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On the link between different U–V pairs and related finite-gap solutions of the stationary axisymmetric Einstein equation

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Abstract. An explicit link between finite-gap solutions of the stationary axisymmetric Einstein equation found by Korotkin and Matveev is obtained.

Recent progress in exact solutions of stationary axially symmetric (SAS) solutions of the Einstein equation was initiated in 1978 by Belinskii and Zakharov [4] and Maison [5]. The U–V pairs found there allowed one to consider this equation in the framework of the inverse scattering method. For example, standard 'dressing' procedure gives the multisoliton solutions describing the interaction of a few Kerr–NUT objects on an arbitrary background. The U–V pair of Belinskii and Zakharov (BZ) is related to the SAS Einstein equation written in terms of the metric coefficients while the U–V pair of Maison corresponds to the Ernst formulation, when the metric is expressed in terms of one complex-valued function—the Ernst potential; once the Ernst potential is known, the metric coefficients may be found in quadratures. In a slightly different form the U–V pair of Maison was obtained in 1979 by Neugebauer [6]; we shall use this formulation, calling it the Maison–Neugebauer (MN) U–V pair.

The method of finite-gap (algebro-geometric) integration allowing us to get a generalization of multisoliton solutions in terms of multidimensional theta-functions (see for the basic material and references [7–11]) was applied to the SAS Einstein equation in [2] (in the Ernst formulation) and in [3] (in the metric formulation); some properties of algebro-geometric solutions were investigated in [12, 13]. These finite-gap solutions constitute the most general class of exact solutions of the SAS Einstein equation found so far.

The natural question arising here is about the relationship between BZ and MN linear systems and related finite-gap solutions. On the level of equations this link is obvious: this is the relation between the Ernst potential and metric coefficients in terms of quadratures. On the level of associated U–V pairs the link is less evident; it was first established by Cosgrove [14] in a rather complicated form. Finally, on the level of finite-gap solutions this link is essentially more subtle as we should get explicit correspondence between axiomatic and analytical properties of Ψ-functions solving associated linear systems.

Here we get an explicit 'dressing' transformation between BZ and MN linear systems as a reduction of the Bäcklund transformation of Corrigan *et al* [1] between the $SU(1,1)$ and $SU(2)$ self-dual Yang–Mills fields which was described in [16] on the level of related

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linear systems. It allows one to establish explicit one-to-one correspondence between the finite-gap solutions of the SAS Einstein equation in the Ernst formulation and the 'metric' formulation.

The starting point is the following form of the line element of stationary axisymmetric space-time:

$$\mathrm{d}s^2 = h(\mathrm{d}\rho^2 + dz^2) + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \qquad i, j = 1, 2 \tag{1}$$

where the real symmetric matrix g with signature $(1, 1)$ obeying the condition $\det g = -\rho^2$ and conformal factor h depend only on (ρ, z).

In terms of g_{ij} the Einstein equation may be written as follows:

$$(\rho g_\rho g^{-1})_\rho + (\rho g_z g^{-1})_z = 0 \qquad \det g = -\rho^2 . \tag{2}$$

Once (2) is solved, the factor h may be obtained in quadratures.

Choosing in (1) $x^1 = t$ ('time' coordinate) and $x^2 = \phi$ ('angle' coordinate), we rewrite the line element as follows:

$$\mathrm{d}s^2 = f^{-1}[\mathrm{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\phi^2] - f(\mathrm{d}t + A\mathrm{d}\phi)^2$$

where f, k and A are real-valued functions of (ρ, z); matrix g takes the following form:

$$g = \begin{pmatrix} f & Af \\ Af & A^2 f - \rho^2 f^{-1} \end{pmatrix} . \tag{3}$$

An alternative form of (2) may be obtained introducing the complex-valued function (Ernst potential) $\mathcal{E}(\rho, z)$ related to f, A and k as follows:

$$f = \mathrm{Re}\,\mathcal{E} \qquad A_\xi = 2\rho\frac{(\mathcal{E} - \bar{\mathcal{E}})_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2} \qquad k_\xi = \sqrt{2}\mathrm{i}\rho\frac{\mathcal{E}_\xi\bar{\mathcal{E}}_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2} \tag{4}$$

where $\xi = (1/\sqrt{2})(z + \mathrm{i}\rho)$, $\bar{\xi}$ are new complex coordinates; subscript ξ denotes the partial derivative in ξ. Choice of the boundary conditions for the metric is equivalent to the choice of integrability constants in (4).

In terms of $\mathcal{E}$ (2) takes the form of the Ernst equation:

$$(\mathcal{E} + \bar{\mathcal{E}})\left(\mathcal{E}_{\rho\rho} + \frac{1}{\rho}\mathcal{E}_\rho + \mathcal{E}_{zz}\right) = 2(\mathcal{E}_z^2 + \mathcal{E}_\rho^2) . \tag{5}$$

The connection matrices in the zero-curvature condition

$$U_{\bar{\xi}} - V_\xi + [U, V] = 0 \tag{6}$$

for (2) found by Belinskii and Zakharov look as follows (we use the form with 'variable spectral parameter'; for the relation with the original U–V pair with derivative in the spectral parameter see [17]):

$$U_1 = \frac{(\xi - \bar{\xi})g_\xi g^{-1}}{2\sqrt{\lambda - \xi}\left(\sqrt{\lambda - \bar{\xi}} - \sqrt{\lambda - \xi}\right)} \qquad V_1 = \frac{(\bar{\xi} - \xi)g_{\bar{\xi}} g^{-1}}{2\sqrt{\lambda - \bar{\xi}}\left(\sqrt{\lambda - \xi} - \sqrt{\lambda - \bar{\xi}}\right)} \tag{7}$$

where $\lambda \in \mathbf{C}$ is a parameter called 'spectral'.

So (2) is a compatibility condition of the following linear system:

$$\Psi_{1\xi} = U_1 \Psi_1 \qquad \Psi_{1\bar{\xi}} = V_1 \Psi_1 \tag{8}$$

where $\Psi_1(\lambda, \xi, \bar{\xi})$ is 2×2 matrix-valued function.

For the Ernst equation (5) the connection matrices are the following:

$$\begin{aligned} U_2 &= \begin{pmatrix} X_2 & 0 \\ 0 & Y_2 \end{pmatrix} + \sqrt{\frac{\lambda - \bar{\xi}}{\lambda - \xi}} \begin{pmatrix} 0 & X_2 \\ Y_2 & 0 \end{pmatrix} \\ V_2 &= \begin{pmatrix} \bar{Y}_2 & 0 \\ 0 & \bar{X}_2 \end{pmatrix} + \sqrt{\frac{\lambda - \xi}{\lambda - \bar{\xi}}} \begin{pmatrix} 0 & \bar{Y}_2 \\ \bar{X}_2 & 0 \end{pmatrix} \end{aligned} \tag{9}$$

where

$$X_2 = \frac{\mathcal{E}_\xi}{\mathcal{E} + \bar{\mathcal{E}}} \qquad Y_2 = \frac{\bar{\mathcal{E}}_\xi}{\mathcal{E} + \bar{\mathcal{E}}}. \tag{10}$$

So (5) is a compatibility condition of the linear system

$$\Psi_{2\xi} = U_2 \Psi_2 \qquad \Psi_{2\bar{\xi}} = V_2 \Psi_2 \tag{11}$$

where $\Psi_2(\lambda, \xi, \bar{\xi})$ is new 2×2 matrix-valued function.

Functions Ψ_1 and Ψ_2, solving (8) and (11) respectively, play the central role in the application of the inverse scattering technique to (2) and (5). The canonical way to get the finite-gap (algebro-geometric) solutions is the following [10, 8]: first we present the system of axioms for the Ψ-function which provides the proper structure of its logarithmic derivatives $\Psi_\xi \Psi^{-1}$ and $\Psi_{\bar{\xi}} \Psi^{-1}$ according to (7) or (9). Then one should realize this axiomatic in some way. In particular, to get the algebro-geometric solutions we construct Ψ as a function (in λ) on some special algebraic curve; then the formulae for solutions of (2) or (5) may be extracted from the explicit expression for Ψ in terms of the theta-functions of this curve.

In the Ernst formulation (5), (11) this approach was developed in [2], and in the 'metric' formulation in [3], but the relationship between the final results—the expressions for the Ernst potential and expressions for components of matrix g in terms of theta-functions—was not clear. To fill this gap one should first establish an explicit link between the linear systems (8) and (11) (i.e. find a λ-dependent gauge transformation between connections U_1, V_1 and U_2, V_2), then extend this link on the level of axiomatics of Ψ_1 and Ψ_2 and, eventually, on the level of related finite-gap solutions.

Note that the Ernst equation (5) may be written in the form (2) if instead of matrix g (3) we take

$$g_2 = \frac{1}{\mathcal{E} + \bar{\mathcal{E}}} \begin{pmatrix} 2 & \mathcal{E} - \bar{\mathcal{E}} \\ \bar{\mathcal{E}} - \mathcal{E} & 2\mathcal{E}\bar{\mathcal{E}} \end{pmatrix}. \tag{12}$$

This matrix is Hermitian and $\det g_2 = 1$; so it sets some stationary axisymmetric solution of $SU(2)$ self-duality equation in the Yang formulation. Matrix $(1/\rho)g$ where g is set by (3) is Hermitian too, but $\det((1/\rho)g) = -1$; therefore, it sets some solution of $SU(1, 1)$ self-duality equation. The one-to-one correspondence between $SU(2)$ and $SU(1, 1)$ self-dual fields is given by the Bäcklund transformation of Corrigan *et al* [1]; (4) is a partial case of this correspondence for SAS solutions of special structure (3) and (12). This transformation looks especially simple in a triangle gauge and, as was shown in [16], is equivalent to a simple λ-dependent gauge ('dressing') transformation of an associated linear system. After reduction to the SAS case it gives the transformation which may be extracted from [14]; below we give a more transparent form of it.

First, it is convenient to transform $\Psi_1(P)$ as follows:

$$\tilde{\Psi}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \Psi_1 \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix} . \tag{13}$$

It solves the linear system (8) with

$$\tilde{g} = \begin{pmatrix} f & -\mathrm{i}Af \\ \mathrm{i}Af & A^2 f - \rho^2 f^{-1} \end{pmatrix} .$$

It is easy to verify that function

$$\Psi_1' \equiv \rho^{-1} \begin{pmatrix} \mathrm{i}(A + \rho/f) & -1 \\ \mathrm{i}(A - \rho/f) & -1 \end{pmatrix} \tilde{\Psi}_1 \tag{14}$$

obeys the following linear system:

$$\Psi'_{1\xi} = U_1' \Psi_1' \qquad \Psi'_{1\bar{\xi}} = V_1' \Psi_1' \tag{15}$$

where

$$\begin{aligned} U_1' &= \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} + \sqrt{\frac{\lambda - \bar{\xi}}{\lambda - \xi}} \begin{pmatrix} 0 & X_1 \\ Y_1 & 0 \end{pmatrix} \\ V_1' &= \begin{pmatrix} \bar{Y}_1 & 0 \\ 0 & \bar{X}_1 \end{pmatrix} + \sqrt{\frac{\lambda - \xi}{\lambda - \bar{\xi}}} \begin{pmatrix} 0 & \bar{Y}_1 \\ \bar{X}_1 & 0 \end{pmatrix} \end{aligned} \tag{16}$$

and

$$X_1 = \frac{1}{2(\xi - \bar{\xi})} - \frac{\bar{\mathcal{E}}_\xi}{\mathcal{E} + \bar{\mathcal{E}}} \qquad Y_1 = \frac{1}{2(\xi - \bar{\xi})} - \frac{\mathcal{E}_\xi}{\mathcal{E} + \bar{\mathcal{E}}} . \tag{17}$$

Define the rational algebraic curve $\mathcal{L}_0$ set by the following equation:

$$\omega^2 = (\lambda - \xi)(\lambda - \bar{\xi}) . \tag{18}$$

Now, starting from some solution Ψ_2 of the linear system (11) associated to the Ernst equation (5) we can construct a new function

$$\frac{\sqrt{\xi - \bar{\xi}}}{\mathcal{E} + \bar{\mathcal{E}}} M^{-1} T M \Psi_2 \tag{19}$$

where

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} T = \begin{pmatrix} 0 & \mu \\ \mu^{-1} & 0 \end{pmatrix}$$
$$\mu = \sqrt{\frac{2\gamma}{\xi - \bar{\xi}}} \qquad \gamma = \sqrt{(\lambda - \xi)(\lambda - \bar{\xi})} - \lambda + \frac{\xi + \bar{\xi}}{2}. \tag{20}$$

Function μ changes the sign on some contour l on $\mathcal{L}_0$ between ∞^+ and ∞^- ($\infty^\pm$ are infinite points on different sheets of $\mathcal{L}_0$ where $\omega = \pm\lambda + O(1)$ respectively); we shall specify the choice of this contour below.

Direct calculation shows that (19) obeys the same linear system (15) as Ψ_1'; hence we can put

$$\Psi_1' = \frac{\sqrt{\xi - \bar{\xi}}}{\mathcal{E} + \bar{\mathcal{E}}} M^{-1} T M \Psi_2. \tag{21}$$

So we can formulate the following.

Statement 1. Formulae (13), (14), (20) and (21) set the relationship between the auxiliary linear systems (8) and (11) for the SAS Einstein equation.

Now consider this link on the level of axiomatics of Ψ-functions and finite-gap solutions. Note that the linear systems for functions Ψ_1' and Ψ_2 have very similar structure of connection matrices (9) and (16); the only difference is the form of functions X_1, Y_1 set by (17) and X_2, Y_2 set by (10).

The axiomatic for function Ψ_2 was formulated in [2]; a slightly modified version presented in [13] is the following.

Statement 2. Let function $\Psi_2(P)$ ($P = (\omega, \lambda) \in \mathcal{L}_0$) obey the following conditions:

(a) Logarithmic derivatives $\Psi_{2\xi}\Psi_2^{-1}$ and $\Psi_{2\bar{\xi}}\Psi_2^{-1}$ are holomorphic on $\mathcal{L}_0$ except points $\lambda = \xi$ and $\lambda = \bar{\xi}$ respectively.

(b) $\Psi_2(P)$ is holomorphic and invertible on $\mathcal{L}_0$ at points $\lambda = \xi$ and $\lambda = \bar{\xi}$ (the difference between holomorphicity on the λ-plane and on $\mathcal{L}_0$ at branch points, for instance at $\lambda = \xi$, is in the local parameters $\lambda - \xi$ and $\sqrt{\lambda - \xi}$ respectively).

(c)
$$\Psi_2(P^\sigma) = \sigma_3 \Psi_2(P) \sigma_2 \tag{22}$$

where σ is an involution on $\mathcal{L}_0$ interchanging the sheets; σ_j , $j = 1, 2, 3$ are Pauli matrices.

(d) Normalization condition

$$\Psi_2(\lambda = \infty^+) = \begin{pmatrix} \mathcal{E} & \mathrm{i} \\ \bar{\mathcal{E}} & -\mathrm{i} \end{pmatrix} \tag{23}$$

where $\mathcal{E}$ is some function of $(\xi, \bar{\xi})$.

Then Ψ_2 solves the linear system (11) with matrices U_2, V_2 set by (9) and, therefore, function $\mathcal{E}(\xi, \bar{\xi})$ obeys the Ernst equation (5).

The proof is not difficult (see [2]); in particular, expressions (10) obviously come from (23).

Look at function Ψ_1' set by (21). It is easy to verify that Ψ_1' obeys items (a) and (b) of statement 2 ((b) is obvious; (a) follows from the same requirement for Ψ_2 and normalization condition (23) which provides regularity at $\lambda = \infty$). Besides that, if we assume

$$\mu^{-1}(\lambda) = \mu(\lambda^\sigma) \tag{24}$$

then

$$M^{-1}T(\lambda^\sigma)M = \sigma_3 M^{-1}T(\lambda)M\sigma_3$$

and, taking into account (22),

$$\Psi_1'(\lambda^\sigma) = \sigma_3\Psi_1'(\lambda)\sigma_2 .$$

So Ψ_1' obeys item (c) too. To provide condition (24) we have to choose contour l between ∞^+ and ∞^- setting the function $\mu(\lambda)$ on $\mathcal{L}_0$ in some special way. Namely, condition (24) implies that in the $(\gamma/\xi - \bar{\xi})$-plane contour l coincides with negative reals; thus if we realize $\mathcal{L}_0$ as a two-sheeted covering of the λ-plane, it should have the form shown in figure 1.

Figure 1. Curve $\mathcal{L}_0$ is a two-sheeted covering of the λ-plane with branch points at $\lambda = \xi$ and $\lambda = \bar{\xi}$.

Item (d) certainly should change as X_1, Y_1 differ from X_2, Y_2. Calculating Ψ_1' at $\lambda = \infty^+$ according to (21), we get

$$\Psi_1' \sim \left[\begin{pmatrix} \mathrm{i} & (\xi - \bar{\xi})/(\mathcal{E} + \bar{\mathcal{E}}) - 2\mathrm{i}(\varphi_0 + \psi_0)/(\mathcal{E} + \bar{\mathcal{E}}) \\ -\mathrm{i} & (\xi - \bar{\xi})/(\mathcal{E} + \bar{\mathcal{E}}) + 2\mathrm{i}(\varphi_0 + \psi_0)/(\mathcal{E} + \bar{\mathcal{E}}) \end{pmatrix} + \mathrm{o}(1)\right] \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/(\sqrt{\lambda}) \end{pmatrix} \tag{25}$$

where φ_0 and ψ_0 are coefficients in asymptotical expansion of the first column of function Ψ_2 at $\lambda = \infty^+$:

$$(\Psi_2)_{12} = \mathrm{i} + \frac{\varphi_0}{\lambda} + \mathrm{o}(\lambda^{-1}) \tag{26}$$

$$(\Psi_2)_{22} = -\mathrm{i} + \frac{\psi_0}{\lambda} + \mathrm{o}(\lambda^{-1}) . \tag{27}$$

It is possible to rewrite asymptotics (25) using the formula obtained in [13] for coefficient A in terms of the derivative of function Ψ_2 in the spectral parameter. This formula arises from the simple identity [15]

$$(\Psi^{-1}\Psi_\delta)_\xi = \Psi^{-1}(\Psi_\xi\Psi^{-1})_\delta\Psi \tag{28}$$

where $\delta \equiv 1/\lambda$. Substituting normalization condition (23), (9) and the expression for coefficient A (4) in (28), we get

$$A = A_0 + 2\sqrt{2}(\Psi_{2\delta}\Psi_2^{-1})_{12} \tag{29}$$

where A_0 is an arbitrary constant (in [13] we used the inverse order of columns in (23); thus there we got 21 element in the right-hand side of (29)). Using (26), (27) one obtains from (29)

$$A = A_0 + 2\sqrt{2}\frac{\varphi_0 + \psi_0}{\mathcal{E}+\bar{\mathcal{E}}}.$$

Choosing $A_0 = 0$ we can rewrite (25) as follows:

$$\Psi_1' \sim \left[\begin{pmatrix} \mathrm{i} & (\xi-\bar{\xi})/(\mathcal{E}+\bar{\mathcal{E}}) - \mathrm{i}A/\sqrt{2} \\ -\mathrm{i} & (\xi-\bar{\xi})/(\mathcal{E}+\bar{\mathcal{E}}) + \mathrm{i}A/\sqrt{2}\end{pmatrix} + \mathrm{o}(1)\right]\begin{pmatrix}\sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda}\end{pmatrix} \tag{30}$$

at $\lambda \sim \infty^+$.

So we have established the link between functions Ψ_2 and Ψ_1' on the level of axiomatic: we can claim that if Ψ_2 obeys items (a)–(d) of statement 2, then the function Ψ_1' set by (21) obeys the same conditions (a)–(c) and normalization condition (30) if contour l setting $\mu(\lambda)$ in (21) is chosen according to figure 1.

Besides that, functions Ψ_2 and Ψ_1' related by (21) have the same set of regular singularities (i.e. singularities where the logarithmic derivatives are holomorphic) except $\lambda = \infty$ where components of Ψ_0' have additional poles and zeros of degree $\frac{1}{2}$.

This allows us to establish an explicit link between related classes of finite-gap solutions. These solutions were obtained first in [2, 3]; the natural character of this construction is explained in [18] for the more general case of the self-duality equation.

In our SAS situation the basic algebraic curve $\hat{\mathcal{L}}$ is a two-sheeted covering of $\mathcal{L}_0$, i.e. a four-sheeted covering of the λ-plane [2, 3]. Due to reduction restriction (22) involution σ on $\hat{\mathcal{L}}$ should be inherited from $\mathcal{L}_0$; so the set of branch points of $\hat{\mathcal{L}}$ (besides $\lambda = \xi$ and $\lambda = \bar{\xi}$) should be invariant under σ:

$$E_j,\ F_j,\ E_j^\sigma,\ F_j^\sigma \qquad j = 1, \ldots, g\,.$$

To provide item (a) of statement 2, one should take all E_j, F_j independent of $(\xi, \bar{\xi})$.

Numbering sheets of $\hat{\mathcal{L}}$ by $1, \ldots, 4$ we assume that the first copy of $\mathcal{L}_0$ consists of the sheets 1 and 3 of $\hat{\mathcal{L}}$, and the second copy of the sheets 2 and 4. Then sheets 1 and 3, as 2 and 4, are glued at $\lambda = \xi$ and $\lambda = \bar{\xi}$; sheets 1 and 2 are glued at $E_j, F_j,\ j = 1, \ldots, g$ and sheets 3 and 4 at $E_j^\sigma, F_j^\sigma,\ j = 1, \ldots, g$. The Hurvitz diagram of $\hat{\mathcal{L}}$ is presented in [2, 3].

According to the basic ansatz of algebro-geometric construction [10, 11, 18], the columns of matrix $\Psi(\lambda)$ solving (8) or (11) should be values of some vector-function on different sheets of $\hat{\mathcal{L}}$ (i.e. on different copies of $\mathcal{L}_0$). So on the first copy of $\mathcal{L}_0$ consisting of sheets

1 and 3 of $\hat{\mathcal{L}}$ (it is enough to construct $\Psi(P)$ only on one copy of $\mathcal{L}$), $\Psi(P)$ may be written in the following form:

$$\Psi(P) = \begin{pmatrix} \varphi(P) & \varphi(P^*) \\ \psi(P) & \psi(P^*) \end{pmatrix} \tag{31}$$

where φ and ψ are two scalar-valued functions on $\hat{\mathcal{L}}$, point P lies on the first or third sheet of $\hat{\mathcal{L}}$; $*$ is an involution on $\hat{\mathcal{L}}$ interchanging the copies of $\mathcal{L}_0$, i.e. interchanging the sheets $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ of $\hat{\mathcal{L}}$. This form is the same for Ψ_2 and Ψ_1', but functions φ, ψ should certainly be different.

It is easy to verify [2, 3] that the regularity of logarithmic derivatives of $\Psi(\lambda)$ set by (31) implies that functions φ and ψ have the same set of singularities (including the orders of poles); besides that φ and ψ should have common zeros at the zeros of $\det\Psi$ which do not coincide with branch points E_j, F_j.

Reduction (22) (the same for Ψ_1' and Ψ_2) may be rewritten in terms of φ and ψ as follows:

$$\begin{aligned} \varphi(\lambda^{(3)}) &= \mathrm{i}\varphi(\lambda^{(2)}) \qquad & \psi(\lambda^{(3)}) &= -\mathrm{i}\psi(\lambda^{(2)}) \\ \varphi(\lambda^{(4)}) &= -\mathrm{i}\varphi(\lambda^{(1)}) \qquad & \psi(\lambda^{(4)}) &= \mathrm{i}\psi(\lambda^{(1)}) \end{aligned} \tag{32}$$

where by $\lambda^{(j)}$ we denote the point on the jth sheet of $\hat{\mathcal{L}}$ having projection λ on $\mathbf{C}$. Relations (32) allow us to construct φ and ψ first on the hyperelliptic curve $\mathcal{L}$ consisting of sheets 1 and 2 of $\hat{\mathcal{L}}$, and to extend them after that on $\hat{\mathcal{L}}$ according to (32). Curve $\mathcal{L}$ is set by equation

$$\omega^2 = (\lambda - \xi)(\lambda - \bar{\xi}) \prod_{j=1}^{g} (\lambda - E_j)(\lambda - F_j) \,.$$

The genus of $\mathcal{L}$ is equal to g; it has one moving branch cut $[\xi, \bar{\xi}]$ and g immovable branch cuts $[E_j, F_j]$, $j = 1, \ldots, g$.

Denote functions φ, ψ on $\mathcal{L}$ giving Ψ_2 after substitution in (31) by φ_2, ψ_2 and functions giving Ψ_1', by φ_1, ψ_1. According to (21) φ_2, ψ_2 and φ_1, ψ_1 have the same set of singularities on $\mathcal{L}$ except point $\lambda = \infty^{(1,2)}$ where ψ_2, φ_2 are holomorphic while φ_1, ψ_1 have a pole of degree $\frac{1}{2}$ at $\lambda = \infty^{(1)}$ and a zero of degree $\frac{1}{2}$ at $\lambda = \infty^{(2)}$.

First write explicit expressions for φ_2, ψ_2. To provide (23) it is enough to take

$$\bar{\varphi}_2(\bar{\lambda}) = \psi_2(\lambda) \,. \tag{33}$$

As the set of singularities of φ_2 and ψ_2 should be the same, condition (33) implies the invariance of this set under complex conjugation. In particular, it entails the reality of curve $\mathcal{L}$, i.e. for any j one should have

$$E_j = \bar{F}_j \qquad \text{or} \qquad E_j, F_j \in \mathbf{R} \,.$$

Choose the canonical basis of cycles (a_j, b_j), $j = 1, \ldots, g$ on $\mathcal{L}$ as shown in figure 2. Normalizing the dual basis of holomorphic 1-forms $\mathrm{d}U_j$ on $\mathcal{L}$ according to

$$\oint_{a_j} \mathrm{d}U_k = \delta_{ij}$$

we introduce the matrix of b-periods of $\mathcal{L}$

$$B_{jk} = \oint_{b_j} \mathrm{d}U_k$$

and the Abel map $U : \mathcal{L} \to \mathbf{C}^g$

$$U_j(P) = \int_{P_0}^{P} \mathrm{d}U_j$$

where P is variable and P_0 is some fixed point of $\mathcal{L}$. Note that all these objects depend on $(\xi, \bar{\xi})$.

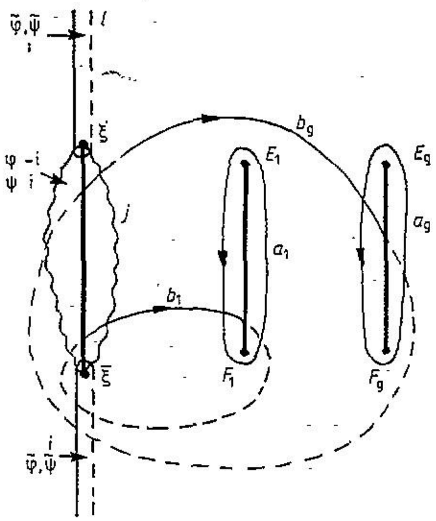

Figure 2. Contours $l, \bar{l}, s$ and canonical basis of cycles on curve $\mathcal{L}$. Continuous contours lie on the first sheet, dotted on the second sheet.

Finally, introducing multidimensional theta-function $\Theta(x|B)$ $(x \in \mathbf{C}^g)$ associated to $\mathcal{L}$ (we shall use the brief notation $\Theta(x)$; for more details see [2, 3, 13, 11]) we can write the expression for $\varphi_2(P)$ as follows:

$$\varphi_2(P) = C_{\varphi 2} \frac{\Theta(U(P) - U(D) + (n/4) + b_W - K)}{\Theta(U(P) - U(D) - K)} \exp\{W(P)\} \tag{34}$$

where K is a vector of Riemann constants of $\mathcal{L}$ (depending on $(\xi, \bar{\xi})$ and P_0); $D = D_1 + \cdots + D_g$ is a real $(D = \bar{D})$ non-special divisor on $\mathcal{L}$ independent of $(\xi, \bar{\xi})$; $n_j = 1$, $j = 1, \ldots, g$. The constant $C_{\varphi 2}(\xi, \bar{\xi})$ should be chosen according to

$$\varphi_2(\infty^{(2)}) = \mathrm{i}\,. \tag{35}$$

Integral $W(P)$ with vector of b-periods $2\pi i b_W$ is a normalized (all a-periods are zero) linear combination of the integrals of second and third kinds with poles and related singular parts independent of $(\xi, \bar{\xi})$; it should obey the reality condition

$$\bar{W}(\bar{P}) = W(P)$$

(as $W(P)$ is an indefinite integral, we understand this equation up to an arbitrary constant).

According to (33), the expression for ψ_2 differs from (34) only by change of the sign before $n/4$ and by the normalization constant.

Functions $\varphi_2(P)$ and $\psi_2(P)$ are discontinuous on contour s between ξ and $\bar{\xi}$ (figure 2) where φ_2 multiplies on $-\mathrm{i}$ and ψ_2 on i according to (32).

It is not difficult to verify (see [2, 3, 13]) that functions φ_2 and ψ_2 set by (34), (33) and (35) define function Ψ_2 obeying items (a)–(d) of Statement 1 for general position of point ξ.

Expression for the Ernst potential may be obtained from (34) according to $\mathcal{E} = \varphi_2(\infty^{(1)})$; choosing the path between $\infty^{(1)}$ and $\infty^{(2)}$ coinciding with contour l (figure 2), we have

$$\mathcal{E} = \frac{\Theta(U(\infty^{(1)}) - U(D) + (n/4) + b_W - K)\Theta(U(\infty^{(2)}) - U(D) - K)}{\Theta(U(\infty^{(2)}) - U(D) + (n/4) + b_W - K)\Theta(U(\infty^{(1)}) - U(D) - K)} \times \exp\{W(\infty^{(1)}) - W(\infty^{(2)})\}. \tag{36}$$

Consider Ψ_1'. According to (21) functions φ_1 and ψ_1 setting Ψ_1' by (31) should have in comparison with φ_2 and ψ_2 an additional pole of degree $\frac{1}{2}$ at $\infty^{(1)}$ and zero of degree $\frac{1}{2}$ at $\infty^{(2)}$; the related contour l between $\infty^{(1)}$ and $\infty^{(2)}$ where φ_1 and ψ_1 change the sign is induced on $\mathcal{L}$ (figure 2) from $\mathcal{L}_0$ (figure 1); we use for these two contours the same notation l.

So the explicit formula for $\varphi_1(P)$ should differ from the expression for φ_2 by inserting in the exponential factor the normalized integral of the third kind $\frac{1}{2}W^l_{\infty^2\infty^1}$ having residue $\frac{1}{2}$ at $P = \infty^{(1)}$ and $-\frac{1}{2}$ at $P = \infty^{(2)}$ connected by contour l. To keep the proper behaviour of φ_1, ψ_1 on contour s (i.e. in respect to a round about b-cycles), one has to add also in the argument of the theta-function in the numerator vector of b-periods of $\frac{1}{2}W^l_{\infty^2\infty^1}$ up to factor $1/2\pi\mathrm{i}$, i.e. $\frac{1}{2}(U(\infty^{(2)} - U(\infty^{(1)}))^l$ (upper index l shows that we calculate the Abel map between $\infty^{(1)}$ and $\infty^{(2)}$ along contour l). As a result we have

$$\varphi_1(P) = C_{\varphi 1}\frac{\Theta(U(P) - U(D) + (n/4) + b_W + \frac{1}{2}(U(\infty^{(2)}) - U(\infty^{(1)}))^l - K)}{\Theta(U(P) - U(D) - K)} \times \exp\{W(P) + \tfrac{1}{2}W^l_{\infty^2\infty^1}\} \tag{37}$$

where all objects are the same as in (34); $C_{\varphi 1}(\xi, \bar{\xi})$ is a normalization constant which should be chosen according to (25), i.e.

$$\varphi(\lambda \sim \infty^{(1)}) \sim \mathrm{i}\sqrt{\lambda}(1 + \mathrm{o}(1)).$$

The formula for $\psi_1(P)$ again differs from φ_1 by changing the sign before $n/4$ and by the normalization constant.

However, the form (37) of $\varphi_1(P)$ does not suit us well because the reality of $\mathrm{i}\sqrt{\lambda}\varphi_1(P)$ at $\lambda = \infty^{(2)}$ which is provided by our previous treatment is not quite obvious from (37) itself. To make this reality apparent let us represent integral $\frac{1}{2}W^l_{\infty^2\infty^1}$ as follows:

$$\tfrac{1}{2}W^l_{\infty^2\infty^1} = W_+ + W_-$$

where

$$W_+(P) = \tfrac{1}{4}(W^l_{\infty^2\infty^1} + W^{\bar{l}}_{\infty^2\infty^1})$$
$$W_-(P) = \tfrac{1}{4}(W^l_{\infty^2\infty^1} - W^{\bar{l}}_{\infty^2\infty^1}).$$

Integral $\frac{1}{4}W^{l}_{\infty^2\infty^1}$ induces multiplication of φ_1, ψ_1 on i on contour l, and integral $\frac{1}{4}W^{\bar{l}}_{\infty^2\infty^1}$ on contour $\bar{l}$ as shown in figure 2 (choice of the sign of this multiplication is induced by orientation: if we consider an arbitrary integral of the third kind W_{QR} with residue $+1$ at Q and -1 at R and go from Q to R along related path l, then the value of W_{QR} on the right-hand side is equal to its value on the left-hand side plus $2\pi\mathrm{i}$; the simple example is $W_{QR} = \ln\lambda$ on CP^1).

Integral $W_+(P)$ is 'real', i.e.

$$\bar{W}_+(\bar{P}) = W_+(P)$$

(as before, we understand this condition up to an arbitrary constant); its vector of b-periods is equal to

$$2\pi\mathrm{i}b_+ = \pi\mathrm{i}\,\mathrm{Im}(U(\infty^{(2)}) - U(\infty^{(1)}))^l .$$

Integral W_- has no singularities at $P = \infty^{(1,2)}$; its vector of b-periods is

$$2\pi\mathrm{i}b_- = \frac{\pi\mathrm{i}n}{2} .$$

Factor $\exp W_-$ induces multiplication of φ_1 on $-\mathrm{i}$ on the contour l according to figure 2, and on contour $\bar{l}$ in inverse direction in comparison with figure 2 as $\frac{1}{4}W^{\bar{l}}_{\infty^2\infty^1}$ is inserted in W_- with a minus sign; thus $\exp W_-$ induces multiplication on $-\mathrm{i}$ on the contour $l-\bar{l}$. Let us continuously deform contour $l-\bar{l}$ into the homotopic contour s through the left-hand side half-plane $\mathrm{Re}\,l \leqslant \mathrm{Re}\,\xi$. Functions $\tilde{\varphi}_1$ and $\tilde{\psi}_1$ which we get in this way differ from φ_1 and ψ_1 respectively in the half-plane $\mathrm{Re}\,l \leqslant \mathrm{Re}\,\xi$ by factor $-\mathrm{i}$ on the first sheet and i on the second sheet; certainly $\tilde{\varphi}_1$ and $\tilde{\psi}_1$ set the same solution of (2). Arising additional multiplication on i on contour s compensates multiplication of φ_1 and ψ_1 on $-\mathrm{i}$ and i respectively on this contour. So $\tilde{\psi}_1$ is continuous on s and $\tilde{\varphi}_1$ its sign changes on s.

All singularities of $\tilde{\varphi}_1$ and $\tilde{\psi}_1$ are symmetric under complex conjugation; together with normalization on $\infty^{(1)}$ in (25) this allows us to claim that

$$\bar{\tilde{\varphi}}_1(\bar{P}) = -\tilde{\varphi}_1(P) \qquad \bar{\tilde{\psi}}_1(\bar{P}) = -\tilde{\psi}_1(P) .$$

This provides the necessary reality of metric coefficients according to the second column in (25).

An explicit expression for $\tilde{\varphi}_1(P)$ may be written as follows:

$$\tilde{\varphi}_1(P) = \tilde{C}_{\varphi 1}\frac{\Theta(U(P) - U(D) + b_W + b_+ + (n/2) - K)}{\Theta(U(P) - U(D) - K)}\exp\{W(P) + W_+(P)\} . \tag{38}$$

The expression for $\tilde{\psi}_1$ differs by the $n/2$ in the argument of the theta-function in the numerator (and certainly by the normalization contant).

Notice that we can add in the argument of the theta-function in the numerator in (34) an arbitrary vector m consisting of 0 and $\frac{1}{2}$ (i.e. to take instead of an ordinary theta-function the theta-function with half-integer characteristics $[0, m]$). Then to keep the proper behaviour on contour s we have to assume that φ_1, ψ_1 change their sign on contour a_j if $m_j = \frac{1}{2}$. Addition of vector m is equivalent to the choice $n_j = -\frac{1}{4}$ if $m_j = \frac{1}{2}$. So in the expression for the Ernst potential we can insert an arbitrary vector n consisting of ± 1. The addition of vector m (i.e. the jump on contour a_j if $m_j = \frac{1}{2}$) is obviously invariant under transformation (21); so the related expression for $\tilde{\varphi}_1$ will differ from (38) only by insertion of vector m in the argument of the theta-function in the numerator (and by the normalization constant).

Taking into account simplicity of this generalization, we proceed to choose $n_j = 1$ (i.e. $m = 0$).

Expression (38) and the analogous expression for $\tilde{\psi}_1$ coincide with formulae which were obtained in [3] for finite-gap solutions in the formalism of the BZ U–V pair if we denote the whole integral $W(P) + W_+(P)$ by $W(P)$.

From (38) we easily obtain explicit expressions for metric coefficients in the BZ formalism. Fix the behaviour of W_+ at $\lambda = \infty^{(1)}$ by

$$W_+(P) \sim -\tfrac{1}{2}\ln\lambda + o(1) \tag{39}$$

(values of $\ln\lambda$ should agree with positions of contours l and $\bar{l}$ setting W_+); then

$$W_+(P) \sim \tfrac{1}{2}\ln\lambda + \alpha + o(1) \qquad \alpha \in \mathbf{R} \qquad \text{at} \qquad \lambda \sim \infty^{(2)}. \tag{40}$$

According to (39) and (40) we get from (38)

$$\begin{aligned}\frac{i(\bar{\xi} - \xi)}{\mathcal{E} + \bar{\mathcal{E}}} - \frac{A}{\sqrt{2}} = {} & [\Theta(U(\infty^{(2)}) - U(D) + b_W + b_+ + (n/2) - K) \\ & \times \Theta(U(\infty^{(1)}) - U(D) - K)] \\ & /[\Theta(U(\infty^{(1)}) - U(D) + b_W + b_+ + (n/2) - K) \\ & \times \Theta(U(\infty^{(2)}) - U(D) - K)] \\ & \times \exp\{W(\infty^{(2)}) - W(\infty^{(1)}) + \alpha\}\end{aligned} \tag{41}$$

and from the analogous expression for $\tilde{\psi}_1$

$$\begin{aligned}\frac{i(\xi - \bar{\xi})}{\mathcal{E} + \bar{\mathcal{E}}} + \frac{A}{\sqrt{2}} = {} & \frac{\Theta(U(\infty^{(2)}) - U(D) + b_W + b_+ - K)\Theta(U(\infty^{(1)}) - U(D) - K)}{\Theta(U(\infty^{(1)}) - U(D) + b_W + b_+ - K)\Theta(U(\infty^{(2)}) - U(D) - K)} \\ & \times \exp\{W(\infty^{(2)}) - W(\infty^{(1)}) + \alpha\}.\end{aligned} \tag{42}$$

Expressions (41) and (42) may be simplified by choosing $P_0 = \xi$. Taking into account the relation $dU(P) = -dU(P^*)$ ($*$ is the involution on $\mathcal{L}_0$ interchanging the sheets) we get in this case

$$b_+ = \frac{1}{2}(U(\infty^{(2)}) - U(\infty^{(1)}))^t - \frac{n}{4}$$

and

$$b_+ + U(\infty^{(1)}) = -\frac{n}{4} \qquad b_+ + U(\infty^{(2)}) = 2U(\infty^{(2)}) - \frac{n}{4}.$$

So

$$\frac{i(\xi - \bar{\xi})}{\mathcal{E} + \bar{\mathcal{E}}} - \frac{A}{\sqrt{2}} = [\Theta(2U(\infty^{(2)}) - U(D) + (n/4) + b_W - K)$$

$$\begin{aligned}
&\times \Theta(U(\infty^{(1)}) - U(D) - K)] \\
&/[\Theta(-U(D) + (n/4) + b_W - K)\Theta(U(\infty^{(2)}) - U(D) - K)] \\
&\times \exp\{W(\infty^{(2)}) - W(\infty^{(1)}) + \alpha\}
\end{aligned} \tag{43}$$

$$\begin{aligned}
\frac{\mathrm{i}(\xi - \bar{\xi})}{\mathcal{E} + \bar{\mathcal{E}}} + \frac{A}{\sqrt{2}} = {}&[\Theta(2U(\infty^{(2)}) - U(D) - (n/4) + b_W - K) \\
&\times \Theta(U(\infty^{(1)}) - U(D) - K)] \\
&/[\Theta(-U(D) - (n/4) + b_W - K)\Theta(U(\infty^{(2)}) - U(D) - K)] \\
&\times \exp\{W(\infty^{(2)}) - W(\infty^{(1)}) + \alpha\}\,.
\end{aligned} \tag{44}$$

As a result we can formulate the following.

Statement 3. Let expression (36) set some finite-gap solution of the Ernst equation. Then metric coefficients A and f related to the Ernst potential (36) by (4) are set by (43) and (44).

Let us make some final remarks:

(1) Expressing $f = \frac{1}{2}(\mathcal{E} + \bar{\mathcal{E}})$ in two different ways, from (36) and from (43), (44), we can get the factor $\exp\alpha$ in terms of theta-functions; substituting it into (43), (44), we obtain the formula for coefficient A in terms of theta-functions only. (Certainly $\exp\alpha$ may be expressed in theta-functions in the standard way using the so-called prime form (see [11]); however, our simple complex-analytic treatment is more straightforward.)
(2) If we change in (36) the sign of some n_j from $+1$ to -1 (or, equivalently, insert some half-integer characteristic $[0, m]$ in the related theta-function), then the same characteristics should be inserted in the first theta-functions in the numerators and denominators of (43), (44).
(3) We can look at the expression for A coming from (43), (44) as an explicit integration of the link (4) between A and $\mathcal{E}$, where $\mathcal{E}$ is set by (36). The important problem which seems to be essentially more difficult is an explicit integration of the link (4) between k and $\mathcal{E}$.
(4) The interplay of the transformation between the BZ and MN linear systems with the gauge transformations was used in [19] to generate the infinite-dimensional symmetry group (the Geroch group) of the SAS Einstein equation.

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