Reliable Decentralized Stabilization of Feedforward and Feedback Interconnected Systems

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Abstract—Synthesis of decentralized controllers that are reliable in the presence of breakdowns in the interconnections are considered for plants obtained via feedforward and feedback type interconnection schemes. The main objective is to determine the precise conditions on the plant which assure such reliable synthesis. It is shown that the central property for reliable decentralized stabilization is always an appropriate *parity interlacing property* among the poles of the interconnection subplants and certain blocking zeros of the overall plant. The conditions under which the stabilization of a two-channel plant is affected by the stabilization of its diagonal subplants are also obtained. The results also find application in *multiple controller reliable synthesis problems*. The discussion is restricted to two-channel plants.

I. INTRODUCTION AND PRELIMINARIES

AN important objective in the design of feedback control systems is to assure reliable performance in the face of deviations of the closed-loop parameters from their nominal values. There are many alternative ways of modeling the changes in the parameters. These may roughly be classified as continuous changes such as in robust control or discrete (or on-off) changes such as in the simultaneous stabilization of finitely many plants. Reliable stabilization in the face of “breakdowns” of certain subplants falls into the second category and it has attracted considerable attention in the synthesis of controllers for large-scale systems [10], [9], [16], [11], [8], [14], [1], [5]. A further significance of reliable decentralized stabilization is the application it finds in some reliable full-feedback (centralized) stabilization problems [10], [8]. Our objective is to determine the precise conditions for the existence of decentralized stabilizing compensators which are reliable against breakdowns in the plant inter-

Fig. 1.

connection issues. We consider two different types of interconnection schemes. In Fig. 1, the main subplants Σ_{11} and Σ_{22} are interconnected via feedforward paths comprising the interconnection subplants Σ_{12} and Σ_{21} . In Fig. 2, on the other hand, the main subplants Σ_{11} and Σ_{22} are combined via feedback paths comprising the interconnection subplants Σ_{12} and Σ_{21} .

If we let Z_{ij} and T_{ij} denote the transfer matrices associated with the subplants Σ_{ij} and $\bar{\Sigma}_{ij}$, respectively, for $i, j \in 2$, the overall transfer matrices for the resulting two-channel plants of Figs. 1 and 2 are

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \quad (1)$$

and

$$T = \begin{bmatrix} (I - T_{11}T_{12}T_{22}T_{21})^{-1}T_{11} & (I - T_{11}T_{12}T_{22}T_{21})^{-1}T_{11}T_{12}T_{22} \\ (I - T_{22}T_{21}T_{11}T_{12})^{-1}T_{22}T_{21}T_{11} & (I - T_{22}T_{21}T_{11}T_{12})^{-1}T_{22} \end{bmatrix} \quad (2)$$

connections and (in some cases) in the external feedback paths. We restrict our study to two-channel plants and to

provided T_{ij} for some $i, j \in 2$ is strictly proper to assure that the interconnection scheme of Fig. 2 is well defined. The same decentralized feedback scheme is considered for the two plants as shown in Fig. 3. To assure that the closed-loop system of Fig. 3 is well defined for both cases, the transfer matrices Z_{11} , Z_{22} , T_{11} , and T_{22} are all restricted to be strictly proper.

The main problems we consider in this paper are given below.

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Fig. 2.

Fig. 3.

P1: Given the feedforward interconnected plant Σ , determine local compensators Σ_{c1} and Σ_{c2} such that: i) $(\Sigma_{11}, \Sigma_{c1})$, ii) $(\Sigma_{22}, \Sigma_{c2})$, and iii) $(\Sigma, \text{diag}\{\Sigma_{c1}, \Sigma_{c2}\})$ are all internally stable.

P2: Given the feedback interconnected plant $\bar{\Sigma}$, determine local compensators Σ_{c1} and Σ_{c2} such that: i) $(\bar{\Sigma}_{11}, \Sigma_{c1})$, ii) $(\bar{\Sigma}_{22}, \Sigma_{c2})$, and iii) $(\bar{\Sigma}, \text{diag}\{\Sigma_{c1}, \Sigma_{c2}\})$ are all internally stable.

The following table summarizes the type of reliability that follows from the synthesis according to P1 and P2. The reliability properties of P1 listed in the table summarizes the discussion in [8].

An alternative motivation for P1 is obtained by noting that it is the problem of *determining conditions under which stabilization of the overall plant is affected by the stabilization of the main subplants Σ_{11} and Σ_{22}* . For such applications, it is more suitable to define an input/output version of P1 which takes the overall two-channel plant as its starting point rather than the subplants.

P1_{i/o}: Given Z of (1), determine local compensators of transfer matrices Z_{c1} and Z_{c2} such that: i) (Z_{11}, Z_{c1}) , ii) (Z_{22}, Z_{c2}) , and iii) $(Z, \text{diag}\{Z_{c1}, Z_{c2}\})$ are all internally stable.

The only difference between P1 and P1_{i/o} is that, in P1_{i/o}, the unstabilizable and undetectable modes (centralized hidden modes) of Σ which may be created by the feedforward interconnections are not pertinent to solvability. With this alternative motivation in mind, P1_{i/o} has been extensively investigated in the literature and there are many different sufficient solvability conditions. We can cite [6], [18], and the references therein which use

quasi block-diagonal dominance methods. Recently, in [1], P1_{i/o} has been studied and a solvability condition is given. The main result of [1] is, however, essentially a problem transformation result and is only comparable to our Proposition 2.2. The problem P2 can be viewed as the input-output interconnection counterpart of the state interconnection problem of [11] which in particular considers *static* interconnection subplants T_{12} and T_{21} . The problem has also been considered in [9] with static interconnections T_{12} , T_{21} and local state feedback. We should mention that the sufficient conditions obtained in [9] and [11] apply to general N -channel plants.

It turns out that each of the three problems can be transformed either to a simultaneous stabilization problem of two plants, one of which has a parametric representation or to a stabilization problem of a two-channel plant by a stable decentralized compensator. In Section II, we solve P1_{i/o} through the first transformation and specialize the obtained result to P1. In Section II, we solve P2 through a transformation of the second type. The main results of the paper are Theorems 2.2, 2.3, and 3.1 which, respectively, yield solutions to P1_{i/o}, P1, and P2. The solvability condition for P1_{i/o} in Theorem 2.2 is in terms of some arbitrary but fixed initial compensators applied at both channels. The main solvability condition for this problem is a parity interlacing condition among the poles and certain blocking zeros of the initial closed-loop system obtained. The solvability conditions of Theorems 2.3 and 3.1 on the other hand are purely in terms of the problem data. The main conditions for both cases is a parity interlacing condition among the poles of the interconnection subplants and certain blocking zeros associated with the overall plant. There is a noteworthy duality between the conditions obtained for P1 and P2 which is a consequence of an inherent duality in the problem definitions. In Corollaries 2.1–2.4 and 3.1, 3.2, the main results are specialized to some cases of independent interest. In Corollary 2.1 and 3.1, the case where the main subplants are stable is examined and it is shown that the main requirement for solvability is that there are an even number of poles between each pair of real positive blocking zeros of the off-diagonal transfer matrices in (1) and (2), respectively. In the case where the interconnection subplants are stable, it is shown in Corollaries 2.2 and 3.2 that, all three problems are solvable. The Corollaries 2.3 and 2.4 concern the application of P1_{i/o} to some full-feedback reliable synthesis problems among which the *multiple controller reliable synthesis problem* of Šiljak [10] can be mentioned. (Also see [16], [8], [5] and [1].) All these results do provide intuition into the structure of feedforward and feedback interconnected systems in spite of the fact that our consideration is limited to two-channel cases and some general statements can be made. These are listed in the section on conclusions. We also note that although our discussion below is restricted to continuous-time systems, the generalizations to discrete-time systems and to systems of arbitrary stability sets is straightforward by the results of [15].

TABLE I

	Type of Failure Assured Stability	Subplant interconnection failures modeled as P1: $Z_{12} = 0$ and/or $Z_{21} = 0$ P2: $T_{12} = 0$ and/or $T_{21} = 0$	Feedback interconnection failures modeled as $Z_{11} = 0$ or $Z_{c2} = 0$
P1	Stability of the diagonal subplants	Σ_{11} and Σ_{22} remain stabilized	The diagonal subplant which is connected to the functional compensator remains stabilized
	Stability of the overall closed-loop system	The overall closed-loop plant remains stabilized provided the resulting open-loop plant has no unstable decentralized fixed modes	The set of the unstable poles of the overall closed-loop plant is a subset of the unstable poles of the open-loop plant
P2	Stability of the diagonal subplants	$\bar{\Sigma}_{11}$ and $\bar{\Sigma}_{22}$ remain stabilized	The diagonal subplant which is connected to the functional compensator remains stabilized
	Stability of the overall closed-loop system	The overall closed-loop plant remains stabilized provided the resulting open-loop plant has no unstable decentralized fixed modes	Not assured

We now briefly mention some additional notation to the ones already used. By $\mathcal{C}$ and $\mathcal{R}$, we denote the fields of complex and real numbers, respectively. The symbol $\mathcal{C}_{+e}$ denotes the points on the extended unstable region of the complex plane including infinity and $\mathcal{R}_{+e} := \mathcal{R} \cap \mathcal{C}_{+e}$. Thus, $\mathcal{R}_{+e}$ is the set of real positive numbers including infinity. The set of transfer matrices are denoted by P and the set of stable transfer matrices by S . A complex number z_0 is a *blocking zero* of $Z \in P^{p \times r}$ if $Z(z_0) = 0$. If Z is stable, then the unstable blocking zeros are the unstable zeros of the *smallest invariant factor* (SIF) of Z over S . Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two finite collections of numbers in $\mathcal{R}_{+e}$, in which some numbers may occur more than once. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint, then we say that the ordered pair $(\mathcal{S}_1, \mathcal{S}_2)$ has *parity interlacing property* if there are an even number of elements from $\mathcal{S}_1$ between each pair of elements from $\mathcal{S}_2$. The terminology is borrowed from [19] in which $\mathcal{S}_1$ and $\mathcal{S}_2$ are, respectively, the poles (with multiplicity) and the blocking zeros of a transfer matrix. Note that, if $\mathcal{S}_1$ is the set of $\mathcal{R}_{+e}$ zeros with multiplicity of $a \in S$, then $a(z)$ takes the same sign at all elements $z \in \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2)$ has the parity interlacing property. If q is an integer, $\mathbf{q}$ denotes the ordered set $\{1, 2, \dots, q\}$. By I_s , we denote the identity matrix of size S , the subscript is dropped if the size is clear from the context. The determinant of a square matrix A is denoted by $|A|$ and the transpose of a matrix B by B' . Occasionally, the abbreviation “iff” is used for “if and only if.” For all other undefined terminology and notation pertaining to the algebraic and topological structure of the ring S and for matrices over S , we refer the reader to [15] and [7].

II. FEEDFORWARD INTERCONNECTED SYSTEMS

We first consider $P1_{i/o}$, a solution to which immediately yields a solution to P1. The solutions to $P1_{i/o}$ and

P1 are obtained by exploiting some recent results in [7]. The main idea is to build the solution upon the characterization result in [7] of all solutions to the single channel canonicity problem with internal stability (SCCPIS). It turns out that $P1_{i/o}$ is equivalent to simultaneously stabilizing Z_{11} and *any one* of the closed-loop transfer matrices attainable by a solution to SCCPIS (see Theorem 2.1 below). The standard result of [15] on simultaneous stabilization and some genericity arguments then yield the main result Theorem 2.2.

Let the overall open-loop transfer matrix Z be written in a bicoprime fractional representation over S as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} Q_{11}^{-1} [R_1 \ R_2] \quad (3)$$

where $P_i \in S^{p_i \times q}$, $R_i \in S^{q \times r_i}$, $i = 2$ and $Q_{11} \in S^{q \times q}$ is nonsingular. It is well known that the decentralized stabilization problem (DSP) [17], [2], [3], [7], [12] is solvable for Z , that is for some $\bar{Z}_{c1} \in P^{r_1 \times p_1}$ and $\bar{Z}_{c2} \in P^{r_2 \times p_2}$ (Z , $\text{diag}\{\bar{Z}_{c1}, \bar{Z}_{c2}\}$) is internally stable, if and only if $(P_1, Q_{11}, R_2, 0)$ and $(P_2, Q_{11}, R_1, 0)$ are complete quadruples, in other words, for all $z \in \mathcal{C}_{+e}$, it holds that

$$\text{rank} \begin{bmatrix} Q_{11} & R_2 \\ -P_1 & 0 \end{bmatrix} \geq q = \text{size}(Q_{11}),$$

$$\text{rank} \begin{bmatrix} Q_{11} & R_1 \\ -P_2 & 0 \end{bmatrix} \geq q = \text{size}(Q_{11}).$$

The completeness conditions are equivalent to the condition that Z has no unstable decentralized fixed modes. Since by the problem definition, the solvability of $P1_{i/o}$ requires the solvability of DSP, we assume that the completeness conditions above hold. It is further assumed that Z is strongly connected [2], [7], i.e., both Z_{12} and Z_{21} are nonzero. If Z is not strongly connected, the plant transfer

matrix is upper and/or lower triangular, and any Z_{c1} , Z_{c2} such that (Z_{11}, Z_{c1}) and (Z_{22}, Z_{c2}) are internally stable, also solves $P1_{i/o}$.

We now summarize certain relevant results of [7].

Given (3), let $C = \text{gcrf}(P_2, Q_{11})$ so that $Q_{11} = Q_2 C$, and $P_2 = PC$ for a right-coprime pair (P, Q_2) where "gcrf" is an abbreviation for "a greatest common right factor." Also let $D = \text{glf}(Q_2, R_2)$ so that $Q_2 = DQ$ and $R_2 = DR$ for a left-coprime pair of matrices (Q, R) , where "glf" stands for "a greatest common left factor." Consider the generalized Bezout identities

$$\begin{bmatrix} E & F \\ -R_l & Q_l \end{bmatrix} \begin{bmatrix} Q & -F_r \\ P & E_r \end{bmatrix} = I, \begin{bmatrix} Q & R \\ -H_l & G_l \end{bmatrix} \begin{bmatrix} G & -Pr \\ H & Q_r \end{bmatrix} = I \quad (4)$$

for some matrices $E, F, E_r, F_r, Q_l, R_l, G, H, G_l, H_l, Q_r$, and P_r over S and let

$$P_{c2}(X) := HF_r + Q_r X, Q_{c2}(X) := E_r + PGF_r - PP_r X \quad (5)$$

for $X \in S^{r_2 \times p_2}$. It follows by the standard Youla-Bongiorno-Jabr-Kučera parameterization that (Z_{22}, Z_{c2}) is internally stable if and only if $Z_{c2} = P_{c2}(X)Q_{c2}(X)^{-1}$ for some $X \in S^{r_2 \times p_2}$ (see [15], [7] for details). Also note that for all $X \in S^{r_2 \times p_2}$, $(P_{c2}(X), Q_{c2}(X))$ is a coprime pair. Applying the compensator $Z_{c2} = P_{c2}(X)Q_{c2}(X)^{-1}$ to the second channel, the closed-loop transfer matrix around the first channel, denoted by $\hat{Z}_{11}(X)$, becomes

$$\hat{Z}_{11}(X) = \begin{bmatrix} P_1 & 0 \end{bmatrix} \begin{bmatrix} DQC & DRP_{c2}(X) \\ -PC & Q_{c2}(X) \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}. \quad (6)$$

A compensator Z_{c2} is a solution to SCCPIS if and only if there exists X such that $Z_{c2} = P_{c2}(X)Q_{c2}(X)^{-1}$ and the fractional representation in (6) is bicoprime. In this case, by [7, theorem 32], $\text{diag}\{Z_{c1}, Z_{c2}\}$ solves DSP if and only if $(\hat{Z}_{11}(X), Z_{c1})$ is internally stable. On the other hand, [7, theorem 33] yields a characterization of all solutions to SCCPIS. The crucial feature of this result is that almost all choices for X yields a solution to SCCPIS, provided the original plant is free of unstable decentralized fixed modes. These results of [7, theorems 3.2, 3.3] are summarized in the following proposition in which the precise description of $\mathcal{X}$ is left out as it is not needed for the subsequent considerations.

Proposition 2.1: Let the plant Z have no unstable decentralized fixed modes. Consider $Z_{c2} = P_{c2}(X)Q_{c2}(X)^{-1}$, where $P_{c2}(X)$ and $Q_{c2}(X)$ are given by (5).

a) $\text{diag}\{Z_{c1}, Z_{c2}\}$ solves DSP for some Z_{c1} if and only if (6) is a bicoprime fraction and $(\hat{Z}_{11}(X), Z_{c1})$ is internally stable.

b) There exists an open and dense subset $\mathcal{X} \subseteq S^{r_2 \times p_2}$ such that (6) is a bicoprime fraction if and only if $X \in \mathcal{X}$.

We are now able to formulate $P1_{i/o}$ as a simultaneous stabilization problem.

Theorem 2.1: Let Z have no unstable decentralized fixed modes. The problem $P1_{i/o}$ is solvable if and only if there exists $X \in \mathcal{X}$ such that Z_{11} and $\hat{Z}_{11}(X)$ are simultaneously stabilizable.

Proof: [Only If] If Z_{c1}, Z_{c2} solve $P1_{i/o}$, by ii) of $P1_{i/o}$, (Z_{22}, Z_{c2}) is internally stable, i.e., $Z_{c2} = P_{c2}(X)Q_{c2}(X)^{-1}$ for some $X \in S^{r_2 \times p_2}$. By Proposition 2.1, the solvability of DSP implies that (6) is bicoprime and hence $X \in \mathcal{X}$. By i) and iii), respectively, (Z_{11}, Z_{c2}) and $(\hat{Z}_{11}(X), Z_{c1})$ are also internally stable. Hence $\hat{Z}_{11}(X)$ and Z_{11} are simultaneously stabilizable. [If] We have (Z_{22}, Z_{c2}) is internally stable by the choice $Z_{c2} = P_{c2}(X)Q_{c2}(X)^{-1}$ where $X \in \mathcal{X}$. So, ii) of the problem statement is satisfied. Let Z_{c1} be such that (Z_{11}, Z_{c1}) and $(\hat{Z}_{11}(X), Z_{c1})$ are internally stable. In this case i) is also satisfied. To show $(Z, \text{diag}\{Z_{c1}, Z_{c2}\})$ is internally stable, it is enough to show (6) is bicoprime, since then, the result follows by part a) of Proposition 2.1. But this follows by part b) of Proposition 2.1 and the fact that $X \in \mathcal{X}$. $\square$

We now make use of [15, theorem 5.4.2] to reduce our simultaneous stabilization problem to a parity interlacing condition. For this purpose, let

$$Z_{11} = D_l^{-1}N_l = N_r D_r^{-1}$$

be coprime fractional representations of Z_{11} so that there exist matrices $\bar{\Psi}_1, \bar{\Psi}_2, \Psi_1, \Psi_2$ satisfying

$$\begin{bmatrix} \Psi_1 & \Psi_2 \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -\bar{\Psi}_2 \\ N_r & \bar{\Psi}_1 \end{bmatrix} = I. \quad (7)$$

Also let

$$D^{-1}R_1 = \bar{R}_1 \bar{D}^{-1}$$

for some right-coprime pair $(\bar{R}_1, \bar{D})$. Consider the matrices

$$\Theta := \begin{bmatrix} C & (E + GF_r R_l) \bar{R}_1 \\ -\Psi_2 P_1 & \Psi_1 \bar{D} \end{bmatrix}, \Gamma_1 := \begin{bmatrix} 0 \\ I_{p_1} \end{bmatrix} \quad \Lambda_1 := \begin{bmatrix} D_l P_1 & N_l \bar{D} \end{bmatrix} \quad (8)$$

$$\Gamma_2 := \begin{bmatrix} P_r \\ 0 \end{bmatrix}, \Lambda_2 := \begin{bmatrix} 0 & R_l \bar{R}_1 \end{bmatrix}.$$

Let us also define

$$\Pi := \begin{bmatrix} C & (E + GF_r R_l) \bar{R}_1 \\ D_l P_1 & N_l \bar{D} \end{bmatrix}, \Gamma_0 := \begin{bmatrix} P_r \\ 0 \end{bmatrix}$$

where the matrices Γ_2 and Γ_0 are the same except for the size of their zero blocks, i.e., Γ_2 is of size $(q + r_1) \times r_2$ whereas Γ_0 is of size $(q + p_1) \times r_2$.

Proposition 2.2: Let $X \in \mathcal{X}$. Then, Z_{11} and $\hat{Z}_{11}(X)$ are simultaneously stabilizable if and only if $|\Theta - \Gamma_0 X \Lambda_2|$ takes the same sign at $z \in \mathcal{S}(X)$, where

$$\mathcal{S}(X) := \{z \in \mathcal{R}_{+e} : \text{rank}[\Pi - \Gamma_2 X \Lambda_2](z) = q\}. \quad (9)$$

Proof: Let $X \in \mathcal{X}$ be fixed. Note from (4)–(6) that

$$\hat{Z}_{11}(X) = P_1 C^{-1} [(E + GF_r R_l) \bar{R}_1 - P_r X R_l \bar{R}_1] \bar{D}^{-1}.$$

Let

$$C^{-1} [(E + GF_r R_l) \bar{R}_1 - P_r X R_l \bar{R}_1] = L(X) K(X)^{-1}$$

for some right-coprime $(L(X), K(X))$. Since $X \in \mathcal{X}$, by Proposition 2.1-b, it follows that $[P_1 L(X)] [\bar{D} K(X)]^{-1}$ is a right-coprime fraction of $\hat{Z}_{11}(X)$. Employing [15, theorem 5.4.2], the pair of plants $Z_{11}, \hat{Z}_{11}(X)$ are simultaneously stabilizable if and only if $|\Theta(X)|$ takes the same sign at $\mathcal{R}_{+e}$ -blocking zeros of $\Psi(X)$, where

$$\begin{bmatrix} \Theta(X) \\ \Psi(X) \end{bmatrix} = \begin{bmatrix} \Psi_1 & \Psi_2 \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} \bar{D} K(X) \\ P_1 L(X) \end{bmatrix}.$$

It remains to show that $|\Theta(X)| = |\Theta - \Gamma_2 X \Lambda_2|$ and that $\mathcal{R}_{+e}$ -blocking zeros of $\Psi(X)$ are given by the set $\mathcal{S}(X)$. In fact

$$\begin{aligned} |\Theta(X)| &= \left| \begin{bmatrix} I_q & L(X) \\ 0 & \Theta(X) \end{bmatrix} \right| = \left| \begin{bmatrix} I_q & L(X) \\ -\Psi_2 P_1 & \Psi_1 \bar{D} K(X) \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} C & E \bar{R}_1 + (GF_r - P_r X) R_l \bar{R}_1 \\ -\Psi_2 P_1 & \Psi_1 \bar{D} \end{bmatrix} \right| \\ &= |\Theta - \Gamma_2 X \Lambda_2|. \end{aligned}$$

On the other hand, observe that the set of $\mathcal{R}_{+e}$ -blocking zeros of $\Psi(X)$ are the same as $\{z \in \mathcal{R}_{+e} \mid \text{rank} \begin{bmatrix} I_q & 0 \\ 0 & \Psi(X) \end{bmatrix} (z) = q\}$. By simple manipulations and utilizing (4), it is not difficult to see that

$$\begin{aligned} \text{rank} \begin{bmatrix} I_q & 0 \\ 0 & D_l P_1 L(X) - N_l \bar{D} K(X) \end{bmatrix} &= q \text{ iff} \\ \text{rank} \begin{bmatrix} C & E \bar{R}_1 + (GF_r - P_r X) R_l \bar{R}_1 \\ D_l P_1 & N_l \bar{D} \end{bmatrix} &= q \end{aligned}$$

for all $z \in \mathcal{R}_{+e}$. The antecedent of the above proposition is precisely $\text{rank} [\Pi - \Gamma_0 X \Lambda_2] = q$ and the result follows. $\square$

Remark 2.1: The introduction of the matrices in (8) can be further motivated by transforming $P1_{i/o}$ to an equivalent problem of *decentralized strong stabilization*. Note that any compensator which internally stabilizes Z_{11} is of the form $Z_{c1} = (\Psi_1 - YN_l)^{-1}(\Psi_2 + YD_l)$ for some $Y \in S^{r_1 \times p_1}$. Applying Z_{c1} to $\hat{Z}_{11}(X)$, one obtains an autonomous plant of the denominator matrix

$$\hat{\Phi}(X, Y) := \begin{bmatrix} DQC & R_l & DR(HF_r + Q_r X) \\ -(\Psi_2 + YD_l)P_1 & \Psi_1 - YN_l & 0 \\ -PC & 0 & E_r + PGF_r - PP_r X \end{bmatrix}.$$

By simple manipulations, it is easy to see that $|\hat{\Phi}(X, Y)| = |\Theta - \Gamma_1 Y \Lambda_1 - \Gamma_2 X \Lambda_2|$, where the matrices on the right-hand side are as defined in (8). It follows that the problem $P1_{i/o}$ has a solution if and only if there exist $Y \in S^{r_1 \times p_1}$ and $X \in S^{r_2 \times p_2}$ such that $\Theta - \Gamma_1 Y \Lambda_1 - \Gamma_2 X \Lambda_2$ is unimodular over $\mathcal{S}$. In other words, $P1_{i/o}$ is solvable iff there exists a stable decentralized compensator which stabilizes the subsidiary two-channel plant of transfer matrix

$$\begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} \\ \hat{Z}_{21} & \hat{Z}_{22} \end{bmatrix} = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \Theta^{-1} [\Gamma_1 \quad \Gamma_2].$$

The matrices $[\Pi \quad \Gamma_0]$ and $[\Pi' \quad \Lambda_2']$ that occur in the description of the set $\mathcal{S}$ of Proposition 2.3 below, on the other hand, are (in essence) the system matrices associated with $[\hat{Z}_{11} \quad \hat{Z}_{12}]$ and $[\hat{Z}_{21}' \quad \hat{Z}_{22}']$, respectively. $\triangle$

The next step is determining the precise conditions under which an $X \in \mathcal{X}$ exists such that $|\Theta - \Gamma_0 X \Lambda_2|$ takes the same sign at all $z \in \mathcal{S}$. For this purpose, we first identify the elements of $\mathcal{S}(X)$ which are independent of X .

Consider the connectivity condition

$$\text{rank}(Z_{12}) \geq 2 \text{ or } \text{rank}(Z_{21}) \geq 2. \quad (10)$$

Proposition 2.3: Let

$$\begin{aligned} \mathcal{S} &:= \left\{ z \in \mathcal{R}_{+e} : \text{rank} [\Pi \quad \Gamma_0](z) \right. \\ &\quad \left. = q \text{ or } \text{rank} \begin{bmatrix} \Pi \\ \Lambda_2 \end{bmatrix} (z) = q \right\}. \end{aligned} \quad (11)$$

- i) For all $X \in \mathcal{X}$, the inclusion $\mathcal{S} \subset \mathcal{S}(X)$ holds.
- ii) If (10) holds, then there exists an open and dense subset $\tilde{\mathcal{X}}$ of $\mathcal{X}$ such that $\mathcal{S} = \mathcal{S}(X)$ for all $X \in \tilde{\mathcal{X}}$.

Proof: See the Appendix. $\square$

Thus, the elements of the set $\mathcal{S}$ are the $\mathcal{R}_{+e}$ blocking zeros of $\mathcal{S}(X)$ that are independent of $X \in \mathcal{X}$. If $\text{rank } Z_{12} \geq 2$ or $\text{rank } Z_{21} \geq 2$, then the set of $\mathcal{R}_{+e}$ -blocking zeros of $\mathcal{S}(X)$ is precisely $\mathcal{S}$ for almost all elements of $\mathcal{X}$. In other words, when (10) holds the following is true: If $X_0 \in \mathcal{X}$ is such that $\mathcal{S}(X_0) = \mathcal{S}$, then the equality still holds under sufficiently small perturbations on X_0 . On the other hand, if for some $X_0 \in S^{r_2 \times p_2}$, $\mathcal{S}(X_0)$ contains some zeros different than those in $\mathcal{S}$, there exists an arbitrarily small perturbation on X_0 such that these zeros are eliminated, so that $\mathcal{S}(X_{0p}) = \mathcal{S}$, where $X_{0p} \in \mathcal{X}$ is the suitable perturbation of X_0 .

Consider the set

$$\mathcal{T} := \{z \in \mathcal{R}_{+e} : (\Lambda_2 \Theta^{-1} \Gamma_2)(z) = 0\} \quad (12)$$

which is the set of real positive blocking zeros of the transfer matrix $\hat{Z}_{22}$ of Remark 2.1. We can now give the main result of this section.

Theorem 2.2: Suppose (10) holds. Then, $P1_{i/o}$ is solvable if and only if Z has no unstable decentralized fixed modes and $|\Theta|$ takes the same sign at $z \in \mathcal{S} \cap \mathcal{T}$, where $\Theta, \mathcal{S}$, and $\mathcal{T}$ are as defined by (8), (11), and (12).

Proof: Let $\Omega_l := \text{gclf}(\Theta, \Gamma_2)$ so that $\Theta = \Omega_l \bar{\Theta}$, $\Gamma_2 = \Omega_l \hat{\Gamma}$ for some left coprime pair of matrices $(\bar{\Theta}, \hat{\Gamma})$. Also let $\Omega_r := \text{gcrf}(\bar{\Theta}, \Lambda_2)$ so that $\bar{\Theta} = \hat{\Omega}_r \Lambda_2$, $\Lambda_2 = \hat{\Lambda}_r \Omega_r$ for some right coprime pair $\hat{\Omega}_r, \hat{\Lambda}_r$. Then, $\Lambda_2 \Theta^{-1} \Gamma_2 = \hat{\Lambda}_r \hat{\Omega}_r^{-1} \hat{\Gamma}$, where the latter is a bicoprime fractional representation. Moreover

$$|\Theta - \Gamma_2 X \Lambda_2| = |\Omega_l| |\hat{\Theta} - \hat{\Gamma} X \hat{\Lambda}| |\Omega_r|$$

and if $A^{-1}B$ is a left-coprime fractional representation of $\Lambda_2 \Theta^{-1} \Gamma_2$, then $|\Theta - \Gamma_2 X \Lambda_2|$ and $|\Omega_l| |A - BX| |\Omega_r|$ are associates, i.e., they are equal up to a multiplication by a unit in $\mathcal{S}$. Since this is true for all X and for $X = 0$ in particular, $|\Theta|$ and $|\Omega_l| |\Omega_r| |A|$ as well as $|\Omega_l| |\hat{\Theta}| |\Omega_r|$ are associates. Finally, note that $z \in \mathcal{T}$ if and only if $B(z) = 0$.

[Only If] By Proposition 2.2, if $P1_{i/o}$ has a solution, then there exists $X \in \mathcal{X}$ such that $|\Theta - \Gamma_2 X \Lambda_2|$ takes the same sign at all $z \in \mathcal{S}(X)$. By Proposition 2.3-i), for all $X \in \mathcal{X}$ one has $\mathcal{S} \subset \mathcal{S}(X)$. Hence, $|\Theta - \Gamma_2 X \Lambda_2|$ has the same sign at all elements of $\mathcal{S}$. On the other hand, at the elements in $\mathcal{T}$, $|\Theta - \Gamma_2 X \Lambda_2|$ has constant sign if and only if the sign of $|\Theta|$ is constant since $\mathcal{T} = \{z \in \mathcal{R}_{+e} : B = 0\}$. Therefore, the solvability of $P1_{i/o}$ implies that the sign of $|\Theta|$ is the same for all $z \in \mathcal{S} \cap \mathcal{T}$.

[If] Say $|\Omega_l| |\Omega_r| |A| > 0$ for all $z \in \mathcal{S} \cap \mathcal{T}$. If the reverse inequality holds, the proof below can be modified accordingly. Let $\bar{\mathcal{S}}$ be the complement of $\mathcal{S} \cap \mathcal{T}$ in $\mathcal{S}$, i.e., $\bar{\mathcal{S}} := \mathcal{S} - \mathcal{S} \cap \mathcal{T}$. Note that, if we denote $\sigma := \text{sign}$, then $\sigma(z) \neq 0$ for all $z \in \bar{\mathcal{S}}$ since $\bar{\mathcal{S}}$ does not contain any elements from $\mathcal{T}$. Let $\bar{\mathcal{S}}$ consist of $\{z_i : i \in \nu\}$. We will now construct $X \in \mathcal{X}$ such that $|\Theta - \Gamma_2 X \Lambda_2|$ has the same sign at all elements of $\mathcal{S}$. For each $z_i \in \bar{\mathcal{S}}$, there exists $d_i \in \mathcal{R}$ such that $|\Omega_l| |\Omega_r| (z_i) |A(z_i) - \sigma(z_i) d_i| > 0$. One can construct $x \in \mathcal{S}$ by any interpolation technique such that $x(z_i) = d_i$, $i \in \nu$. Hence $|\Omega_l| |\Omega_r| |A| - \sigma x > 0$ whenever $z \in \bar{\mathcal{S}}$. By [15, theorem 4.4.2], there exists $X \in \mathcal{S}^{r_2 \times p_2}$ such that $|A| - \sigma x = |A - BX|$. That is, there exist X such that $|\Omega_l| |\Omega_r| |A - BX| > 0$, whenever $z \in \bar{\mathcal{S}}$. Noting that $|\Omega_l| |\Omega_r| |A - BX|$ and $|\Theta - \Gamma_2 X \Lambda_2|$ are associates, the latter takes the same sign at $z \in \bar{\mathcal{S}}$ with this construction of X . Since $\mathcal{X}$ is open and dense in $\mathcal{S}^{r_2 \times p_2}$, X can be assumed to be contained in $\mathcal{X}$, possibly with an arbitrarily small perturbation on X , which does not change the sign of $|\Theta - \Gamma_2 X \Lambda_2|$ at $z \in \bar{\mathcal{S}}$. By Proposition 2.3-ii) we can also assure that the $\mathcal{R}_{+e}$ -blocking zeros of $\Pi - \Gamma_0 X \Lambda_2$ are precisely the elements of $\mathcal{S}$. Otherwise, there exists an arbitrarily small perturbation X_p of X such that this property holds. In addition, the perturbation can be restricted to be sufficiently small such that $|\Theta - \Gamma_2 X_p \Lambda_2|$ still has the same sign at all $z \in \mathcal{S}$. The result now follows by Proposition 2.2 and Theorem 2.1. $\square$

We now use the result of Theorem 2.2 to get a solution to P1.

Let

$$Z_{ij} = D_{ij}^{-1} N_{ij} = \bar{N}_{ij} \bar{D}_{ij}^{-1}$$

be some left- and right-coprime fractional representations for $i, j \in 2$. It is easy to verify that the overall transfer matrix Z has the following fractional representation in terms of the left-coprime fractional representations for the subplants Z_{ij} above

$$Z = \begin{bmatrix} I & I & 0 & 0 \\ 0 & 0 & I & I \end{bmatrix} \begin{bmatrix} D_{11} & 0 & 0 & 0 \\ 0 & D_{12} & 0 & 0 \\ 0 & 0 & D_{21} & 0 \\ 0 & 0 & 0 & D_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} N_{11} & 0 \\ 0 & N_{12} \\ N_{21} & 0 \\ 0 & N_{22} \end{bmatrix}. \quad (13)$$

Let

$$R_1 := \begin{bmatrix} N_{11} \\ 0 \\ N_{21} \\ 0 \end{bmatrix}, R_2 := \begin{bmatrix} 0 \\ N_{12} \\ 0 \\ N_{22} \end{bmatrix}$$

$$P_1 := [I \ I \ 0 \ 0], P_2 := [0 \ 0 \ I \ I]$$

and $Q_{11} := \text{diag}\{D_{11}, D_{12}, D_{21}, D_{22}\}$. The decentralized stabilization problem for Σ is solvable if and only if the fractional representation is bicoprime, i.e., (Q_{11}, R_1, R_2) is left and (P_1, P_2, Q_{11}) is right coprime, and the quadruples $(P_1, Q_{11}, R_2, 0)$ and $(P_2, Q_{11}, R_1, 0)$ are both complete. It is easy to see that the fraction in (13) is bicoprime if and only if

$$(D_{21}, D_{22}), (D_{12}, D_{11}) \text{ are right-coprime pairs} \quad (14)$$

and

$$(\bar{D}_{21}, \bar{D}_{11}), (\bar{D}_{12}, \bar{D}_{22}) \text{ are left-coprime pairs.} \quad (15)$$

Let $(\tilde{D}_{11}, \tilde{D}_{21})$ and $(\tilde{D}_{22}, \tilde{D}_{12})$ be some left-coprime pairs of matrices satisfying

$$\bar{D}_{21}^{-1} \bar{D}_{11} = \tilde{D}_{11} \tilde{D}_{21}^{-1}, \bar{D}_{12}^{-1} \bar{D}_{22} = \tilde{D}_{22} \tilde{D}_{12}^{-1}.$$

The completeness conditions turn out to be equivalent to the rank inequalities

$$\begin{aligned} \text{rank} \begin{bmatrix} D_{21} & N_{21} \bar{D}_{11} & 0 \\ D_{22} & 0 & 0 \\ 0 & 0 & D_{12} \end{bmatrix} &\geq p_1 + p_2 \quad \text{and} \\ \text{rank} \begin{bmatrix} D_{12} & N_{12} \bar{D}_{22} & 0 \\ D_{11} & 0 & 0 \\ 0 & 0 & D_{21} \end{bmatrix} &\geq p_1 + p_2 \end{aligned} \quad (16)$$

holding for all $z \in \mathcal{R}_{+e}$. The coprimeness conditions (14), (15) and the rank conditions in (16) are thus necessary

conditions for the solvability of P1. Let the matrices G_i , H_i , $\bar{G}_i$, $\bar{H}_i$ satisfy

$$\begin{bmatrix} D_{ii} & N_{ii} \\ -\bar{H}_i & \bar{G}_i \end{bmatrix} \begin{bmatrix} G_i & -\bar{N}_{ii} \\ H_i & \bar{D}_{ii} \end{bmatrix} = I, \quad i \in 2$$

where by left coprimeness of $(D_{11}, N_{11}\bar{D}_{21})$ and of $(D_{22}, N_{22}\bar{D}_{12})$ one can in particular choose $H_1 = \bar{D}_{21}\hat{H}_1$ and $H_2 = \bar{D}_{12}\hat{H}_2$, for some $\hat{H}_1$ and $\hat{H}_2$. If Σ has no unstable centralized fixed modes, then the problem P1 is solvable if and only if P1_{i/o} is solvable for the plant (13) and we can make use of Theorem 2.2 with Q_{11} , P_i , R_i , $i \in 2$ as defined above. Note that with

$$E := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I + D_{22} & I & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, F_r := \begin{bmatrix} 0 \\ -I \\ I \\ 0 \end{bmatrix},$$

$$G := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & G_2 & 0 \\ 0 & 0 & -N_{12}H_2 & I \end{bmatrix}$$

$$R_l = [0 \quad D_{22} \quad I \quad 0]$$

(4) is satisfied. With these choices, the matrix Θ is such that

$$|\Theta| = \left| \begin{bmatrix} D_{12} & -N_{12}H_2D_{22} \\ N_{21}H_1D_{11} & D_{21} \end{bmatrix} \right|$$

$$= |D_{12}||D_{21}||I - \bar{N}_{21}\hat{H}_1D_{11}\bar{N}_{12}\hat{H}_2D_{22}|$$

$$= |D_{12}||D_{21}||I - \bar{N}_{21}\bar{D}_{11}\bar{H}_1\bar{N}_{12}\bar{D}_{22}\bar{H}_2|.$$

The set $\mathcal{S}$ is easily shown to simplify to

$$\mathcal{S} = \left\{ z \in \mathcal{R}_{+e} \mid \text{rank} \begin{bmatrix} D_{12} & N_{12}\bar{D}_{22} \\ D_{11} & 0 \end{bmatrix} (z) = p_1 \text{ or} \right.$$

$$\left. \text{rank} \begin{bmatrix} D_{21} & N_{21}\bar{D}_{11} \\ D_{22} & 0 \end{bmatrix} (z) = p_2 \right\}$$

on taking $D_l = D_{11}$, $N_l = N_{11}$. By the coprimeness conditions (14), (15), the set $\mathcal{S}$ has an even simpler description given by

$$\mathcal{S} = \{ \mathcal{R}_{+e}\text{-blocking zeros of } D_{11}\bar{N}_{12}\bar{D}_{22} \}$$

$$\cup \{ \mathcal{R}_{+e}\text{-blocking zeros of } D_{22}\bar{N}_{21}\bar{D}_{11} \}$$

$$= \{ \mathcal{R}_{+e}\text{-blocking zeros of } D_{11}Z_{12}\bar{D}_{22} \}$$

$$\cup \{ \mathcal{R}_{+e}\text{-blocking zeros of } D_{22}Z_{21}\bar{D}_{11} \}.$$

On the other hand, the set $\mathcal{T}$ becomes

$$\mathcal{T} = \left\{ \mathcal{R}_{+e}\text{-blocking zeros of } \begin{bmatrix} 0 & D_{22} \end{bmatrix} \right.$$

$$\cdot \left. \begin{bmatrix} D_{12} & -N_{12}H_2D_{22} \\ N_{21}H_1D_{11} & D_{21} \end{bmatrix}^{-1} \begin{bmatrix} N_{12}\bar{D}_{22} \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \mathcal{R}_{+e}\text{-blocking zeros of } (I - D_{22}\bar{N}_{21}\hat{H}_1D_{11}\bar{N}_{12}\hat{H}_2)^{-1} \right.$$

$$\cdot D_{22}\bar{N}_{21}\hat{H}_1D_{11}D_{12}^{-1}N_{12}\bar{D}_{22} \left. \right\}.$$

Noting that $\hat{H}_1D_{11} = \bar{D}_{21}^{-1}\bar{D}_{11}\bar{H}_1$ and $\hat{H}_2D_{22} = \bar{D}_{12}^{-1}\bar{D}_{22}\bar{H}_2$, we have $|I - D_{22}\bar{N}_{21}\hat{H}_1D_{11}\bar{N}_{12}\hat{H}_2| = |I - D_{22}Z_{21}\bar{D}_{11}\bar{H}_1\bar{N}_{12}\bar{H}_2| = |I - \bar{N}_{21}\hat{H}_1D_{11}\bar{N}_{12}\hat{H}_2D_{22}| = |I - \bar{N}_{21}\hat{H}_1D_{11}Z_{12}\bar{D}_{22}\bar{H}_2|$, it follows that $z \in \mathcal{S}$ implies $z \in \mathcal{T}$, i.e., $\mathcal{S} \cap \mathcal{T} = \mathcal{S}$. Moreover, at all zeros in $\mathcal{S}$, it is clear that $|\Theta| = |D_{12}||D_{21}|$. We therefore have the following solution to P1.

Theorem 2.3: Suppose that either $\text{rank } Z_{12} \geq 2$ or $\text{rank } Z_{21} \geq 2$. There exists a solution to P1 if and only if the system Σ is free of unstable decentralized fixed modes (equivalently, the coprimeness conditions (14), (15), and the rank conditions in (16) hold) and $|D_{12}||D_{21}|$ takes the same sign at the set

$$\mathcal{S} = \{ \mathcal{R}_{+e}\text{-blocking zeros of } D_{11}Z_{12}\bar{D}_{22} \}$$

$$\cup \{ \mathcal{R}_{+e}\text{-blocking zeros of } D_{22}Z_{21}\bar{D}_{11} \}.$$

Remark 2.2: i) The connectivity condition (10) is only required in the “if” parts of the proofs of Theorems 2.2 and 2.3. Hence, all plants for which the problems have solutions satisfy the given related parity interlacing conditions. ii) although the solvability condition given by Theorem 2.2 depends on some initial compensators at the first and second channels, the solvability condition of Theorem 2.3 is purely in terms of the problem data. This is mainly due to the fact that, in problem P1, a suitable choice for the initial compensators that avails more transparent solvability conditions is quite easy to see, whereas in P1_{i/o} it is not. This might have been quite natural to expect since P1_{i/o} is in fact a more encompassing problem definition than P1. iii) an alternative proof of Theorem 2.3 can also be given in a manner similar to the proof of Theorem 3.1 below. The above proof is however preferable as it exploits and clarifies the connection between P1 and P1_{i/o}. $\triangle$

We now consider some other specializations of the result of Theorem 2.3.

Case 1: Z_{11} and Z_{22} are stable transfer matrices. This is the case where the original plants are stable and are combined according to Fig. 1 via possibly unstable interconnections. We observe that by the stability of Z_{11} and Z_{22} the problems P1 and P1_{i/o} are equivalent. We specialize the result of Theorem 2.3 by making the particular choices $D_{11} = I$, $D_{22} = I$, $\bar{D}_{11} = I$, $\bar{D}_{22} = I$, $H_1 = 0$, H_2

= 0. Theorem 2.3 immediately yields the following result provided the connectivity condition (10) holds. With a slight modification in the proof of Theorem 2.2, however, it can be shown that this assumption can be discarded in this special case. We thus have the following extension of a result in [8] which considers two-input, two-output plants.

Corollary 2.1: Let Z_{11} and Z_{22} be stable transfer matrices. Then, P1 is solvable if and only if Σ is free of unstable decentralized fixed modes (equivalently

$$\text{rank} \begin{bmatrix} D_{12} & 0 \\ 0 & N_{21} \end{bmatrix} \geq p_1, \text{rank} \begin{bmatrix} D_{21} & 0 \\ 0 & N_{12} \end{bmatrix} \geq p_2$$

for all $z \in \mathcal{C}_{+e}$) and $|D_{12}| |D_{21}|$ takes the same sign at the union of the real unstable blocking zeros of Z_{12} and Z_{21} .

Case 2: Z_{12} and Z_{21} are stable transfer matrices. This is the case where the original plants Z_{11} and Z_{22} are interconnected according to Fig. 1 with stable interconnection subplants. As in Case 1, there are no unstable hidden modes of Σ and hence the problems P1 and $P1_{i/o}$ are equivalent. Moreover, by hypothesis $|D_{12}|$ and $|D_{21}|$ are both units so that the rank conditions (16) and the parity interlacing condition of Theorem 2.3 are automatically satisfied. We hence have the following result.

Corollary 2.2: Let Z_{12} and Z_{21} be stable transfer matrices and suppose that either $\text{rank } Z_{12} \geq 2$ or $\text{rank } Z_{21} \geq 2$. Then, P1 is solvable.

Case 3: Z is stabilizable and detectable from both Channels 1 and 2. This is the case where in (3), $P_1 Q_{11}^{-1} R_1$ and $P_2 Q_{11}^{-1} R_2$ are both bicoprime fractional representations of Z_{11} and Z_{22} . The significance of this case is the application it finds in the multiple controller reliable synthesis problem of [10] (see Case 4 below).

Corollary 2.3: Let $P_1 Q_{11}^{-1} R_1$ and $P_2 Q_{11}^{-1} R_2$ be bicoprime fractions and suppose that (10) holds. Then, $P1_{i/o}$ is solvable if and only if $|Q_{11}|$ takes the same sign at the union of the real unstable blocking zeros of Z_{12} and Z_{21} .

Proof: The proof is obtained by a specialization of the result of Theorem 2.2. We only outline the main steps. By the hypothesis that $P_2 Q_{11}^{-1} R_2$ is bicoprime, we can let $C = I$, $D = I$, $\bar{D} = I$, and $\bar{R}_1 = R_1$. It follows that

$$\Lambda_2 \Theta^{-1} \Gamma_2 = R_1 R_1 [\Psi_1 + \Psi_2 P_1 (E + GF_r R_1) R_1]^{-1} \Psi_2 P_1 P_r.$$

Using the fact that $P_1 Q_{11}^{-1} R_1$ is bicoprime and by various other coprimeness conditions, it is not difficult to show that the right-hand side of the above equality is also bicoprime. The next step is to establish the set equality

$$\mathcal{S} = \{z \in \mathcal{R}_{+e} | P_1 P_r = 0 \text{ or } R_1 R_1 = 0\}.$$

This can be shown by first noting that $P_1 Q_{11}^{-1} R_1 = D_r^{-1} N_r = N_r D_r^{-1}$, in which each fraction is bicoprime, implies that $D_r P_1 = \bar{P} Q_{11}$, $N_r = \bar{P} R_1$, $R_1 D_r = Q_{11} \bar{R}$, and $N_r = P_1 \bar{R}$ for some matrices $\bar{P}$, $\bar{R}$ over $\mathcal{S}$. Now, using (4) and

(7), one can show that

$$\text{rank} [\Pi \quad \Gamma_0](z) = \begin{bmatrix} I_q & (E + GF_r R_1) R_1 & P_r \\ D_r P_1 & N_r & 0 \end{bmatrix}(z) = q$$

if and only if

$$\text{rank} \begin{bmatrix} I_q & 0 & 0 \\ 0 & P_1 P_r H_r \bar{R} & P_1 P_r \end{bmatrix}(z) = q \quad (17)$$

for any $z \in \mathcal{C}_{+e}$ which is the case if and only if $[P_1 P_r](z) = 0$. Similarly, one can show that $\text{rank} [\Pi' \Lambda_2'](z) = q$ if and only if $[R_1 R_1](z) = 0$. This establishes the expression for $\mathcal{S}$ given above. It immediately follows that if $z \in \mathcal{S}$, then $\Lambda_2 \Theta^{-1} \Gamma_2(z) = 0$ by the bicoprime expression obtained above for the transfer matrix $\Lambda_2 \Theta^{-1} \Gamma_2$. Hence, $\mathcal{S} \cap \mathcal{T} = \mathcal{S}$. Finally, by an easy computation, we have

$$\begin{aligned} |\Theta| &= |\Psi_1 + \Psi_2 P_1 (G + P_r H_r E) R_1| \\ &= |\bar{\Psi}_1 + P_1 (E + GF_r R_1) R_1 \bar{\Psi}_2| \end{aligned}$$

and

$$\begin{aligned} [\Psi_1 + \Psi_2 P_1 (G + P_r H_r E) R_1] D_r &= I - \Psi_2 P_1 P_r H_r \bar{P} \bar{R}, \\ D_r [\bar{\Psi}_1 + P_1 (E + GF_r R_1) R_1 \bar{\Psi}_2] &= I - \bar{P} R H_r R_1 \bar{\Psi}_2. \end{aligned}$$

Noting that $|D_r|$, $|D_r|$, and $|Q_{11}|$ are associates, it follows that $|\Theta|$ takes the same sign at $\mathcal{S}$ if and only if so does $|Q_{11}|$. The result now follows by Theorem 2.2. $\square$

Case 4: Applications to full-feedback synthesis. The reliability of the closed-loop stability of a centralized (full) feedback system due to compensator breakdowns can be improved by using multiple stabilizing compensators in a decentralized structure [10]. The following problem is the *multiple controller reliable synthesis problem* (MCRSP) of Šiljak [10] in our notation.

MCRSP: Let $Z = [Z_{ij}]$, where $Z_{ij} \in \mathcal{P}^{p_i \times r_j}$, $i, j \in \{1, 2\}$, be the transfer matrix of a two-channel plant. Determine compensators $Z_{ci} \in \mathcal{P}^{r_i \times p_i}$, $i \in \{1, 2\}$ such that: a) $(Z, \text{diag}\{Z_{c1}, Z_{c2}\})$ is internally stable; b) $(Z, \text{diag}\{Z_{c1}, 0\})$ is internally stable; and c) $(Z, \text{diag}\{0, Z_{c2}\})$ is internally stable.

The motivation of the problem is that (see also [10]) in the case of a possible breakdown of any one of the compensators, the other one continues to stabilize the plant Z . It is easily seen that the solution of the problem requires that Z is stabilizable and detectable from both Channels 1 and 2 which implies the equivalence of the conditions b), c) above and i), ii) in the definition of $P1_{i/o}$. Consequently, MCRSP and $P1_{i/o}$ where Z is stabilizable and detectable from Channels 1 and 2 are equivalent problems. We can then employ Corollary 2.3 to solve MCRSP.

Corollary 2.4: Suppose that either $\text{rank } Z_{12} \geq 2$ or $\text{rank } Z_{21} \geq 2$ and Z_{11} , Z_{22} are strictly proper. Then, MCRSP is solvable if and only if Z is stabilizable and detectable from both Channels 1 and 2, and has an even

number of real poles between each pair of zeros in the union of the sets of $\mathcal{R}_{+e}$ -blocking zeros of Z_{12} and Z_{21} .

It is possible to define and solve several different versions of the multiple controller reliable synthesis problem as applications of Corollary 2.3. As an example, we consider the following *reliable full-feedback synthesis problem* (RFFSP) in which the plant Z has two input channels and only one output channel.

RFFSP: Let $G = [G_1 \ G_2]$, where $G_i \in \mathbf{P}^{l \times s_i}$, $i \in 2$, be the transfer matrix of a two-input-channel, one-output-channel plant. Find a compensator $Z_c = [Z'_{c1} \ Z'_{c2}]'$, $Z_{c1} \in \mathbf{P}^{s_1 \times l}$, $i \in 2$ such that: a) (G, Z_c) is internally stable; b) $(G, [Z'_{c1} \ 0]')$ is internally stable; and c) $(G, [0 \ Z'_{c2}]')$ is internally stable.

Let $Q_0^{-1}[S_1 \ S_2]$ be a left-coprime fractional representation of G . For RFFSP to be solvable, it is clear that G must be stabilizable from both of the input channels separately, in other words (Q_0, S_1) and (Q_0, S_2) must be left coprime pairs. Now consider the two-input, two-output expanded plant

$$G_e = \begin{bmatrix} G_1 & G_2 \\ G_1 & G_2 \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} Q_0^{-1} [S_1 \ S_2]. \quad (18)$$

Assume that either $\text{rank } G_1 \geq 2$ or $\text{rank } G_2 \geq 2$. Applying Corollary 2.3 to $Z = G_e$ with $Z_{11} = G_1$, $Z_{12} = G_2$, $Z_{21} = G_1$, and $Z_{22} = G_2$, one can obtain Z_{c1} , Z_{c2} which solve P1 for the expanded plant G_e , if $|Q_0|$ takes the same sign at $\{z \in \mathcal{R}_{+e} | G_1 = 0 \text{ or } G_2 = 0\}$. By the fact that (Q_0, S_1) and (Q_0, S_2) are coprime pairs in (18), it holds that: i) $(G_e, \text{diag}\{Z_{c1}, 0\})$ is internally stable; ii) $(G_e, \text{diag}\{0, Z_{c2}\})$ is internally stable; and iii) $(G_e, \text{diag}\{Z_{c1}, Z_{c2}\})$ is internally stable. It can be verified that i), ii), and iii) imply a), b), and c). As a result, we have the following.

Suppose that either $\text{rank } G_1 \geq 2$ or $\text{rank } G_2 \geq 2$, and G is strictly proper. Then, a sufficient condition for RFFSP to be solvable is that G has an even number of real poles between each pair of zeros in the union of the sets of $\mathcal{R}_{+e}$ -blocking zeros of G_1 and G_2 .

III. FEEDBACK INTERCONNECTED SYSTEMS

Given the feedback interconnected system $\bar{\Sigma}$ of Fig. 2, where $T_{11} \in \mathbf{P}^{p_1 \times r_1}$, $T_{22} \in \mathbf{P}^{p_2 \times r_2}$, $T_{12} \in \mathbf{P}^{r_1 \times p_2}$, $T_{21} \in \mathbf{P}^{r_2 \times p_1}$ with T_{11}, T_{22} strictly proper, let

$$T_{ij} = D_{ij}^{-1} N_{ij} = \bar{N}_{ij} \bar{D}_{ij}^{-1}$$

be some left- and right-coprime fractional representations of the subplant transfer matrices for $i, j \in 2$. Then the transfer matrix T of $\bar{\Sigma}$ has the fractional representation

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} D_{11} & -N_{11} & 0 & 0 \\ 0 & D_{12} & 0 & -N_{12} \\ -N_{21} & 0 & D_{21} & 0 \\ 0 & 0 & -N_{22} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & N_{22} \end{bmatrix}. \quad (19)$$

Let

$$Q_{11} := \begin{bmatrix} D_{11} & -N_{11} & 0 & 0 \\ 0 & D_{12} & 0 & -N_{12} \\ -N_{21} & 0 & D_{21} & 0 \\ 0 & 0 & -N_{22} & D_{22} \end{bmatrix} \quad R_1 := \begin{bmatrix} N_{11} \\ 0 \\ 0 \\ 0 \end{bmatrix}, R_2 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ N_{22} \end{bmatrix} \quad (20)$$

$$P_1 := [I \ 0 \ 0 \ 0], P_2 := [0 \ 0 \ 0 \ I].$$

The system $\bar{\Sigma}$ is free of unstable hidden modes if and only if the fraction (19) is bicoprime, i.e., (Q_{11}, R_1, R_2) is left coprime and (P_1, P_2, Q_{11}) is right coprime, which in turn is the case if and only if

$$(N_{22}, D_{21}), (N_{11}, D_{12}) \text{ are right-coprime pairs} \quad (21)$$

and

$$(\bar{D}_{21}, \bar{N}_{11}), (\bar{D}_{12}, \bar{N}_{22}) \text{ are left-coprime pairs.} \quad (22)$$

Let

$$\bar{D}_{21}^{-1} \bar{N}_{11} = \hat{N}_{11} \hat{D}_{21}^{-1}, \bar{D}_{12}^{-1} \bar{N}_{22} = \hat{N}_{22} \hat{D}_{12}^{-1}$$

for some right-coprime pairs $(\hat{N}_{11}, \hat{D}_{21})$ and $(\hat{N}_{22}, \hat{D}_{12})$. The system $\bar{\Sigma}$ is free of unstable decentralized fixed modes if and only if the coprimeness conditions (21), (22) hold and for all $z \in \mathcal{E}_{+e}$ it holds that

$$\begin{aligned} \text{rank} \begin{bmatrix} D_{12} & N_{12} \bar{N}_{22} & 0 \\ -N_{11} & 0 & 0 \\ 0 & 0 & D_{21} \end{bmatrix} &\geq r_1 + r_2 \\ \text{rank} \begin{bmatrix} D_{21} & N_{21} \bar{N}_{11} & 0 \\ -N_{22} & 0 & 0 \\ 0 & 0 & D_{12} \end{bmatrix} &\geq r_1 + r_2. \end{aligned}$$

Exploiting the fact that $(\bar{N}_{22}, \bar{D}_{12})$ and $(\bar{N}_{11}, \bar{D}_{21})$ are left-coprime pairs by (22), it can be seen that these rank inequalities hold if and only if for all $z \in \mathcal{E}_{+e}$ the simpler inequalities

$$\begin{aligned} \text{rank} \begin{bmatrix} D_{21} & 0 \\ 0 & N_{11} \bar{N}_{12} \hat{N}_{22} \end{bmatrix} &\geq r_2 \\ \text{rank} \begin{bmatrix} D_{12} & 0 \\ 0 & N_{22} \bar{N}_{21} \hat{N}_{11} \end{bmatrix} &\geq r_1 \end{aligned} \quad (23)$$

hold. The main result of this section can now be given. Consider the condition

$$\text{rank}(T_{11} T_{12} T_{22}) \geq 2 \text{ or } \text{rank}(T_{22} T_{21} T_{11}) \geq 2. \quad (24)$$

Theorem 3.1: Suppose that (24) holds. Then P2 has a solution if and only if $\bar{\Sigma}$ has no unstable decentralized fixed modes (equivalently, the coprimeness conditions (21), (22) and the rank conditions (23) hold) and $|D_{12}| |D_{21}|$

takes the same sign at the set

$$\begin{aligned} \bar{\mathcal{S}} = \{ \mathcal{R}_{+e}\text{-blocking zeros of } N_{11}T_{12}\bar{N}_{22} \} \\ \cup \{ \mathcal{R}_{+e}\text{-blocking zeros of } N_{22}T_{21}\bar{N}_{11} \}. \end{aligned}$$

Proof: Observe that $T_{ii}T_{ij}T_{jj} = D_{ii}^{-1}N_{ii}\bar{N}_{ij}\hat{N}_{jj}(\bar{D}_{jj}\hat{D}_{ij})^{-1}$ for $i, j \in \mathbf{2}$, $i \neq j$ and hence by (24) we have

$$\text{rank}(N_{11}\bar{N}_{12}\hat{N}_{22}) \geq 2 \text{ or } \text{rank}(N_{22}\bar{N}_{21}\hat{N}_{11}) \geq 2. \quad (25)$$

Also let $K_i, L_i, \bar{K}_i, \bar{L}_i$, $i \in \mathbf{2}$ be such that

$$\begin{aligned} \begin{bmatrix} D_{11} & N_{11} \\ -\bar{L}_1 & \bar{K}_1 \end{bmatrix} \begin{bmatrix} \bar{D}_{21}K_1 & -\bar{N}_{11} \\ L_1 & \bar{D}_{11} \end{bmatrix} &= I, \\ \begin{bmatrix} D_{22} & N_{22} \\ -\bar{L}_2 & \bar{K}_2 \end{bmatrix} \begin{bmatrix} \bar{D}_{12}K_2 & -\bar{N}_{22} \\ L_2 & \bar{D}_{22} \end{bmatrix} &= I. \end{aligned} \quad (26)$$

Such matrices exist by left coprimeness of the pairs $(D_{11}\bar{D}_{21}, N_{11})$ and $(D_{22}\bar{D}_{12}, N_{22})$. The set of all internally stabilizing compensators $Z_{c1}(Z_{c2})$ of $T_{11}(T_{22})$ can then be written as

$$\begin{aligned} Z_{c1} &= (L_1 + \bar{D}_{11}X_1)(\bar{D}_{21}K_1 - \bar{N}_{11}X_1)^{-1} \\ Z_{c2} &= (L_2 + \bar{D}_{22}X_2)(\bar{D}_{12}K_2 - \bar{N}_{22}X_2)^{-1} \end{aligned} \quad (27)$$

for some $X_i \in \mathcal{S}^{r_i \times p_i}$, $i \in \mathbf{2}$. Moreover, for any such Z_{ci} , the pair of plants $(T, \text{diag}(Z_{c1}, Z_{c2}))$ is internally stable if and only if

$$\Theta := \begin{bmatrix} D_{12} & -N_{12}(\bar{D}_{12}K_2 - \bar{N}_{22}X_2)N_{22} \\ -N_{21}(\bar{D}_{21}K_1 - \bar{N}_{11}X_1)N_{11} & D_{21} \end{bmatrix}$$

is a unimodular matrix for some X_1 and X_2 . This follows on performing simple unimodular operations on the closed-loop denominator matrix

$$\begin{bmatrix} Q_{11} & R_1P_{c1} & R_2P_{c2} \\ -P_1 & Q_{c1} & 0 \\ -P_2 & 0 & Q_{c2} \end{bmatrix}$$

where the various matrices are as given in (20) and (27). Summarizing our considerations so far if $\bar{\Sigma}$ is free of unstable decentralized fixed modes, then P2 has a solution if and only if there exist matrices X_1 and X_2 such that the matrix Θ is unimodular. We now find an equivalent matrix $\tilde{\Theta}$ to Θ which is in a more convenient form. For this purpose, let $(\tilde{D}_{12}, \tilde{N}_{11})$ and $(\tilde{D}_{21}, \tilde{N}_{22})$ be some right-coprime pairs such that $|D_{12}| = |\tilde{D}_{12}|$, $|D_{21}| = |\tilde{D}_{21}|$ and

$$N_{11}D_2^{-1} = \tilde{D}_{12}^{-1}\tilde{N}_{11}, N_{22}D_1^{-1} = \tilde{D}_{21}^{-1}\tilde{N}_{22}.$$

Consider

$$\tilde{\Theta} := \begin{bmatrix} \tilde{D}_{12} & -\tilde{N}_{11}N_{12}(\bar{D}_{12}K_2 - \bar{N}_{22}X_2) \\ -\tilde{N}_{22}N_{21}(\bar{D}_{21}K_1 - \bar{N}_{11}X_1) & \tilde{D}_{21} \end{bmatrix}.$$

It easily follows that

$$\text{diag}\{\tilde{N}_{11}, \tilde{N}_{22}\}\Theta = \tilde{\Theta}\text{diag}\{N_{11}, N_{22}\}.$$

From the various coprimeness conditions above, it also follows that $\tilde{\Theta}$ is unimodular if and only if Θ is unimodular.

[Only If] If P2 has a solution, then $\bar{\Sigma}$ is free of decentralized fixed modes so that the coprimeness conditions (21), (22) as well as the rank conditions (23) hold. Moreover, the compensator transfer matrices are necessarily of the form (27) for some X_1 and X_2 , since the compensators separately satisfy the conditions i) and ii) of the problem definition. By the condition iii), the compensators also achieve decentralized stabilization so that X_1 and X_2 are such that $|\tilde{\Theta}|$ is a unit, or equivalently, it has constant sign at all points of $\mathcal{R}_{+e}$. By unimodularity of $\tilde{\Theta}$, the pair $(\tilde{D}_{12}, \tilde{N}_{11}N_{12}(\bar{D}_{12}K_2 - \bar{N}_{22}X_2))$ is left coprime which implies that the pair $(\tilde{D}_{12}, X_2)$ is left coprime. Let

$$\hat{D}_{12}^{-1}X_2 = B_2A_{12}^{-1}$$

for some right-coprime pair (B_2, A_{12}) such that $|A_{12}| = |\tilde{D}_{12}|$. The following alternative expression for $|\tilde{\Theta}|$ is now easy to verify also using (26)

$$|\tilde{\Theta}| = |\tilde{D}_{21}A_{12} - \tilde{N}_{22}N_{21}\bar{N}_{11}(\bar{K}_1 - X_1N_{11}) \cdot \bar{N}_{12}(K_2A_{12} - \hat{N}_{22}B_2)|.$$

When this is evaluated at any $\mathcal{R}_{+e}$ -blocking zero of $\tilde{N}_{22}N_{21}\bar{N}_{11}$, one has $|\tilde{\Theta}| = |\tilde{D}_{21}||A_{12}| = |\tilde{D}_{21}||\tilde{D}_{12}|$. This

$$\begin{bmatrix} -N_{12}(\bar{D}_{12}K_2 - \bar{N}_{22}X_2)N_{22} \\ D_{21} \end{bmatrix}$$

yields that $|D_{21}||\tilde{D}_{12}|$ has constant sign at all $\mathcal{R}_{+e}$ -blocking zeros of $\tilde{N}_{22}N_{21}\bar{N}_{11}$. By a completely dual argument, one can establish that at any $\mathcal{R}_{+e}$ -blocking zero of $\tilde{N}_{11}N_{12}\bar{N}_{22}$ one has $|\tilde{\Theta}| = |\tilde{D}_{12}||\tilde{D}_{21}|$. Consequently, at all zeros in the set

$$\begin{aligned} \bar{\mathcal{S}} := \{ \mathcal{R}_{+e}\text{-blocking zeros of } \tilde{N}_{11}N_{12}\bar{N}_{22} \} \\ \cup \{ \mathcal{R}_{+e}\text{-blocking zeros of } \tilde{N}_{22}N_{21}\bar{N}_{11} \} \end{aligned}$$

the sign of $|\tilde{D}_{12}||\tilde{D}_{21}|$ is constant. Since the latter is an associate of $|D_{12}||D_{21}|$, the sign of $|D_{12}||D_{21}|$ remains constant for all zeros in $\bar{\mathcal{S}}$. Finally note that

$$N_{11}\bar{N}_{12}\hat{N}_{22}\hat{D}_{12}^{-1} = \tilde{D}_{12}^{-1}\tilde{N}_{11}N_{12}\bar{N}_{22}$$

$$N_{22}\bar{N}_{21}\hat{N}_{11}\hat{D}_{21}^{-1} = \tilde{D}_{21}^{-1}\tilde{N}_{22}N_{21}\bar{N}_{11}$$

where each fraction is coprime. Consequently, the $\mathcal{R}_{+e}$ -blocking zeros of $N_{11}\bar{N}_{12}\hat{N}_{22}$ (respectively, $N_{22}\bar{N}_{21}\hat{N}_{11}$)

are the same as the $\mathcal{R}_{+e}$ -blocking zeros of $\tilde{N}_{11}N_{12}\tilde{N}_{22}$ (respectively, $\tilde{N}_{22}N_{21}\tilde{N}_{11}$). In other words, the set $\mathcal{S}$ has the alternative descriptions

$$\begin{aligned} \bar{\mathcal{S}} := \{ & \mathcal{R}_{+e}\text{-blocking zeros of } N_{11}\tilde{N}_{12}\hat{N}_{22} \} \\ & \cup \{ \mathcal{R}_{+e}\text{-blocking zeros of } N_{22}\tilde{N}_{21}\hat{N}_{11} \} \end{aligned}$$

or the one in the theorem statement. The necessity of the parity interlacing condition in the theorem statement is thus established.

[If] Let the coprimeness conditions (21), (22), and the rank conditions (16) hold. We need to show that the parity interlacing condition of the theorem, under the hypothesis (25) implies the existence of X_1, X_2 such that $\tilde{\Theta}$ is unimodular. We first show that there exists X_1 satisfying the following three conditions, where $Y_1 := \tilde{D}_{21}K_1 - \tilde{N}_{11}X_1$.

C1: $(\tilde{D}_{21}, \tilde{N}_{22}N_{21}Y_1)$ is left coprime.

C2: $(\tilde{N}_{22}N_{21}Y_1, \tilde{D}_{12})$ is left coprime.

C3: The set of $\mathcal{R}_{+e}$ -blocking zeros of $\tilde{N}_{22}N_{21}Y_1 N_{11}\tilde{N}_{12}\tilde{N}_{22}$ is the set $\mathcal{S}$.

To show that such an X_1 exists, we show that the set of X_1 satisfying each condition is open and dense in $S^{r_1 \times p_1}$. Note, by using $\tilde{D}_{21}^{-1}\tilde{N}_{22}N_{21} = N_{22}\tilde{N}_{21}\tilde{D}_{21}^{-1}$ and $\tilde{D}_{21}^{-1}\tilde{N}_{11} = \tilde{N}_{11}\tilde{D}_{21}^{-1}$ where each fraction is coprime, that C1 holds iff $(\tilde{D}_{21}, X_1)$ is left coprime. This last condition holds for almost all $X_1 \in S^{r_1 \times p_1}$. On the other hand, the first rank inequality of (23) holds for all $z \in \mathcal{E}_{+e}$ iff for all such z one has

$$\text{rank} \begin{bmatrix} \tilde{D}_{12} & \tilde{N}_{22}N_{21}\tilde{D}_{21}K_1 \\ 0 & \tilde{N}_{22}N_{21}\tilde{N}_{11} \end{bmatrix} \geq p_1$$

as the two relevant matrices can be shown to be equivalent over S . This last rank condition is equivalent to the quadruple $(0, \tilde{D}_{12}, \tilde{N}_{22}N_{21}\tilde{D}_{21}K_1, \tilde{N}_{22}N_{21}\tilde{N}_{11})$ being complete. Applying [7, lemma 2.6], it now follows that C2 also holds for almost all $X_1 \in S^{r_1 \times p_1}$. Finally, by Lemma A.2 of the Appendix and by the hypothesis (25), it follows that the condition C3 is satisfied for almost all $X_1 \in S^{r_1 \times p_1}$. Therefore, there exists an open and dense subset of $S^{r_1 \times p_1}$ every element X_1 of which satisfies all three conditions C1–C3. Fix one such X_1 and Y_1 and let $A, B, \bar{A}, \bar{B}$ be some matrices satisfying $|A| = |\tilde{D}_{12}|$ and

$$\begin{bmatrix} \bar{A} & -\bar{B} \\ B & A \end{bmatrix} \begin{bmatrix} \tilde{D}_{12} \\ -\tilde{N}_{22}N_{21}Y_1 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

which is possible by C2. On multiplying $\tilde{\Theta}$ on the left by the first matrix above, it is clear that

$$|\tilde{\Theta}| = |A\tilde{D}_{21} - B\tilde{N}_{11}N_{12}(\tilde{D}_{12}K_2 - \tilde{N}_{22}X_2)|$$

or noting that $B\tilde{N}_{11}N_{12}\tilde{D}_{12} = A\tilde{N}_{22}N_{21}Y_1N_{11}\tilde{N}_{12}$ and $Y_1N_{11} = \tilde{N}_{11}\bar{Y}_1$ with $\bar{Y}_1 := \bar{K}_1 - X_1N_{11}$

$$\begin{aligned} |\tilde{\Theta}| &= |\hat{A} + \hat{B}X_2| \\ &= |A(\tilde{D}_{21} - \tilde{N}_{22}N_{21}\tilde{N}_{11}\bar{Y}_1\tilde{N}_{12}K_2) + B\tilde{N}_{11}N_{12}\tilde{N}_{22}X_2| \end{aligned}$$

where the matrices $\hat{A}, \hat{B}$ have the obvious definitions. By C1 and various other left coprimeness conditions, it follows that $(\hat{A}, \hat{B})$ is left coprime. From (26) it follows that $\bar{K}_2 = \hat{D}_{12}\hat{K}_2$, for some $\hat{K}_2$. On noting that $(\tilde{D}_{21} - \tilde{N}_{22}N_{21}\tilde{N}_{11}\bar{Y}_1\tilde{N}_{12}K_2)^{-1}\tilde{N}_{22} = N_{22}(D_{21} - N_{21}Y_1N_{11}\tilde{N}_{12}\tilde{N}_{22}\hat{K}_2)^{-1}$ where both fractions are coprime, and $|\tilde{D}_{21}| = |\hat{D}_{21}|$, it follows that $|\hat{A}|$ takes the same sign at all $z \in \mathcal{S}$. Also from C3, the set of $\mathcal{R}_{+e}$ -blocking zeros of $\hat{B}$ is precisely the set $\mathcal{S}$. The parity interlacing condition of the theorem statement now implies that $|\hat{A}|$ has constant sign at all $\mathcal{R}_{+e}$ -blocking zeros of $\hat{B}$ which, by the standard result on strong stabilization, yields the fact that there exists X_2 such that $\tilde{\Theta}$ is unimodular. $\square$

Remark 3.1: The connectivity assumption of the theorem is only used in the ‘If’ part of the proof. Hence, all systems $\tilde{\Sigma}$ for which P2 has a solution satisfy the given parity interlacing condition. $\triangle$

Remark 3.2: On comparing the solvability conditions to problems P1 and P2, it is clear that the poles and zeros of the main subplants play a dual role. This is certainly not a coincidence and is a consequence of a deeper duality in the feedforward and feedback interconnections of Figs. 1 and 2, or in the problem definitions P1 and P2. We have at present no precise formulation of this latter duality but can offer the following crude version. Assuming the main subplant transfer matrices Z_{ii}, T_{ii} for $i \in 2$ are all nonsingular, it is easy to verify, via (1) and (2), that

$$T^{-1} = Z[T_{11}^{-1}, -T_{12}, T_{22}^{-1}, -T_{21}] = \begin{bmatrix} T_{11}^{-1} & -T_{12} \\ -T_{21} & T_{22}^{-1} \end{bmatrix}$$

where the notation $Z[Z_{11}, Z_{12}, Z_{22}, Z_{21}]$ is a more explicit representation for Z as a function of its subplant transfer matrices. (Similarly, $Z^{-1} = T[Z_{11}^{-1}, -Z_{12}, Z_{22}^{-1}, -Z_{21}]$.) Neglecting all the properness requirements and assuming that both compensators employed are nonsingular, it follows that (Z_{c1}, Z_{c2}) is a solution to P2 for T if and only if $(Z_{c1}^{-1}, Z_{c2}^{-1})$ is a solution to P1 for T^{-1} . Consequently, P2 is solvable for T if and only if P1 is solvable for T^{-1} , and similarly, P1 is solvable for Z if and only if P2 is solvable for Z^{-1} . A refinement of the formulation of this type of duality between P1 and P2 will be useful in unifying the reliable synthesis problems for N -channel ($N \geq 2$) interconnected plants. $\triangle$

We now specialize the result of Theorem 3.1 to two cases.

Case 1: T_{11} and T_{22} are stable transfer matrices. In this case, we can let $\bar{D}_{ii} = I, \bar{D}_{ii} = I$ for $i \in 2$ so that $N_{11}\tilde{N}_{12}\tilde{N}_{22}$ and $T_{11}T_{12}T_{22}$ have the same set of $\mathcal{R}_{+e}$ -blocking zeros. Similarly for their duals.

Corollary 3.1: Let T_{11} and T_{22} be stable transfer matrices and suppose that (24) holds. Then, P2 has a solution if and only if the conditions (21), (22), and (23) hold and $|D_{12}||D_{21}|$ takes the same sign at the union of the $\mathcal{R}_{+e}$ -blocking zeros of $T_{11}T_{12}T_{22}$ and $T_{22}T_{21}T_{11}$.

Case 2: T_{12} and T_{21} are stable transfer matrices. In this

case $|D_{12}|, |D_{21}|$ are units and the parity interlacing condition as well as the coprimeness and rank conditions (21), (22), and (23) are all satisfied.

Corollary 3.2: Let T_{12} and T_{21} be stable transfer matrices and suppose that (24) holds. Then, P2 is solvable.

IV. CONCLUSIONS

We have obtained conditions for the solvability of problems P1, P1_{i/o}, P2, and a number of other reliable decentralized stabilization problems under mild connectivity assumptions (10) and (24). The connectivity conditions arise as a consequence of our particular technique of compensator construction and are not believed to be inherently related to solvability. The fact that the central requirement for reliable decentralized stabilization in all these problems is a parity interlacing property indicates that reliable synthesis is generically easy to achieve. The main results obtained in the paper also indicate some interesting features of feedforward and feedback interconnected plants that admit reliable decentralized synthesis. As far as reliability is concerned:

- 1) stable interconnections cause no extra problems;
- 2) stable main subplants lead to considerable simplification in P1;
- 3) nonsingular and minimum-phase main subplants lead to considerable simplifications in P2;
- 4) periodically time-varying feedbacks should help since these lead to an elimination of blocking zeros (see [4]);
- 5) the roles played by the poles and zeros of the main subplants are dual in feedforward and feedback interconnected systems.

We note that one can think of many quite different extensions of the problems we considered to the case of N -channels for $N > 2$. In all these extensions, the general features noted in 1)–5) are anticipated to find a fundamental use.

APPENDIX

In this Appendix we prove Proposition 2.3. We need the following technical lemmata.

Lemma A.1: Let $\tilde{D} \in S^{\tilde{p} \times \tilde{r}}$, $\tilde{E} \in S^{\tilde{p} \times \tilde{n}}$, $\tilde{F} \in S^{\tilde{m} \times \tilde{r}}$, and $X_0 \in S^{\tilde{n} \times \tilde{m}}$ where $\tilde{p} \geq 2$, $\tilde{r} \geq 2$. Let q_0 be an integer satisfying $0 < q_0 < \min(\tilde{p}, \tilde{r})$ such that $\text{rank}(\tilde{D} + \tilde{E}X_0\tilde{F}) \geq q_0$, for all $z \in \mathcal{E}_{+e}$. Then, given $z_0 \in \mathcal{E}_{+e}$, the class of Δ for which $\text{rank}(\tilde{D} + \tilde{E}(X_0 + \Delta)\tilde{F})(z_0) > q_0$ is dense in $S^{\tilde{n} \times \tilde{m}}$ if and only if

$$\text{rank}[\tilde{D} \quad \tilde{E}](z_0) > q_0 \text{ and } \text{rank}[\tilde{D}' \quad \tilde{F}'](z_0) > q_0.$$

Lemma A.2: Let $\tilde{A} \in S^{\tilde{p} \times \tilde{r}}$, $B \in S^{\tilde{p} \times \tilde{n}}$, and $C \in S^{\tilde{m} \times \tilde{r}}$ be such that either $\text{rank} \tilde{B} \geq 2$ or $\text{rank} \tilde{C} \geq 2$. Then, for $z \in \mathcal{E}_{+e}$, $(\tilde{A} + \tilde{B}X\tilde{C}) = 0$ only if $[\tilde{A} \quad \tilde{B}] = 0$ or $[\tilde{A}' \quad \tilde{C}'] = 0$, for almost all $X \in \mathcal{X}$.

Proof of Proposition 2.3: Proof of i) is immediate, and is therefore omitted. To prove ii), define $\mathcal{Q}_{11} = \{z \in \mathcal{E}_{+e} | Q_{11}| = 0\}$ and $\tilde{\mathcal{Q}}_{11} = \mathcal{Q}_{11} \cap \mathcal{S}$. We will first show that

if $X_0 \in \mathcal{X}$ is such that $\mathcal{S} \not\subseteq \mathcal{S}(X_0)$, there exists a perturbation on X_0 , which provides $\mathcal{S} = \mathcal{S}(X_p)$, where $X_p = X_0 + \Delta$ for some Δ . Moreover, the norm of Δ can be chosen smaller than any prespecified positive number. Let $z_0 \in \mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}$. Then, $\text{rank}[\Pi \quad \Gamma_0] > q$ and $\text{rank}[\Pi' \quad \Lambda_2'] > q$. Applying Lemma A.1, with $\tilde{D} = \Pi$, $\tilde{E} = \Gamma_0$, and $\tilde{F} = \Lambda_2$, we have that the class of X satisfying $z_0 \notin \mathcal{S}(X)$ is dense in $\mathcal{X}$. (Recall that $\mathcal{X}$ is dense in $\mathcal{S}^{r_2 \times p_2}$.) It can also be verified that this class is open. Hence, if $\mathcal{S}(X_0)$ contains any z from $\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}$, there exists Δ_0 with arbitrarily small norm satisfying $\mathcal{S}(X_0 + \Delta_0) \cap (\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}) = \emptyset$, i.e., $\mathcal{S}(X_0 + \Delta_0)$ does not contain any zero from $\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}$.

Define $A(X) = E + (GF_r - P_r X)R_l$. Let $D_l P_l C^{-1} = T_l^{-1} M_l$, for a left-coprime pair of matrices (T_l, M_l) . Note that $\text{rank} Z_{21} = \text{rank} R_l \bar{R}_1$ and $\text{rank} Z_{12} = \text{rank} M_l P_r$, and we have that either $\text{rank} R_l \bar{R}_1 \geq 2$ or $\text{rank} M_l P_r \geq 2$. For any $X \in \mathcal{X}$, $\mathcal{N}(X) = \{z \in \mathcal{E}_{+e} - \mathcal{Q}_{11} | (-T_l N_l \bar{D} + M_l A(X) \bar{R}_1) = 0\} = \mathcal{S}(X) - \mathcal{Q}_{11}$. Observe that, if $z \in \mathcal{E}_{+e} - \mathcal{Q}_{11}$ satisfies

$$[-T_l N_l \bar{D} + M_l(E + GF_r R_l) \bar{R}_1 - M_l P_r] = 0 \quad (28)$$

or

$$\begin{bmatrix} -T_l N_l \bar{D} + M_l(E + GF_r R_l) \bar{R}_1 \\ R_l \bar{R}_1 \end{bmatrix} = 0 \quad (29)$$

then, $z \in \mathcal{N}(X)$, independent of X . Conversely, Lemma A.2 implies that for almost $X \in \mathcal{X}$, $\mathcal{N}(X)$ consists only of those z , for which (28) and (29) hold. On the other hand, for $z \in \mathcal{E}_{+e} - \mathcal{Q}_{11}$, (28) holds only if $\text{rank}[\Pi \quad \Gamma_0] = q$ and (29) holds only if $\text{rank}[\Pi' \quad \Lambda_2'] = q$. That is, there exists Δ_1 with arbitrarily small norm such that $z \in \mathcal{S}(X_0 + \Delta_0 + \Delta_1) - \mathcal{Q}_{11}$ implies $\text{rank}[\Pi \quad \Gamma_0] = q$ or $\text{rank}[\Pi' \quad \Lambda_2'] = q$ holds. The norm of Δ_1 can be chosen sufficiently small still to have $\mathcal{S}(X_0 + \Delta_0 + \Delta_1) \cap (\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}) = \emptyset$. These results imply $\mathcal{S}(X_0 + \Delta) = \mathcal{S}$, where $\Delta = \Delta_0 + \Delta_1$. Also, the norm of Δ can be chosen smaller than any prespecified positive number, by restricting the norms of Δ_0 and Δ_1 suitably.

It will now be shown that if $X_0 \in \mathcal{X}$ satisfies $\mathcal{S}(X_0) = \mathcal{S}$, there exists an open ball around X_0 such that for all X belonging to this ball $\mathcal{S}(X) = \mathcal{S}$. First, one can verify that the class of X satisfying $\mathcal{S}(X) \cap (\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}) = \emptyset$ is open in $\mathcal{X}$. Clearly, $\mathcal{S}(X_0) \cap (\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}) = \emptyset$, and there exists an open ball around X_0 such that for all $X \in \mathcal{X}$ in this ball, $\mathcal{S}(X) \cap (\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}) = \emptyset$. Also, $\mathcal{S}(X_0) - \mathcal{Q}_{11}$ consists only of those z for which (28) or (29) holds. From Lemma A.2, there exists an open ball around X_0 such that for all $X \in \mathcal{X}$ in this ball, $\mathcal{S}(X_0) - \mathcal{Q}_{11}$ consists only of those z for which (28) or (29) holds. Since the intersection of open sets is open, we conclude that there is an open ball around X_0 , such that for all X in this ball $\mathcal{S}(X) \cap (\mathcal{Q}_{11} - \tilde{\mathcal{Q}}_{11}) = \emptyset$ and $\mathcal{S}(X) - \mathcal{Q}_{11}$ contains only those z for which (28) or (29) holds, i.e., $\mathcal{S}(X) = \mathcal{S}$. $\square$

The simple proof of Lemma A.1 is omitted.

Proof of Lemma A.2: We will prove the following statement, which directly implies Lemma A.2.

Let $\tilde{A} \in S^{\hat{p} \times \hat{r}}$, $\tilde{B} \in S^{\hat{p} \times \hat{n}}$, and $\tilde{C} \in S^{\hat{m} \times \hat{r}}$ be such that the smallest invariant factor (sif) of $[\tilde{A} \ \tilde{B}]$ and the sif of $[\tilde{A}' \ \tilde{C}']$ are units, and either $\text{rank } \tilde{B} \geq 2$ or $\text{rank } \tilde{C} \geq 2$. Then, for almost all $X \in S^{\hat{n} \times \hat{m}}$, $\text{sif}(\tilde{A} + \tilde{B}X\tilde{C})$ is unit.

Proof: We can assume neither $\tilde{B}$ nor $\tilde{C}$ equals zero, because otherwise $\text{sif} \tilde{A}$ is unit, and the lemma holds trivially. If X_0 is such that $\text{sif}(\tilde{A} + \tilde{B}X_0\tilde{C})$ is unit, by choosing the norm of Δ small enough, $\text{sif}(\tilde{A} + \tilde{B}(X_0 + \Delta)\tilde{C})$ is still a unit, since the set of units are open in S . To show that the class of such X is dense, assume X_0 is such that $\text{sif}(\tilde{A} + \tilde{B}X_0\tilde{C})$ is not a unit of S . Let U_l and U_r be unimodular matrices of suitable size, such that

$$U_l \tilde{B} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \text{ and } \tilde{C} U_r = [\hat{C} \ 0]$$

where $\hat{B} \in S^{\hat{p} \times \hat{n}}$ and full-row rank, and $\hat{C} \in S^{\hat{m} \times \hat{r}}$ and full-column rank. By assumption, either $\text{rank } \hat{B} \geq 2$ or $\text{rank } \hat{C} \geq 2$. We assume $\text{rank } \hat{B} \geq 2$. Otherwise $\text{rank } \hat{C} \geq 2$ and the dual of the proof below follows. Clearly, $\text{rank } \hat{B} \geq 2$ implies $\hat{p} \geq 2$. Let b and c be the smallest invariant factors of $\hat{B}$ and $\hat{C}$, respectively. Define $B_l = \hat{B}/b$ and $C_r = \hat{C}/c$. There exist unimodular matrices V_l and V_r such that $V_l B_l = B$, $C_r V_r = C$, where the first row of B is left unimodular and the first column of C is right unimodular. Further define

$$\bar{A} = \begin{bmatrix} V_l & 0 \\ 0 & I \end{bmatrix} U_l \tilde{A} U_r \begin{bmatrix} V_r & 0 \\ 0 & I \end{bmatrix}.$$

Partition $\bar{A}$ as follows:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

where $\bar{A}_{11} \in S^{\hat{p} \times \hat{r}}$, $\bar{A}_{12} \in S^{\hat{p} \times \hat{r} - \hat{r}}$, $\bar{A}_{21} \in S^{\hat{p} - \hat{p} \times \hat{r}}$, and $\bar{A}_{22} \in S^{\hat{p} - \hat{p} \times \hat{r} - \hat{r}}$. Clearly, $\text{sif}(\bar{A} + \tilde{B}(X_0 + \Delta)\tilde{C})$ equals the sif of

$$\bar{A} + bc \begin{bmatrix} B \\ 0 \end{bmatrix} (X_0 + \Delta) \begin{bmatrix} C & 0 \end{bmatrix}$$

for any $\Delta \in S^{\hat{n} \times \hat{m}}$. Define $A = \bar{A}_{11} + bcBX_0C$. Let us assume, without loss of generality that the first column of A is nonzero, because otherwise there exists a perturbation Δ_1 on X_0 with arbitrarily small norm such that the first column of A in nonzero with X_0 is replaced by $X_0 + \Delta_1$ (This is guaranteed by the fact that B and the first column of C are nonzero.) Also note that for any $z \in \mathcal{E}_{+e}$, $(bc) = 0$ and $\bar{A}_{11} = 0$ imply

$$\begin{bmatrix} 0 & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \neq 0$$

because of the hypothesis that $\text{sif}[\tilde{A} \ \tilde{B}]$ and $\text{sif}[\tilde{A}' \ \tilde{C}']$ are units. Let $\Sigma_{i=1}^{\hat{n}} \beta_i b_{ji} = 1$, and $\Sigma_{i=1}^{\hat{m}} c_{ij} \alpha_i = 1$, for some α_i , $i = 1, \dots, \hat{m}$ and β_i , $i = 1, \dots, \hat{n}$, where b_{ji} , $i = 1, \dots, \hat{n}$ denote the first-row elements of B , and c_{ij} , $i = 1, \dots, \hat{m}$ denote the first-column elements of C . Define

$\theta_j = \Sigma_{i=1}^{\hat{n}} \beta_i b_{ji}$, $j = 1, \dots, \hat{p}$, and $\gamma_j = \Sigma_{i=1}^{\hat{m}} c_{ij} \alpha_i$, $j = 1, \dots, \hat{r}$, where b_{ji} denotes the (j, i) th element of B and c_{ij} denotes the (i, j) th element of C . Note that, $\theta_1 = 1$ and $\gamma_1 = 1$. By the fact that $\hat{p} \geq 2$, $\gamma_1 = 1$ and the first column of A is nonzero, we can assume that for at least one index pair (i, j) , $a_{ij} \neq \theta_i \gamma_j a_{11}$. (We omit the simple proof of the construction of such β_i and α_j , $i = 1, \dots, \hat{n}$, and $j = 1, \dots, \hat{m}$.) Now let Σ be a nonempty set of index pairs so that $\Sigma = \{(i_1, j_1), (i_2, j_2), \dots, (i_v, j_v)\}$ where $v = \min(\hat{p}, \hat{r})$, satisfying $a_{ij} \neq \theta_i \gamma_j a_{11}$, whenever $(i, j) \in \Sigma$, and $a_{ij} = \theta_i \gamma_j a_{11}$, whenever $(i, j) \notin \Sigma$. Define $q_{ij} = \text{gcf}(a_{ij}, bc)$, $i = 1, \dots, \hat{p}$, $j = 1, \dots, \hat{r}$, such that $a_{ij} = q_{ij} \hat{a}_{ij}$ and $bc = q_{ij} \hat{q}_{ij}$ for coprime pairs $(\hat{a}_{ij}, \hat{q}_{ij})$. If $a_{11} = 0$, let δ satisfy (δ, a_{ij}) are coprime for all $(i, j) \in \Sigma$. If $a_{11} \neq 0$, let δ satisfy $(\hat{a}_{11} + \delta \hat{q}_{11}, a_{ij} - \theta_i \gamma_j a_{11})$ are coprime for all $(i, j) \in \Sigma$. The norm of δ can be chosen arbitrarily small in both cases. By letting

$$\Delta = \begin{bmatrix} \beta_1 \delta \alpha_1 & \cdots & \beta_1 \delta \alpha_{\hat{m}} \\ \vdots & & \vdots \\ \beta_{\hat{n}} \delta \alpha_1 & \cdots & \beta_{\hat{n}} \delta \alpha_{\hat{m}} \end{bmatrix}$$

we have $(A + bcB\Delta C)_{ij} = a_{ij} + \delta bc \theta_i \gamma_j$, $i = 1, \dots, \hat{p}$, $j = 1, \dots, \hat{r}$. If $a_{11} = 0$, the choice of Δ yields

$$\text{gcf}_{i=1, \hat{p}, j=1, \hat{r}}[(A + bcB\Delta C)_{ij}] = \text{gcf}_{(i, j) \in \Sigma}(a_{ij}, bc).$$

In the case that $a_{11} \neq 0$, the choice of Δ yields

$$\text{gcf}_{i=1, \hat{p}, j=1, \hat{r}}[(A + bcB\Delta C)_{ij}] = \text{gcf}_{(i, j) \in \Sigma}(a_{ij}, q_{11}).$$

(This latter statement can be seen more clearly as follows. Observe that whenever $(i, j) \notin \Sigma$, $\text{gcf}[(A + B\Delta Cbc)_{11}, (A + B\Delta Cbc)_{ij}] = a_{11} + bc\delta$. So

$$\begin{aligned} \text{gcf}_{i=1, \hat{p}, j=1, \hat{r}}[(A + bcB\Delta C)_{ij}] &= \text{gcf}(a_{11} + bc\delta, \text{gcf}_{(i, j) \in \Sigma}(A + bcB\Delta C)_{ij}). \\ (\hat{a}_{11} + \delta \hat{q}_{11}) &= 0, z \in \mathcal{E}_{+e} \text{ imply } \delta(z) = -\hat{a}_{11}/\hat{q}_{11}, \text{ where } \hat{q}_{11} \text{ is nonzero because of the coprimeness of } (\hat{a}_{11}, \hat{q}_{11}). \text{ In this case } (a_{ij} + bc\delta\theta_i\gamma_j) = (a_{ij} - \theta_i\gamma_j a_{11}) \neq 0. \text{ Hence} \end{aligned}$$

$$\begin{aligned} &\text{gcf}(a_{11} + bc\delta, \text{gcf}_{(i, j) \in \Sigma}(A + B\Delta Cbc)_{ij}) \\ &= \text{gcf}(q_{11}, \text{gcf}_{(i, j) \in \Sigma}(A + B\Delta Cbc)_{ij}) \\ &= \text{gcf}_{(i, j) \in \Sigma}(q_{11}, a_{ij}). \end{aligned}$$

In both cases $\text{sif}(A + B\Delta Cbc)$ is coprime with

$$\text{sif} \begin{bmatrix} 0 & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}.$$

Since the norm of Δ can be made smaller than any prespecified positive number by choosing δ suitably, the proof is completed. $\square$

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