by using the algorithm described in [26].

The proofs of convexity and closure of the sets $C_{i,l}^m$ and $C_{i,l}^p$ can be established as the proof of $C_{i,l}$. Note that $C_{i,l}^p \cap C_{i,l}^m = C_{i,l}$.

5) C^m : The set of sequences whose Fourier transform magnitude is equal to a prescribed magnitude, $|X_g(\omega)|$, i.e.,

$$C^m \stackrel{\text{def}}{=} \{x: |X(\omega)| = |X_g(\omega)|, x \in l^*\}. \quad (22)$$

The proof of convexity and closure of C^m is given in [16].

In the signal reconstruction algorithm the sets C_e , $C_{i,l}^m$ and $C_{i,l}^p$ are used. Other closed and convex sets in l^* are described in [16], [17].

IV. ITERATIVE ALGORITHM AND SIMULATION EXAMPLES

In this section the sets which are defined in the previous section and Bregman's theorem are used to devise an iterative signal reconstruction algorithm in l^* .

Let us assume that the bispectrum, $B_g(\omega_1, \omega_2)$, of the sequence x_d is available at frequency values $(\omega_{1i}, \omega_{2l})$, $i = 1, 2, \dots, I$; $l = 1, 2, \dots, L$. The sets, $C_{i,l}$, $C_{i,l}^m$, $i = 1, 2, \dots, I$; $l = 1, 2, \dots, L$, which are defined in the previous section are constructed. The desired sequence x_d must be an element of these sets $C_{i,l}$, $C_{i,l}^m$, $i = 1, 2, \dots, I$; $l = 1, 2, \dots, L$, and C_e , if an upper bound, ϵ_o , on the energy of the signal is available. Thus the sequence x_d must be in the intersection set, $C_o = \cap_{i,l} [C_{i,l}^p \cap C_{i,l}^m] \cap C_e$. Any vector in the set C_o is a feasible solution to the problem of signal reconstruction from bispectrum because it satisfies all the constraints and the prior information. Since the above sets are closed and convex in l^* , a sequence of vectors which converges to a vector in the set C_o can be generated by using Bregman's theorem.

The key operation of the method of POCS is the orthogonal projection onto a convex set. Let us describe the projection operation onto the set $C_{i,l}^m$. The projection x_p of any sequence $x_o \in l^*$, ($x_o \notin C_{i,l}^m$) is the solution of the following constrained minimization problem

$$\begin{aligned} &\text{minimize } \|x - x_o\|_* \\ &\text{subject to } [X(\omega_{1i})X(\omega_{2l})X(\omega_{1i} + \omega_{2l})] = |B_g(\omega_{1i}, \omega_{2l})| \end{aligned} \quad (23)$$

where

$$\|x - x_o\|_*^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log X(\omega) - \log X_o(\omega)|^2 d\omega. \quad (24)$$

Let $\hat{X}(\omega) = \log X(\omega)$ and $\hat{X}_o(\omega) = \log X_o(\omega)$. Then,

$$\|x - x_o\|_*^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{X}(\omega) - \hat{X}_o(\omega)|^2 d\omega. \quad (25)$$

Note that $\hat{X}(\omega) = F[\hat{x}]$ where $\hat{x}$ is the cepstrum [24] of the sequence $x \in l^*$. Thus

$$\|x - x_o\|_*^2 = \|\hat{x} - \hat{x}_o\|_2^2. \quad (26)$$

This means that l^* norm of the sequence $x - x_o$ is equal to the l_2 norm of the cepstral sequence $\hat{x} - \hat{x}_o$ (the cepstrum $\hat{x}$ of any $x \in l^*$ exists [23]). Therefore, the above optimization problem (23) can be rephrased in an equivalent way as follows:

$$\begin{aligned} &\text{minimize } \|\hat{x} - \hat{x}_o\|_2 \\ &\text{subj. to } \text{Re} [\hat{X}(\omega_{1i}) + \hat{X}(\omega_{2l}) + \hat{X}(\omega_{1i} + \omega_{2l})] \\ &\quad = \log |B_g(\omega_{1i}, \omega_{2l})|. \end{aligned} \quad (27)$$

An approximate solution of this problem exists and is given by

$$\begin{aligned} \hat{x}_p(n) &\cong \hat{x}_o(n) + \lambda [\cos(\omega_{1i}n) + \cos(\omega_{2l}n) \\ &\quad + \cos((\omega_{1i} + \omega_{2l})n)], \\ n &= -p, -p + 1, \dots, 0, 1, 2, \dots, q \end{aligned} \quad (28)$$

where

$$\lambda = \frac{\log |B_g(\omega_{1i}, \omega_{2l})| - \operatorname{Re} [\hat{X}_o(\omega_{1i}) + \hat{X}_o(\omega_{2l}) + \hat{X}_o(\omega_{1i} + \omega_{2l})]}{\sum_{n=-p}^q [\cos(\omega_{1i}n) + \cos(\omega_{2l}n) + \cos((\omega_{1i} + \omega_{2l})n)]^2}. \quad (29)$$

In general, the complex cepstrum $\hat{y}(n)$ of a finite extent sequence $y(n)$ is of infinite extent. Thus both $\hat{x}_o(n)$ and $\hat{x}_p(n)$ are probably infinite sequences. However, the cepstral sequences decay exponentially with a $1/n$ scaling factor for finite extent sequences [24] (Appendix B), if there are no zeros on the unit circle. In almost all practical problems the signals are finite extent so two large integers $p > 0$ and $q > 0$ can be selected such that the truncated sequence $\hat{x}_T = \{\dots, 0, 0, \hat{x}(-p), \hat{x}(-p+1), \dots, \hat{x}(0), \hat{x}(1), \dots, \hat{x}(q), 0, 0, \dots\}$ approximates the sequence $\hat{x}$ arbitrarily closely. A sufficiently large p and q work quite satisfactorily in practice.

Similarly, the projection $\mathbf{x}_p$ of any sequence $\mathbf{x}_o \in l^*$, ($\mathbf{x}_o \in C_{i,l}^p$) onto the set $C_{i,l}^p$ is the solution of the following constrained minimization problem:

$$\begin{aligned} \min \|\hat{x} - \hat{x}_o\|_2 \\ \text{subject to } \operatorname{Im} [\hat{X}(\omega_{1i}) + \hat{X}(\omega_{2l}) - \hat{X}(\omega_{1i} + \omega_{2l})] \\ = \arg [B_g(\omega_{1i}, \omega_{2l})]. \end{aligned} \quad (30)$$

And the approximate solution is given by

$$\begin{aligned} \hat{x}_p(n) &\equiv \hat{x}_o(n) + \lambda [\sin(\omega_{1i}n) + \sin(\omega_{2l}n) \\ &\quad - \sin((\omega_{1i} + \omega_{2l})n)] \\ n &= -p, -p+1, \dots, 0, 1, 2, \dots, q \end{aligned} \quad (31)$$

where

$$\lambda = \frac{\arg [B_g(\omega_{1i}, \omega_{2l})] - \operatorname{Im} [\hat{X}_o(\omega_{1i}) + \hat{X}_o(\omega_{2l}) - \hat{X}_o(\omega_{1i} + \omega_{2l})]}{\sum_{n=-p}^q [\sin(\omega_{1i}n) + \sin(\omega_{2l}n) - \sin((\omega_{1i} + \omega_{2l})n)]^2}, \quad \omega_{1i} \neq 0, \omega_{2l} \neq 0. \quad (32)$$

The projection $\mathbf{x}_p$ of any sequence $\mathbf{x}_o \in l^*$ onto the set C_e (Appendix B) is given as follows:

$$\hat{x}_p(n) = \begin{cases} \hat{x}_o(n) & \text{if } \|\mathbf{x}_o\|_*^2 = \|\hat{\mathbf{x}}_o\|_2^2 \leq \epsilon_o \\ \left(\frac{\epsilon_o}{\sum_{n=-\infty}^{\infty} \hat{x}_o(n)^2} \right)^{1/2} \times \hat{x}_o(n) & \text{if } \|\hat{\mathbf{x}}_o\|_2^2 \geq \epsilon_o. \end{cases} \quad (33)$$

Let us describe the signal reconstruction algorithm from bispectrum. The algorithm starts with an arbitrary initial estimate $\mathbf{x}_o \in l^*$. First the cepstrum $\hat{\mathbf{x}}_o$ of $\mathbf{x}_o$ is computed. Although the algorithm converges for any $\mathbf{x}_o$ it is convenient to begin with the unit impulse sequence δ in which case $\hat{\delta} = 0$. Next $\hat{\mathbf{x}}_o$ is successively projected onto the sets $C_{i,l}^p, C_{i,l}^m$, $i = 1, 2, \dots, I$; $l = 1, 2, \dots, L$, by using (28) and (31). If the energy information is available then the current iterate is projected onto the set C_e by using (33), too. The order of projections is immaterial. In this

manner the first iteration cycle is completed. The procedure is repeated until a satisfactory level of error difference in successive iterations is obtained. The final sequence is the cepstrum of the desired sequence which is then computed. The flow diagram of the iterative procedure is shown in Fig. 1.

Some simulation examples to show the use of the algorithm are presented. The l_1 norm of the error is chosen as the criteria for comparison between the original signal and the reconstructed signal.

Example 1: Let $\mathbf{x}_d = \{\dots, 0, 0, 1.0, 0.75, \mathbf{x}_d(0) = 1.0, -0.5, 0.5, 1.0, 0, 0, \dots\}$ be the desired signal. $\mathbf{x}_d$ is a nonminimum phase sequence. It is assumed that $B_g(\omega_1, \omega_2) = X(\omega_1)X(\omega_2)X^*(\omega_1 + \omega_2)$ is available. Sets as in (20) and (21) are constructed corresponding to frequency pairs $(2\pi k/64, 2\pi l/64)$, $k = 0, 1, 2, \dots, 32$ and $(2\pi l/64, 0)$, $l = 0, 1, 2, \dots, 32$. After 15 iteration cycles the iterative algorithm is terminated. The absolute error $\|e_{15}\|_1 = \|\mathbf{x}_d - \mathbf{x}_{15}\|_1 = 5 \times 10^{-3}$. The reconstructed signal is $\mathbf{x}_{15} = \{\dots, 0, 2, \times 10^{-3}, 1.0, 0.75, 1.0, -0.499, 0.501, 1.0, 0, 0, \dots\}$. The plot of the l_1 norm of the error e_k versus the number of iteration cycles is shown in Fig. 2.

Example 2: In this example, the original signal is the same as the one in example 1. However, it is assumed that the bispectrum is corrupted by noise. The signal-to-noise ratios, SNRM = 15.56 and SNRP = 16.13. The

signal-to-noise ratios are defined as follows:

$$\text{SNRM} = \frac{\sum_{i,l} |B(\omega_{1i}, \omega_{2l})|^2}{\sum_{i,l} N_m(\omega_{1i}, \omega_{2l})^2} \quad (34)$$

and

$$\text{SNRP} = \frac{\sum_{i,l} \arg [B(\omega_{1i}, \omega_{2l})]^2}{\sum_{i,l} N_p(\omega_{1i}, \omega_{2l})^2} \quad (35)$$

Fig. 1. The flow diagram of the algorithm.

Fig. 2. The l_1 norm of the error versus the number of iteration cycles in example 1.

where $N_m(\omega_{1i}, \omega_{2i})$ and $N_p(\omega_{1i}, \omega_{2i})$ are the additive, independent, uniform noise terms which contaminate the magnitude and phase of the given bispectrum $B_g(\omega_1, \omega_2)$, respectively. The minimum error is achieved after 15 iterations and $\|e_{15}\|_1 = \|x_d - x_{15}\|_1 = 0.28$. It is also assumed that the energy of the sequence x_d , $\|x_d\|_*^2 = 1.647$ is also available. This information is used in the reconstruction algorithm by making a projection onto the set C_e in every iteration cycle. The resultant minimum error is 0.16.

Another bound on the energy, $\|x_d\|_*^2 \leq 1.75$, is considered. This upper bound on the energy of the signal is more relaxed than the previous case. In this case the resultant minimum error is 0.19.

Example 3: In this example, the noise is increased to the level that $\text{SNRM} = 3.89$ and $\text{SNRP} = 4.03$. First,

the signal is reconstructed without using any energy information. In this case the resultant minimum error and the percent error are 0.56 and $(\|e_{15}\|_1 / \|x_d\|_1) \times 100 = 11.9$, respectively. Second, an energy bound of 1.75 is used in the reconstruction, and the resultant minimum error and the percent error are 0.36 and 7.57, respectively.

It was observed in [8] that cepstra-based signal reconstruction methods [6], [8] from bispectrum produce comparable results. The use of energy information improves the quality of the reconstructed signal as it can be observed from examples 2 and 3. Recently, Petropulu and Nikias [27] also used the energy set C_e in an iterative algorithm and observed that the energy information not only stabilizes the iterations but increases the rate of convergence as well.

Our algorithm requires that $X(\omega) \neq 0$ for all ω in $[-\pi, \pi]$. If this condition is not satisfied, then the bispectrum can be computed in a circle on which $X_z(z)$, z transform of x , has no zeros in the z domain. This signal reconstruction method can easily be extended to this case, too.

V. CONCLUSION

In this paper, a procedure to reconstruct a (minimum or nonminimum phase) signal from its bispectrum is devised. The algorithm is iterative and based on using the method of projections onto convex sets. Prior information such as an upper bound on the signal energy can be incorporated into this algorithm. Corresponding to given bispectrum values and other prior information such as energy closed and convex sets are constructed in an inner product space l^* . The iterative algorithm consists of successively projecting an initial guess onto these closed and convex sets. Convergence of the algorithm regardless of the initial guess is assured. An algorithm that employs energy information was found to produce better results in simulation than one that ignores energy constraints.

APPENDIX A

PROOFS OF CONVEXITY OF THE SETS DEFINED IN SECTION III

Result: The set C_e is closed and convex.

Proof: Let us consider two arbitrary sequences x_1 and x_2 in the set C_e . Thus, both of these sequences satisfy the following inequality:

$$(x_k, x_k) \leq \epsilon_0, \quad k = 1, 2. \quad (\text{A.1})$$

In l^* any convex combination $x = t \cdot x_1 + (1 - t) \cdot x_2$, $0 \leq t \leq 1$ of x_1 and x_2 is given by

$$x = (t \odot x_1) * [(1 - t) \odot x_2], \quad 0 \leq t \leq 1. \quad (\text{A.2})$$

The energy of the signal x is given by

$$(x, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log X_1^t(\omega) + \log X_2^{1-t}(\omega)|^2 d\omega \quad (\text{A.3})$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |t \log X_1(\omega) + (1 - t) \log X_2(\omega)|^2 d\omega \quad (\text{A.4})$$

$$\begin{aligned} &\leq t_2(x_1, x_1) + (1-t)^2(x_2, x_2) \\ &\quad + 2t(1-t) \|x_1\|_* \cdot \|x_2\|_* \end{aligned} \quad (\text{A.5})$$

$$\leq \epsilon_o. \quad (\text{A.6})$$

Thus the set $C_e = \{x: (x, x) \leq \epsilon_o\}$ is a convex set.

In order to establish closure of sets C_e , with the inner product space l^* assumed to be completed, consider a sequence of vectors $\{x_n\}$ with $x_n \in C_e$ such that $x_n \rightarrow x$, then $\|x_n\|_* \rightarrow \|x\|_*$ and closure follows.

Result: The set $C_{i,l}$ is closed and convex in l^* .

Proof: Let h_1 and $h_2 \in V$ and let the corresponding bispectrums be given by

$$B_k(\omega_1, \omega_2) = H_k(\omega_1)H_k(\omega_2)H_k^*(\omega_1 + \omega_2), \quad k = 1, 2 \quad (\text{A.7})$$

where $|B_k(\omega_1, \omega_2)|$ is greater than 0. For $0 \leq t \leq 1$ the Fourier transform (FT) of the vector

$$h = (t \odot h_1) * ((1-t) \odot h_2) \quad (\text{A.8})$$

is given by

$$H(\omega) = (H_1(\omega))'(H_2(\omega))^{1-t}. \quad (\text{A.9})$$

The corresponding bispectrum is given by

$$\begin{aligned} B(\omega_1, \omega_2) &= (H_1(\omega_1))'(H_2(\omega_1))^{1-t} \times (H_1(\omega_2))'(H_2(\omega_2))^{1-t} \\ &\quad \times (H_2^*(\omega_1 + \omega_2))'(H_2^*(\omega_1 + \omega_2))^{1-t}. \end{aligned} \quad (\text{A.10})$$

If $B_1(\omega_1, \omega_2)$ and $B_2(\omega_1, \omega_2)$ are equal to $B_g(\omega_{1i}, \omega_{2l})$ at the frequency pair $(\omega_{1i}, \omega_{2l})$ then it follows from (A.8) that

$$B(\omega_1, \omega_2) = B_g(\omega_1, \omega_2), \quad \text{for } (\omega_{1i}, \omega_{2l}). \quad (\text{A.11})$$

Therefore the set

$$C_{i,l} = \{x: X(\omega_{1i})X(\omega_{2l})X^*(\omega_{1i} + \omega_{2l}) = B_g(\omega_{1i}, \omega_{2l})\} \quad (\text{A.12})$$

is a convex set in l^* .

Proof of closure can be established as in [16], [17].

Similarly, the proofs of convexity of the sets $C_{i,l}^m$ and $C_{i,l}^p$ can be established.

APPENDIX B

ORTHOGONAL PROJECTION ONTO THE SETS DEFINED IN SECTION III

1. Orthogonal Projection onto the Set C_e

Orthogonal projection x_p of a vector $x_o \in l^*$ onto the set C_e is obtained as follows: If $\|x_o\|_*^2 \leq \epsilon_o$ then $x_p = x_o$. If $\|x_o\|_*^2 > \epsilon_o$ then the projection x_p is the solution of the following problem:

$$\min_{x \in C_e} \|x - x_o\|_* \quad (\text{B.1})$$

or

$$\begin{aligned} &\min \|\hat{x} - \hat{x}_o\|_2 \\ &\text{subject to } \|\hat{x}\|_2^2 = \epsilon_o. \end{aligned} \quad (\text{B.2})$$

The following Lagrangian is constructed:

$$L = \|\hat{x} - \hat{x}_o\|^2 + \lambda \left(\sum_{n=-\infty}^{\infty} (x(n))^2 - \epsilon_o \right). \quad (\text{B.3})$$

Necessary conditions are

$$\frac{\partial L}{\partial \hat{x}(n)} = 0, \quad \text{for } n = 0, \pm 1, \pm 2, \dots \quad (\text{B.4})$$

and

$$\frac{\partial L}{\partial \lambda} = 0. \quad (\text{B.5})$$

By solving (B.4) and (B.5) the following equation for $\hat{x}_p$ is obtained:

$$\hat{x}_p(n) = \begin{cases} \hat{x}_o(n) & \text{if } \|x_o\|_*^2 = \|\hat{x}_o\|_2^2 \leq \epsilon_o \\ \left(\frac{\epsilon_o}{\sum_{n=-\infty}^{\infty} x_o(n)^2} \right)^{1/2} \times \hat{x}_o(n) & \text{if } \|\hat{x}_o\|_2^2 > \epsilon_o. \end{cases} \quad (\text{B.6})$$

2. Orthogonal Projection onto the Set $C_{i,l}^m$

The orthogonal projection onto the set $C_{i,l}^m$ is obtained as follows: Let $x_o \in l^*$ and let us find its projection x_p onto the set $C_{i,l}^m$. The sequence x_p is defined as the solution of the following problem:

$$\min_{x \in C_{i,l}^m} \|x - x_o\|_*. \quad (\text{B.7})$$

This problem can be rephrased as

$$\begin{aligned} &\min \|x - x_o\|_* \\ &\text{subject to } |X(\omega_{1i})X(\omega_{2l})X^*(\omega_{1i} + \omega_{2l})| = |B_g(\omega_{1i}, \omega_{2l})|. \end{aligned} \quad (\text{B.8})$$

As it has been shown earlier $\|x - x_o\|_* = \|\hat{x} - \hat{x}_o\|_2$. So the problem stated in (B.8) is equivalent to the following problem

$$\begin{aligned} \min \|\hat{x} - \hat{x}_o\|_2 \\ \text{subject to } \operatorname{Re} [\hat{X}(\omega_{1i}) + \hat{X}(\omega_{2i}) + \hat{X}(\omega_{1i} + \omega_{2i})] \\ = \log |B_g(\omega_{1i}, \omega_{2i})|. \end{aligned} \quad (\text{B.9})$$

$$\lambda = \frac{\log |B_g(\omega_{1i}, \omega_{2i})| - \operatorname{Re} [\hat{X}_o(\omega_{1i}) + \hat{X}_o(\omega_{2i}) + \hat{X}_o(\omega_{1i} + \omega_{2i})]}{\sum_{n=-p}^q [\cos(\omega_{1i}n) + \cos(\omega_{2i}n) + \cos((\omega_{1i} + \omega_{2i})n)]^2}. \quad (\text{B.17})$$

where

$$\begin{aligned} \operatorname{Re} [\hat{B}(\omega_{1i}, \omega_{2i})] = \sum_{n=-\infty}^{\infty} \hat{x}(n) \cos(\omega_{1i}n) + \hat{x}(n) \cos(\omega_{2i}n) \\ + \hat{x}(n) \cos(\omega_{1i}n + \omega_{2i}n). \end{aligned} \quad (\text{B.10})$$

It is well known that $\hat{x}$ decays very fast [26], [28] if $x \in V$. The following result due to Goodman gives a measure of the rate of decay:

Lemma [28]: If the Fourier transforms of $x \in l_1(\mathbb{Z})$ is nonzero for all ω , then the cepstrum $\hat{x}$ of x also is an l_1 sequence, i.e., $\sum |\hat{x}(n)| < \infty$.

This lemma holds for multidimensional sequences, too. Therefore the above infinite sum (B.10) can be approximated with a finite sum as

$$\begin{aligned} \operatorname{Re} [\hat{B}(\omega_{1i}, \omega_{2i})] \cong \sum_{n=-p}^q \hat{x}(n) \cos(\omega_{1i}n) + \hat{x}(n) \cos(\omega_{2i}n) \\ + \hat{x}(n) \cos(\omega_{1i}n + \omega_{2i}n) \end{aligned} \quad (\text{B.11})$$

where p, q are sufficiently large integers.

The following Lagrangian is constructed:

$$\begin{aligned} L = \|\hat{x} - \hat{x}_o\|^2 + \lambda \left[\sum_{n=-p}^q \hat{x}(n) \cos(\omega_{1i}n) + \cos(\omega_{2i}n) \right. \\ \left. + \cos(\omega_{1i}n + \omega_{2i}n) - \log(B_g(\omega_{1i}, \omega_{2i})) \right] \end{aligned} \quad (\text{B.12})$$

where

$$\|\hat{x} - \hat{x}_o\|^2 = \sum_{n=-\infty}^{\infty} (\hat{x}_o(n))^2 \cong \sum_{n=-p}^q (\hat{x}(n) - \hat{x}_o(n))^2. \quad (\text{B.13})$$

Necessary conditions are

$$\begin{aligned} \frac{\partial L}{\partial \hat{x}(n)} = 0, \quad \text{for } n = -p, -p+1, \dots, \\ -1, 0, 1, \dots, q \end{aligned} \quad (\text{B.14})$$

and

$$\frac{\partial L}{\partial \lambda} = 0. \quad (\text{B.15})$$

From (B.14) the following relation is obtained:

$$\begin{aligned} \hat{x}_p(n) \cong \hat{x}_o(n) + \lambda [\cos(\omega_{1i}n) + \cos(\omega_{2i}n) \\ + \cos(\omega_{1i} + \omega_{2i})n], \\ n = -p, -p+1, \dots, -1, 0, 1, \dots, q. \end{aligned} \quad (\text{B.16})$$

After substituting (B.16) into the (B.15) λ is solved and

Similarly, the projection operation onto the set $C_{i,l}^p$ can be obtained.

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