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Identification of Nonminimum Phase MA Systems Using Cepstral Operations on Slices of Higher Order Spectra

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Abstract—In this paper, recursive and least squares methods for reconstructing the impulse response of a nonminimum phase, linear, time-invariant (NMP-LTI) MA system from its $(k+1)$ st-order spectrum ($k > 1$) are presented. The methods are nonparametric. They are based on computing the complex cepstrum of the impulse response sequence of the unknown system from the diagonal slice of the $(k+1)$ st-order spectrum. The computational complexity of the system identification problem can be considerably reduced by using slices of higher order spectra.

I. INTRODUCTION

Let $y(n)$ be the output of a linear time-invariant (LTI) system described by

$$y(n) = \sum_i h(i)x(n-i) \quad (1)$$

where $x(n)$ is non-Gaussian, i.i.d., random process with $E\{x(n)\} = 0$ and $(k+1)$ st-order cumulant β_{k+1} . In this paper we present new methods for reconstructing the impulse response $h(n)$ or equivalently the phase and magnitude of the system transfer function $H(\omega)$ of a nonminimum phase (NMP) system given only the output data $\{y(n)\}$. System identification methods are based on homomorphic signal processing techniques. Identification of nonminimum phase systems using complex cepstra of higher order cumulants was first introduced by Pan and Nikias [6]. Their FFT-based method utilizes all samples of the polyspectrum. In this paper, we show how to solve this problem using cepstral operations on a 1-D slice of the $(k+1)$ st-order spectrum of the output signal.

The main motivation behind the use of slices of higher order spectra in system identification problem is to use well-known 1-D signal processing techniques [1], [2]. In this way the computational complexity of the problem is reduced. In the sequel, we define a 1-D sequence $g(n)$ in terms of the $(k+1)$ st-order spectrum. Then, we develop a relationship between $g(n)$ and the complex cepstrum $\hat{h}(n)$ of the system impulse response $h(n)$. The sequence $\hat{h}(n)$ will then be recovered from the sequence $g(n)$ by using *recursive* and *nonrecursive* methods. Both of these methods are nonparametric and have the advantage of being

computationally efficient. For instance the *recursive* method requires three 1-D FFT operations to obtain the complex cepstrum of the underlying system from the third-order cumulants, as opposed to Pan and Nikias' FFT-based method, which requires three 2-D FFT operations [5].

This paper is organized as follows. In Sections II and III, we develop recursive and least squares methods for the reconstruction of the system impulse response using a 1-D slice of the $(k+1)$ st-order spectrum. In Section IV, we discuss some of the properties of the methods described; and in Section V, we present simulation examples demonstrating the effectiveness of our algorithms.

II. RECURSIVE (RC) METHOD

The $(k+1)$ st-order spectrum, $C_{k+1}(z_1, \dots, z_k)$, of the output sequence $y(n)$ is related to the system function, $H(z)$, via the following relationship [3]:

$$C_{k+1}(z_1, \dots, z_k) = \beta_{k+1} H(z_1) \cdots H(z_k) H(z_1^{-1} z_2^{-1} \cdots z_k^{-1}). \quad (2)$$

In this section, we present a recursive method for the reconstruction of 1-D, NMP-LTI systems from a 1-D slice of the output $(k+1)$ st-order spectrum. The method utilizes the 1-D slice, $G(z)$, corresponding to $z_1 = z_2 = \cdots = z_k = z$, i.e.,

$$G(z) = C_{k+1}(z, \dots, z) = \beta_{k+1} H^k(z) H(z^{-k}). \quad (3)$$

In many practical problems, $H(z)$ is a nonminimum phase function; it has zeros both *inside* and *outside* the unit circle. If $h(n)$ has finite length, then its z -transform is given by

$$H(z) = Az^{-r} I(z^{-1}) O(z) \quad (4)$$

where A is a constant, r is an integer, $I(z^{-1})$ is the minimum phase component of $H(z)$, that is,

$$I(z^{-1}) = \prod_{i=1}^{L_1} (1 - a_i z^{-1}), \quad |a_i| < 1 \quad (5)$$

and $O(z)$ is the maximum phase component of $H(z)$, that is,

$$O(z) = \prod_{i=1}^{L_2} (1 - b_i z), \quad |b_i| < 1. \quad (6)$$

By substituting (4)–(6) into (3), we get

$$G(z) = \beta_{k+1} A^{k+1} \prod_{i=1}^{L_1} (1 - a_i z^{-1})^k \prod_{i=1}^{L_2} (1 - b_i z)^k \cdot \prod_{i=1}^{L_1} (1 - a_i z^k) \prod_{i=1}^{L_2} (1 - b_i z^{-k}). \quad (7)$$

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The 1-D slice of the $(k+1)$ st-order spectrum, $G(z)$, is a polynomial of degree $2(L_1 + L_2)k$. It has the following roots: $\{a_i\}_{i=1}^{L_1}$ and $\{1/b_i\}_{i=1}^{L_2}$ repeated k times, $\{\sqrt[k]{a_i}\}_{i=1}^{L_1}$, and $\{1/\sqrt[k]{b_i}\}_{i=1}^{L_2}$. If the roots of $G(z)$ are known, $H(z)$ is reconstructed directly using (4). In practical applications, the received signal is usually a noise corrupted version of the original one. In addition, there is an error associated with the estimation of $G(z)$. As a result, $G(z)$ can not be factored as in (7). Hence the zeros of $H(z)$ can not be identified from the roots of $G(z)$. To circumvent such a problem, we propose a complex cepstrum based identification method which makes use of the 1-D slice $G(z)$, without finding its roots. The method is based on the following observations.

Let us define the sequence, $g(n)$, as the inverse Z-transform of $G(z)$ as follows:

$$g(n) = \mathcal{Z}^{-1}[G(z)]. \quad (8)$$

The slice $G(z)$ is related to the $(k+1)$ st-order cumulants $c_{k+1}(\tau_1, \tau_2, \dots, \tau_k)$ via the following relationship:

$$G(z) = \sum_{\tau_1, \dots, \tau_k} c_{k+1}(\tau_1, \tau_2, \dots, \tau_k) z^{-(\tau_1 + \dots + \tau_k)}. \quad (9)$$

By virtue of (8) and (9), it is straightforward to show that

$$g(n) = \sum_{\tau_2, \dots, \tau_k} c_{k+1}(n - \tau_2 - \dots - \tau_k, \tau_2, \dots, \tau_k). \quad (10)$$

Let $\hat{g}(n) = \mathcal{F}^{-1}[\log G(z)]$ be the complex cepstrum of the sequence $g(n)$ [4]. If $G(z)$ has no zeros on the unit circle, then

$$\hat{g}(n) = \mathcal{F}^{-1}[\log G(\omega)] = \mathcal{F}^{-1}[\log C_{k+1}(\omega, \dots, \omega)] \quad (11)$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform operator. The assumption $G(\omega) \neq 0$ for $\omega \in [-\pi, \pi]$ implies that $H(\omega) \neq 0$ for $\omega \in [-\pi, \pi]$. By substituting (3) into (11), we obtain

$$\begin{aligned} \hat{g}(n) &= \mathcal{F}^{-1}[\log \beta_{k+1} + k \log H(\omega) + \log H(-k\omega)] \\ &= \begin{cases} (k+1)\hat{h}(n) + \log \beta_{k+1} & \text{if } n = 0 \\ k\hat{h}(n) + \hat{h}(-n/k) & \text{if } n = km; |m| = 1, 2, \dots \\ k\hat{h}(n) & \text{otherwise.} \end{cases} \end{aligned} \quad (12)$$

Examining (12), it is clear that the 1-D complex cepstrum $\hat{g}(n)$ computed from the cumulants of the received signal $y(n)$ is only related to the complex cepstrum $\hat{h}(n)$ of the underlying system $h(n)$. If $\hat{g}(n)$ is known, $\hat{h}(n)$ can be determined from (12) using the following recursion.

$$\hat{h}(n) = \begin{cases} (\hat{g}(n) - \log \beta_{k+1})/(k+1) & \text{if } n = 0 \\ (\hat{g}(n) - \hat{h}(-n/k))/k & \text{if } n = km; |m| = 1, 2, \dots \\ \hat{g}(n)/k & \text{otherwise.} \end{cases} \quad (13)$$

Once $\hat{h}(n)$ is determined, the sequence $h(n)$ can be obtained using the definition of the complex cepstrum, i.e.,

$$h(n) = \mathcal{F}^{-1}[\exp(\mathcal{F}[\hat{h}(n)])]. \quad (14)$$

III. LEAST SQUARES (LS) METHOD

The reconstruction method presented in the previous section requires 1-D phase unwrapping to compute $\hat{g}(n)$ from $G(\omega)$. To

avoid such a problem as well as polyspectra estimation techniques required by (3), we develop a direct convolutional relationship between $g(n)$ and the sequence $\hat{h}(n)$ in this section and show how it can be solved using least squares method. The algorithm is based on the following observations.

By computing the derivative of both sides of the equation $\hat{G}(z) = \log G(z)$ with respect to z and after some algebraic manipulations, we get

$$G(z)z \frac{d\hat{G}(z)}{dz} = z \frac{dG(z)}{dz}. \quad (15)$$

By computing the inverse z-transform of (15), we obtain

$$\sum_{m=-\infty}^{\infty} m\hat{g}(m)g(n-m) = ng(n). \quad (16)$$

Since $\hat{g}(n)$ is related to $\hat{h}(n)$ by (12), the following relationship between $g(n)$ and $\hat{h}(n)$ results:

$$ng(n) = \sum_{m=-\infty}^{\infty} km[g(n-m) - g(n+km)]\hat{h}(m). \quad (17)$$

Generally, the complex cepstrum of a finite extent sequence is of infinite extent. Thus, $\hat{h}(n)$ is probably an infinite sequence. However, the cepstral sequences decay exponentially with a $1/n$ scaling factor for finite extent sequences as shown below [5]:

$$\hat{h}(n) = \mathcal{F}^{-1}[\log H(z)] = \begin{cases} -\sum_{i=1}^{L_1} \frac{(a_i)^n}{n} & \text{if } n > 0 \\ \sum_{i=1}^{L_2} \frac{(b_i)^{-n}}{n} & \text{if } n < 0 \end{cases} \quad (18)$$

where (4) is used with $A = 1$ and $r = 0$. Since $|a_i| < 1$ and $|b_i| < 1$, we can select two integers $p, q > 0$ [5] such that the truncated sequence $\hat{h}_t = \{\dots, 0, 0, \hat{h}(-q), \hat{h}(-q+1), \dots, \hat{h}(0), \dots, \hat{h}(p), 0, 0, \dots\}$ approximates the sequence $\hat{h}(n)$ arbitrarily closely. The infinite sum in (17) can now be approximated by a finite sum as follows:

$$ng(n) \approx \sum_{m=-q}^p km[g(n-m) - g(n+km)]\hat{h}(m). \quad (19)$$

Equation (19) allows us to compute the complex cepstrum $\hat{h}(n)$ of the system impulse response from the sequence $g(n)$ by solving a set of linear equations. By evaluating (19) for $n = -K_1, -K_1+1, \dots, 0, \dots, K_2$, one can obtain a matrix relation as follows:

$$d \approx Db \quad (20)$$

where $d = [-K_1 g(-K_1), \dots, 0, \dots, K_2 g(K_2)]^T$, a $(K_1 + K_2 + 1)$ column vector, $b = [\hat{h}(-q), \dots, \hat{h}(-1), \hat{h}(1), \dots, \hat{h}(p)]^T$, a $(p+q)$ column vector, and D is a matrix of size $(K_1 + K_2 + 1) \times (p+q)$ whose entries are expressions of the form $\{km[g(n-m) - g(n+km)]\}$. To solve the above equations, $K_1 + K_2 \geq p+q-1$. Typically, one can choose $K_1 = K_2 = k \max(p, q)$. If $K_1 + K_2 > p+q-1$, then the least squares solution of this overdetermined system of equations is

$$\hat{b} = (D^T D)^{-1} D^T d. \quad (21)$$

We note that a weighted least squares solution of (20) can also be obtained. Once we obtain $\hat{h}(n)$ from (21), $h(n)$ can be

computed either by using (14) or by using the following recursive formulas [5].

$$h_{\min}(n) = \sum_k (k/n) \hat{h}_{\min}(k) h_{\min}(n-k), \quad n > 0 \quad (22)$$

$$h_{\max}(n) = \sum_k (k/n) \hat{h}_{\max}(k) h_{\max}(n-k), \quad n < 0 \quad (23)$$

where $h_{\min}(n)$ and $h_{\max}(n)$ are the minimum and maximum phase components of the sequence $h(n)$, and $\hat{h}_{\min}(n)$ and $\hat{h}_{\max}(n)$ are the complex cepstra of $h_{\min}(n)$ and $h_{\max}(n)$, respectively. Since $h_{\min}(n)$ and $h_{\max}(n)$ are related to the system impulse response by $h(n) = h_{\min}(n) * h_{\max}(n)$. The complex cepstra of the minimum and maximum phase components of the system impulse response $h(n)$ will then be given by $\hat{h}_{\min}(n) = \{\dots, 0, 0, \hat{h}(0)/2, \hat{h}(1), \hat{h}(2), \dots\}$ and $\hat{h}_{\max}(n) = \{\dots, \hat{h}(-2), \hat{h}(-1), \hat{h}(0)/2, 0, 0, \dots\}$.

IV. PROPERTIES OF THE METHODS

In this section, we discuss properties of the reconstruction methods presented in Sections II and III.

1) Since β_{k+1} is generally not known and the complex bicepstra of $h(n)$ and $h(n-\tau)$ where τ is an integer are equal to each other, the reconstructed sequence might be a shifted and scaled version of the original one.

2) The RC algorithm appears to introduce propagating errors when the initial estimates of $\hat{h}(n_i)$, $n_i = \dots, -3, -2, -1, 1, 2, \dots$ are inaccurate. This might happen when there are measurement noise and/or estimation errors. Based on (13), only samples of $\hat{h}(n)$ where $|n/k| = \text{integer}$ are affected. However, it can easily be shown that the effect of error introduced by $\hat{h}(n_i)$ on the sample $\hat{h}(n)$, where $n = ((-k)^m n_i)_{m=0}^{\infty}$, decreases exponentially with a $|n_i|/|n|$ scaling factor as n increases.

3) It is possible to compute the sequence $\hat{g}(n)$ from the sequence $g(n)$ using three FFT operations without performing phase unwrapping. This can be achieved by taking the Fourier transform of both sides of (16). By doing so, we obtain [6]

$$\hat{g}(n) = 1/n \mathcal{F}^{-1}\{\mathcal{F}[ng(n)]/\mathcal{F}[g(n)]\}. \quad (24)$$

4) Given the $(k+1)$ st-order spectrum, the computational complexity of the RC algorithm does not increase with the order of the utilized spectrum. That is, with the same computational complexity, one could recover $h(n)$ from the third- or fourth-order spectra. If $C_3(\omega, \omega)$ is the 1-D slice of the bispectrum then $G(\omega)$ is related to the system transfer function $H(\omega)$ by

$$G(\omega) = \beta_3 H^2(\omega) H(-2\omega) \quad (25)$$

where $G(\omega) = C_3(\omega, \omega)$. Similarly, we can relate the 1-D slice of the trispectrum $C_4(\omega, \omega, \omega)$ to the system transfer function by

$$G(\omega) = \beta_4 H^3(\omega) H(-3\omega) \quad (26)$$

where $G(\omega) = C_4(\omega, \omega, \omega)$. Solving (25) requires the same amount of computations as (26).

5) The methods presented utilize cumulants (spectra) of order greater than two; therefore, they are (theoretically) [8] unaffected by presence of additive Gaussian noise. This is due to the fact that higher order (than two) cumulants of a Gaussian process are identically zero.

6) The RC and LS methods can be extended in a straightforward manner to the reconstruction of two and higher dimensional systems and signals (see Appendix A).

7) The methods presented can reconstruct the phase and magnitude of the system transfer function separately. Let $M(\omega)$

and $O(\omega)$ denote the magnitude and phase of $H(\omega)$, respectively. Therefore,

$$\log H(\omega) = \log M(\omega) + jO(\omega). \quad (27)$$

By virtue of (27), the complex cepstrum $\hat{m}(n)$ of the magnitude component and the complex cepstrum $\hat{o}(n)$ of the phase component are related to $\hat{h}(n)$ by $\hat{h}(n) = \hat{m}(n) + j\hat{o}(n)$. The phase of the Fourier transform of a real sequence is an odd function, and its magnitude is an even function. Therefore, $\hat{m}(\hat{o})$ is an even (odd) sequence. Consequently, $\hat{o}(n) = (\hat{h}(n) - \hat{h}(-n))/2$ and $\hat{m}(n) = (\hat{h}(n) + \hat{h}(-n))/2$. The sequence $\hat{h}(n)$ can be computed either by using (13) or (21). The above equations will play an important role in determining the phase of the system when the magnitude is distorted and visa versa. Reconstruction of the distorted phase/magnitude can be obtained from the recovered magnitude/phase under certain conditions [3], [7].

8) The sequence $g(n)$ can be computed from the $(k+1)$ st-order cumulant sequence, $c_{k+1}(\tau_1, \tau_2, \dots, \tau_k)$, i.e.,

$$g(n) = \sum_{\tau_2, \dots, \tau_k} c_{k+1}(n - \tau_2 - \tau_3 - \dots - \tau_k, \tau_2, \tau_3, \dots, \tau_k). \quad (28)$$

9) We use the $\omega_1 = \omega_2 = \omega_3 \dots$ slice in our algorithms. For example, in the case of bispectrum one expects the diagonal slice $\omega_1 = \omega_2$ to be the slice that carries the highest phase information because the slices $\omega_1 = 0$ and $\omega_2 = 0$ are phase blind.

V. SIMULATION EXAMPLES

In this section, we present some simulation examples. In these examples, the third-order cumulants and bispectrum are used. The LTI systems are driven by noise which is zero-mean, i.i.d., exponentially distributed, and is generated by the GGEXN subroutine of the IMSL library, with $\beta_3 = 1$.

Example 1: In this example, the NMP-LTI system $H(z) = (1 + 0.5505z)(1 + 0.1835z^{-1}) = 0.5505z + 1.1010 + 0.1835z^{-1}$ is used. The third-order cumulant sequence is computed in the region defined by $-L \leq \tau_1, \tau_2 \leq L$ using the indirect estimation method described in [4], with $L = 4$. An FFT of size 128 is implemented to compute $G(\omega)$ at 128 frequency points. The values of p and q are chosen to be 4. Four different lengths of output data have been used for this example: 2×128 ($K = 2$ and $M = 128$), 4×128 , 8×128 , and 16×128 . The impulse response sequence is determined as the sample mean of 50 different output realizations. Table I shows the mean values (μ) and standard deviations (σ) of the coefficients of reconstructed impulse responses using the RC, LS, and Pan and Nikias' (PN) least squares methods. The RC method is implemented using (24). Based on the results obtained, it is clear that the RC method has the highest bias and variance compared to the LS and PN's methods. On the other hand, PN's method has the lowest variance compared to the RC and LS methods. This is intuitively not surprising since PN's method uses all the samples of bispectrum, whereas the RC and LS methods use a slice of it.

Example 2: In this example, the MA system $H(z) = 0.7815z^2 + 0.4689z + 0.1562 - 0.7815z^{-1}$ is used. This system is a mixed phase system with zeros located at 0.7815 and $-0.6907 \pm 0.8958j$, which are close to the unit circle. The values of p , q , and L are selected such that $p = q = 12$, and $L = 5$. Note that the values of p and/or q , increase whenever there is a zero(s) close to the unit circle. This is one drawback of using cepstral methods since their computational complexities are dependent on the values of p and q . If the values of p and q are to be severely underestimated, the system impulse response will not be recovered accurately.

TABLE I
THE RECONSTRUCTED IMPULSE RESPONSE OF SIMULATION EXAMPLE 1 USING THE RC, LS, AND PN METHODS

Approach	True	2×128		4×128		8×128		16×128	
	value	μ	σ	μ	σ	μ	σ	μ	σ
RC	0.5505	0.3401	0.3242	0.4124	0.2587	0.3757	0.2793	0.4247	0.2399
	1.1010	1.0119	0.1078	1.0526	0.1620	1.0420	0.2027	1.0805	0.0529
	0.1835	0.2670	0.2773	0.2644	0.2744	0.3020	0.2543	0.2753	0.2167
LS	0.5505	0.4509	0.2211	0.4876	0.1413	0.4926	0.0975	0.5163	0.0671
	1.1010	1.1152	0.0694	1.1136	0.0749	1.1028	0.0456	1.1040	0.0352
	0.1835	0.1975	0.1601	0.2213	0.1654	0.2007	0.1083	0.1933	0.0690
PN	0.5505	0.5272	0.1218	0.5184	0.0683	0.5277	0.0482	0.5318	0.0357
	1.1010	1.0921	0.0520	1.0989	0.0380	1.0948	0.0306	1.0955	0.0236
	0.1835	0.1565	0.0777	0.1816	0.0786	0.1747	0.0623	0.1752	0.0383

TABLE II
THE RECONSTRUCTED IMPULSE RESPONSE OF SIMULATION EXAMPLE 2 USING THE LS AND PN METHODS

Approach	True	4×128		8×128		16×128	
	value	μ	σ	μ	σ	μ	σ
LS	0.7815	0.5716	0.1682	0.6766	0.0651	0.7011	0.0452
	0.4689	0.6185	0.2171	0.5961	0.1551	0.5749	0.1123
	0.1562	0.4583	0.2299	0.2898	0.1024	0.2492	0.0827
	-0.7815	-0.6857	0.1465	-0.7591	0.0528	-0.7669	0.0362
PN	0.7815	0.6277	0.0997	0.7105	0.0400	0.7376	0.0240
	0.4689	0.5492	0.0776	0.5224	0.0625	0.5058	0.0528
	0.1562	0.3834	0.1639	0.2664	0.0827	0.2247	0.0562
	-0.7815	-0.7252	0.0773	-0.7721	0.0387	-0.7818	0.0250

In this simulation example, we used three different lengths of output data: 4×128 , 8×128 , and 16×128 . Table II shows the means and standard deviations of the coefficients of reconstructed impulse responses using the LS and PN's methods, with 50 output realizations. Based on the results displayed, we observe that PN's method produces better estimates compared to our LS method. On the other hand, we observe that the results obtained by our LS method are *relatively comparable* with those of PN's method as the number of observations increase. The improvement in bias and variance of PN's method over those of our LS method is achieved at the expense of more computations. In this example, the size of the coefficient matrix of PN's method is 312×24 whereas the size is 49×24 in our method.

However, the set of equations that can be drawn from (30) is incapable of recovering the sequence $\{h(0, j)\}$. To circumvent such a problem, we use (30) in addition to (31)

$$ng(m, n) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} jk[g(m-i, n-j) - g(m+ki, n+kj)]\hat{h}(i, j). \quad (31)$$

For exponentially stable impulse responses, the cepstral coefficients may be assumed to be effectively nonzero over a finite region of support. Therefore, the infinite sums present in (30) and (31) can be truncated, and (30) in conjunction with (31) can then be written in a matrix form and solved using the LS method as in the 1-D case described in Section III.

APPENDIX A

In the 2-D case, (13) takes the form ($\beta_3 = 1$)

$$\hat{h}(m, n) = \begin{cases} \hat{g}(m, n)/(k+1) & \text{if } m = n = 0 \\ (\hat{g}(m, n) - \hat{h}(-m/k, -n/k))/k & \text{if } \left|\frac{m}{k}\right| \text{ and } \left|\frac{n}{k}\right| = 1, 2, \dots \\ \hat{g}(m, n)/k & \text{otherwise} \end{cases} \quad (29)$$

where $\hat{h}(m, n)$ and $\hat{g}(m, n)$ are the complex cepstra of the system impulse response and the 2-D slice of the $(k+1)$ st-order spectrum, respectively. For 2-D systems, the $(k+1)$ st-order spectrum is a $2k$ -D function. The use of a slice reduces the dimension of the problem from a $2k$ -D to a 2-D problem.

In 2-D, (17) takes the form

$$mg(m, n) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} ik[g(m-i, n-j) - g(m+ki, n+kj)]\hat{h}(i, j). \quad (30)$$

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