Elliptic solutions of stationary axisymmetric Einstein equation

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Abstract. In this paper we present a more transparent version of an earlier construction of genus g algebraic–geometric solutions of the Einstein equation. For one of the metric coefficients we obtain a new expression that allows us to construct the coefficient in terms of derivatives of the function $\Psi(\lambda)$ (solution of associative linear system). Finally, we proceed with an analysis of the two simplest genus 1 (elliptic) solutions.

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1. Introduction

In the recent works [1, 2] the author together with Matveev proposed a new approach to obtain the exact solutions of the stationary axisymmetric Einstein equation and Einstein–Maxwell system. It is based on the finite-gap (algebraic–geometric) method which was invented by Novikov, Dubrovin, Matveev, Its and Krichever [3, 4, 5] to get the periodic and quasi-periodic solutions of the classical integrable systems such as Korteweg–de Vries (KdV), non-linear Schrödinger (NSch) and others.

In [6, 7] we made some initial steps in clarification of the physical status of the vacuum solutions found in [1]. The properties of these solutions appear radically different from the properties of the finite-gap periodic and quasi-periodic solutions of KdV and so on. Namely the dynamics in the finite-gap solutions of the Einstein equation is set by the deformation of hyperelliptic algebraic curve with marked points in contrast to the linear flow on the Jacobi manifold of the fixed curve in traditional finite-gap solutions. This basic property leads to numerous important corollaries. For example, it appears possible to introduce the functional parameters corresponding to some static background into every algebraic–geometric solution of the Einstein equation. In full analogy with the degenerated (multisoliton) solutions describing the interaction of a few Kerr–NUT black holes, the finite-gap solutions on the flat background are localized. The overall genus g solution on the flat static background is set by $3g$ real parameters: $2g$ of them correspond to the sites of $2g$ immovable branch points on the basic real hyperelliptic curve and additional g parameters set the positions of g points of real pole divisor on this curve. In [6] we have shown that such a solution emerges some properties of multisoliton solutions: if the distances between the one-gap components each consisting of one immovable branch cut and one point of

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the divisor become large then their interaction (in some sense) tends to zero; this property remains true in the presence of the static background.

Here we present a modified version of the general construction found in [1, 2] that makes it more transparent due to more relevant choice of the axiomatic of related Ψ -function; however, the final expression for the Ernst potential in terms of the theta-functions is still the same. Besides that, we get a new formula for the metric coefficient $g_{\phi t}$ in terms of the derivative of the Ψ -function in the spectral parameter that allows us to find $g_{\phi t}$ without contour integration.

In [6] we investigated the topology, mass, angular momentum and the NUT parameter of genus-1 (elliptic) solution on the flat background. Here we present a more complete analysis of its properties in the case of the vanishing NUT parameter. The related Einstein manifold consists of two sheets: the stationary and the non-stationary, separated by the surface of stationary limit (infinite redshift surface) and glued along the 'branch ring' where one of the invariants of the Weyl tensor is singular. On the stationary sheet the mass and angular momentum have non-zero values; the compact event horizons are absent. On the non-stationary sheet the solution possesses translational and boost symmetries. So this solution may be interpreted as a massive singular 'hole' between the stationary and the non-stationary sheet.

In letter [7] we briefly discussed another example of elliptic solution which corresponds to some special choice of the static background. Here we prove rigorously the properties of this solution which were announced in [7]. Its Einstein manifold consists of an infinite number of sheets glued along the singular 'branch ring'. The Ernst potential on the different sheets differs by a positive constant factor. On the every sheet the mass is zero but the NUT parameter (magnetic mass) is non-zero. The compact event horizons are absent; the surface of stationary limit represents on the every sheet an infinite family of converging tori containing one another and the 'branch ring'. Taking into account this feature we named this solution a *toron*.

The paper is organized as follows. In section 2 we present a modified in comparison with [1] version of the general construction of the algebraic-geometric solutions of Ernst equation. The lack of space forces us to omit here some proofs and details which are not necessary to understand the main result. More details about the general construction may be found in [1, 2]. Section 3 contains the description of the physical properties of elliptic (genus-1) solution on the flat static background. Here we proceed the investigation started in [6] and obtain new results about the structure of the Einstein manifold of this solution. In section 4 we consider the 'toron' solution and prove rigorously the properties which were announced in [7]. Finally, in section 5 we discuss the obtained results and the perspectives. In appendix we derive the asymptotical expansion of the solution considered in section 3 near the 'branch ring'.

The ends of the proofs are marked by $\square$.

2. Ernst equation and multidimensional theta-functions

Consider the following standard form of the line element of stationary axisymmetric vacuum spacetime:

$$ds^2 = f^{-1}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2] - f(dt + Ad\phi)^2 \quad (1)$$

where (ρ, z) are canonical Weyl coordinates; functions f, k and A depending on (ρ, z) may be expressed in terms of the complex-valued Ernst potential $\mathcal{E}(\rho, z)$ as follows:

$$f = \operatorname{Re} \mathcal{E} \quad A_{\xi} = 2\rho \frac{(\bar{\mathcal{E}} - \mathcal{E})_{\xi}}{(\mathcal{E} + \bar{\mathcal{E}})^2} \quad k_{\xi} = 2i\rho \frac{\mathcal{E}_{\xi} \bar{\mathcal{E}}_{\xi}}{(\mathcal{E} + \bar{\mathcal{E}})^2} \quad (2)$$

where $\xi = (z + i\rho)$ and $\bar{\xi} = (z - i\rho)$ are new complex coordinates.

Then the Einstein equation for line element (1) reduces to the Ernst form:

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{z\bar{z}} + \rho^{-1}\mathcal{E}_{\rho} + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_z^2 + \mathcal{E}_{\rho}^2)$$

or, equivalently,

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{\xi\bar{\xi}} - \frac{\mathcal{E}_{\xi} - \mathcal{E}_{\bar{\xi}}}{2(\xi - \bar{\xi})}) = 2\mathcal{E}_{\xi}\mathcal{E}_{\bar{\xi}}. \quad (3)$$

The integrability of (3) was established in the pioneer works [8, 9]. However, it is more convenient for us to regard (3) as a compatibility condition for the following system which was found later in [10]:

$$\begin{aligned} \Psi_{\xi} &= \left[\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} + \sqrt{\frac{\lambda - \bar{\xi}}{\lambda - \xi}} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \right] \Psi \\ \Psi_{\bar{\xi}} &= \left[\begin{pmatrix} \bar{Y} & 0 \\ 0 & \bar{X} \end{pmatrix} + \sqrt{\frac{\lambda - \xi}{\lambda - \bar{\xi}}} \begin{pmatrix} 0 & \bar{Y} \\ \bar{X} & 0 \end{pmatrix} \right] \Psi \end{aligned} \quad (4)$$

where

$$X = \frac{\mathcal{E}_{\xi}}{\mathcal{E} + \bar{\mathcal{E}}} \quad Y = \frac{\bar{\mathcal{E}}_{\xi}}{\mathcal{E} + \bar{\mathcal{E}}};$$

$\lambda \in \mathcal{C}$ is a spectral parameter, $\Psi(\xi, \bar{\xi}, \lambda)$ is 2×2 matrix-valued function.

The $U - V$ pair (4) is in fact equivalent to the $U - V$ pair of Maison [8] and may be obtained from the $U - V$ pair of Belinskii and Zakharov [9] by a λ -dependent gauge transformation.

The pair (4) 'lives' on the Riemann surface $\mathcal{L}_0$ set by the equation

$$\omega^2 = (\lambda - \xi)(\lambda - \bar{\xi}).$$

It consists of two sheets and has two branch points at $\lambda = \xi$ and $\lambda = \bar{\xi}$; therefore, its genus is equal to zero. We denote by σ the involution on $\mathcal{L}_0$ interchanging the sheets.

The construction of finite-gap solutions of (3) that we present below differs in some details (which make it more transparent) from the construction presented in [1, 2]; the final expression for the Ernst potential is the same.

We follow the standard axiomatic approach (for the Kadomtsev-Petviashvili equation called the Krichever scheme). In order to construct the finite-gap solutions of integrable systems we proceed as follows: first, we present a system of axioms such that a function Ψ satisfying these axioms will satisfy (4) for some potential $\mathcal{E}$. Then we satisfy this system of axioms writing Ψ and $\mathcal{E}$ in terms of the theta-functions of some algebraic curve.

In our case the axioms of Ψ -function are

A. $\Psi(P)$ ($P \in \mathcal{L}_0$) is holomorphic and invertible at the branch points $\lambda = \xi$ and $\lambda = \bar{\xi}$.

B. All singularities of $\Psi(\lambda)$ on $\mathcal{L}_0$ (such as poles, essential singularities, zeros of $\det \Psi$, branch cuts and branch points) are regular, i.e. the logarithmic derivatives $\Psi_{\xi} \Psi^{-1}$ and $\Psi_{\bar{\xi}} \Psi^{-1}$ should be holomorphic in $\mathcal{L}_0$. In particular, the singularities should be independent of $(\xi, \bar{\xi})$.

C. Function $\Psi(\lambda)$ has to satisfy the reduction restriction

$$\Psi(\lambda^\sigma) = \sigma_3 \Psi(\lambda) \gamma \quad (5)$$

where σ_i , $i = 1, 2, 3$ are the ordinary Pauli matrices; the invertible matrix γ is independent of $(\xi, \bar{\xi})$. Henceforth we take $\gamma = \sigma_2$. (In [1, 2] we used $\gamma = \sigma_1$.)

D. Normalization and 'reality' condition:

$$\Psi(P = \infty^+) = \begin{pmatrix} i & \mathcal{E} \\ -i & \bar{\mathcal{E}} \end{pmatrix} \quad (6)$$

where ∞^+ is an infinite point on the first sheet of $\mathcal{L}_0$, where $\omega = \lambda + O(1)$; $\mathcal{E}(\xi, \bar{\xi})$ is some complex valued function which appears to coincide with the Ernst potential related to $\Psi(\lambda)$ by (4). (In [1, 2] we normalized the first column of function Ψ on vector $(1, -1)$ instead of $(i, -i)$.)

It is easy to prove that $\Psi(P)$ satisfying conditions A–D obeys (4). Namely, conditions A and B provide appropriate analytic structure of the logarithmic derivatives of Ψ on $\mathcal{L}_0$:

$$\Psi_\xi \Psi^{-1} = P \sqrt{\frac{\lambda - \bar{\xi}}{\lambda - \xi}} + Q \quad \Psi_{\bar{\xi}} \Psi^{-1} = R \sqrt{\frac{\lambda - \xi}{\lambda - \bar{\xi}}} + T$$

for some matrices P, Q, R, T independent of λ . The structure of these matrices is determined by C and D

A similar system of axioms corresponding to the multisoliton case was presented in [9, 12] in Belinskii–Zakharov formulation which gave multi-Kerr–NUT solutions with function $\Psi(P)$ having in $\mathcal{L}_0$ only poles and zeros of the determinant. Another well-known class of static (Weyl) solutions corresponds to functions Ψ which solve on $\mathcal{L}_0$ the Riemann–Hilbert problem in $\mathcal{L}_0$ with diagonal 'patching' matrices. Finally, the algebraic–geometric construction allows one to obtain functions $\Psi(P)$ having additional branch cuts on $\mathcal{L}_0$ with off-diagonal patching matrices [1, 2]. This construction is as follows.

Figure 1. The Hurvitz diagram of the curves $\hat{\mathcal{L}}$ and $\mathcal{L}$ and the relation between φ and ψ on different sheets.

Denote by $\hat{\mathcal{L}}$ the two-sheeted covering of $\mathcal{L}_0$ whose Hurvitz diagram is presented in figure 1(a) ($\hat{\mathcal{L}}$ is the four-sheeted covering of the complex plane). The surface $\hat{\mathcal{L}}$ possesses two holomorphic involutions interchanging the sheets: $\sigma : 1 \leftrightarrow 3, 2 \leftrightarrow 4$ inherited from $\mathcal{L}_0$ and $*$: $1 \leftrightarrow 2, 3 \leftrightarrow 4$. The function $\Psi(P)$ ($P \in \hat{\mathcal{L}}$) is set on the first copy of $\mathcal{L}_0$ consisting of sheets 1 and 3 of $\hat{\mathcal{L}}$ by the ansatz

$$\Psi(P) = \begin{pmatrix} \varphi(P) & \varphi(P^*) \\ \psi(P) & \psi(P^*) \end{pmatrix} \quad (7)$$

where $\varphi(P)$ and $\psi(P)$ are scalar functions on $\hat{\mathcal{L}}$.

It is possible to show [15] that the algebraic curve $\hat{\mathcal{L}}$ naturally arises for a rather general class of solutions as a characteristic equation $\det(M(\lambda, \rho, z) - \omega I) = 0$ of the so-called monodromy matrix M ; the columns of Ψ -function (7) are the eigenvectors of this matrix (the main condition distinguishing the algebraic-geometric solutions among the others is a finite genus of the curve setting by this polynomial which is provided by the condition that $M(\lambda, \rho, z)$ be rational in λ).

The behaviour of φ and ψ under the involution σ implied by the reduction restriction is

$$\begin{aligned}\varphi(\lambda^{(3)}) &= i\varphi(\lambda^{(2)}) & \varphi(\lambda^{(4)}) &= -i\varphi(\lambda^{(1)}) \\ \psi(\lambda^{(3)}) &= -i\psi(\lambda^{(2)}) & \psi(\lambda^{(4)}) &= i\psi(\lambda^{(1)})\end{aligned}\quad (8)$$

whereby $\lambda^{(k)}$ is the point on the k th sheet of $\hat{\mathcal{L}}$ having projection λ on the complex plane.

Relation (8) allows to construct the functions φ and ψ first on the auxiliary hyperelliptic curve $\mathcal{L}$ (its Hurvitz diagram is shown in figure 1(b)). Functions φ and ψ should be non-single-valued on $\mathcal{L}$: when the point P crosses the closed contour s they get a multiplier $-i$ and i , respectively, as shown in figure 2. To satisfy conditions (8), functions φ and ψ have to be extended from $\mathcal{L}$ to $\hat{\mathcal{L}}$, in accordance with figure 1.

Figure 2. The branch cuts and the canonical basis of cycles on the curve $\mathcal{L}$; continuous contours lie on the first sheet, and the dotted ones lie on the second. Upon crossing a contour s , functions $\varphi(P)$ and $\psi(P)$ get a multiplier $-i$ and i , respectively.

The hyperelliptic curve $\mathcal{L}$ of genus g playing the central role in our construction is defined by the equation

$$\omega^2 = (\lambda - \xi)(\lambda - \bar{\xi}) \prod_{i=1}^g (\lambda - E_i)(\lambda - F_i) \quad (9)$$

where all points E_i, F_i are independent of ξ and $\bar{\xi}$.

To satisfy condition (6) one has to assume that curve $\mathcal{L}$ is real, i.e. for the any $i = 1, \dots, g$ either $E_i = \bar{F}_i$ or $E_i, F_i \in \mathbb{R}$. However, in this paper we consider only solutions with the vertical positions of immovable branch cuts; so we shall take further $E_i = \bar{F}_i, i = 1, \dots, g$ (figure 2).

Note that the curve $\mathcal{L}$ has two branch points $\lambda = \xi$ and $\lambda = \bar{\xi}$ depending on variables (ρ, z) ; this is, the main difference between the present construction and the traditional finite-gap solutions of KdV, NSch and other standard models; this property is responsible for many new features of the algebraic-geometric solutions of the Ernst equation.

Now we define some canonical objects associated with curve $\mathcal{L}$. The basis of cycles is chosen according to figure 2. The dual basis of holomorphic 1-forms (Abelian differentials of the first kind) consists of $dU_j(P)$, $j = 1, \dots, g$ normalized so that

$$\oint_{a_i} dU_j(P) = \delta_{ij}.$$

The corresponding integrals are denoted by

$$U_j(P) = \int_{P_0}^P dU_j(P) \quad (P, P_0 \in \mathcal{L}, j = 1, \dots, g).$$

The matrix

$$B_{ij} = \oint_{b_i} dU_j(P)$$

with positive imaginary part is the matrix of b -periods associated with $\mathcal{L}$.

The multidimensional theta-function with and without characteristics is defined by the series

$$\Theta[u, v](x|B) = \sum_{m \in \mathbb{Z}^g} \exp\{\pi i(B(m+u), m+u) + 2\pi i(x+v, M+u)\}$$

$$\Theta(x) \equiv \Theta[0, 0](x|B) \quad (10)$$

where $x \in \mathbb{C}^g$, $(\cdot, \cdot)$ is the ordinary scalar product.

For more detailed description of the standard algebraic-geometric objects associated with Riemann surfaces see [13, 14].

Notice that to provide condition (6) it suffices to take

$$\bar{\varphi}(\bar{P}) = \psi(P) \quad (11)$$

(in agreement with (8)) and to normalize $\varphi(P)$ according to

$$\varphi(\infty^{(1)}) = i. \quad (12)$$

Hence it suffices to write an explicit expression only for $\varphi(P)$; then the expression for $\psi(P)$ can be easily obtained from (11). The formula for $\varphi(P)$ is slightly modified in comparison with [1, 2] due to a modification of the reduction restriction:

$$\varphi(P) = C_\varphi \frac{\Theta(U(P) - U(D) + \frac{1}{4}n + b_W - K)}{\Theta(U(P) - U(D) - K)} \exp\{W(P)\} \quad (13)$$

where $D = D_1 + \dots + D_g$ is an arbitrary non-special divisor independent of (ρ, z) and satisfying the 'reality conditions' $D = \bar{D}$, K is a vector of Riemann constants and C_φ is a normalization constant determined from (12). The vector $\frac{1}{4}n$ ($n_i = 1$, $i = 1, \dots, g$) in argument of the theta-functions is added to provide the multiplication of φ by $-i$ when tracing a round along contours b_i . This is equivalent to the multiplication by this factor when crossing contour s . Notice that we can set $n_j = \pm 1$; if $n_j = -1$ for some j , functions φ and ψ should change their sign also on related contour a_j .

The Abelian integral $W(P)$ with vector of b -periods $2\pi b_W$ is an arbitrary linear combination of the normalized (all a -periods are zero) Abelian integrals of the second and the third kind; all singularities and residues of these integrals are independent of $\xi, \bar{\xi}$. The integral $W(P)$ has to satisfy the following 'reality conditions':

$$W(\infty^{(2)}) - W(\infty^{(1)}) \in \mathbb{R}$$

$$2\pi i b_W \in \mathbb{R}.$$

To provide these conditions, it is enough to impose on $W(P)$ the restriction

$$\bar{W}(\bar{P}) = W(P) \quad (14)$$

Since $W(P)$ is an indefinite integral, this condition holds up to an arbitrary constant. This equation implies obvious restrictions for the singularities and residues of $W(P)$ [1, 2].

It can be easily verified that function $\Psi(P)$ defined by (7), (13) and (11) satisfies the axioms A–D of our axiomatic for the general position of moving branch cut $[\xi, \bar{\xi}]$. This function has regular singularities at the points E_i, F_i, D_i and the poles of the integral $W(P)$. Besides that it has regular branch cuts; on the cuts $[E_i, F_i]$ the patching matrices are off-diagonal in contrast to the diagonal patching matrices on the cuts introducing by the integral $W(P)$.

From (13) and (12) we obtain the following expression for the Ernst potential $\mathcal{E} = \varphi(\infty^{(2)})$ related to our Ψ -function:

$$\begin{aligned} \mathcal{E}(\xi, \bar{\xi}) = & \frac{\Theta(U(\infty^{(2)}) - U(D) + \frac{1}{4}n + b_W - K)\Theta(U(\infty^{(1)}) - U(D) - K)}{\Theta(U(\infty^{(1)}) - U(D) + \frac{1}{4}n + b_W - K)\Theta(U(\infty^{(2)}) - U(D) - K)} \\ & \times \exp[W(\infty^{(2)}) - W(\infty^{(1)})] \end{aligned} \quad (15)$$

where the path l connecting $\infty^{(1)}$ and $\infty^{(2)}$ is chosen as shown in figure 2. The dependence on $(\xi, \bar{\xi})$ in (15) is due to the dependence of B , $U(\infty^{(1,2)}) - U(D)$, $W(\infty^{(2)}) - W(\infty^{(1)})$, b_W and K on $(\xi, \bar{\xi})$. So the dynamics in (15) is due to the deformation of our curve $\mathcal{L}$ (with fixed divisor and fixed singularities of $W(P)$) in the moduli space of hyperelliptic algebraic curves with marked points.

The integral $W(P)$ corresponds to the static background of solution (15). Indeed, if we take $g = 0$ then

$$\mathcal{E}(\xi, \bar{\xi}) = \exp[W(\infty^{(2)}) - W(\infty^{(1)})] \quad (16)$$

is real and (3) reduces to the Laplace (Euler–Darboux) equation:

$$\begin{aligned} \Delta(\log \mathcal{E}) &= 0 \\ \Delta &\equiv \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2. \end{aligned} \quad (17)$$

Solutions of this equation constitute static (Weyl) class. Note that the integral $W(P)$ may include functional parameters if the linear combination defining $W(P)$ becomes infinite. The linear structure on the manifold of static solutions by means of $W(P)$ induces linear structure on the manifold of solutions of the form (15) with fixed branch points E_i and F_i .

The open question here is whether all solutions of equation (17) can be represented in the form (16).

Generally, a solution of the form (15) may have three different kinds of singularities:

1. Expression (15) may not satisfy the Ernst equation if ξ coincides with the regular singularities of function Ψ , i.e. the branch points, points of the divisor D or poles of $W(P)$, since in this case axiom A is not true.
2. Function (15) may turn infinite, $\mathcal{E} = i$, at the points where the denominator of (15) vanishes.
3. The metric coefficients defined by (2) may become infinite if $\text{Re} \mathcal{E} = 0$; this condition defines some curve in ξ -plane. Related hypersurface $\mathcal{S}$ on the four-dimensional Einstein manifold is called the *surface of stationary limit* (or *infinite redshift surface*). On this surface the solution changes the character of its symmetries: the axial symmetry becomes boost and the time symmetry becomes translational. Nevertheless, in all known cases the metric coefficients are finite on $\mathcal{S}$ (except, probably, the points where $\mathcal{E} = i$).

We will now discuss the second and third possibilities in more detail.

Condition $\mathcal{E} = \infty$ implies that

$$U(D) - \frac{1}{4}n - b_W - U(\infty^{(1)}) \equiv U(D'_1) + \dots + U(D'_{g-1}) \quad (18)$$

modulus linear combination of the columns of matrices I (unit matrix) and B , where $D' = D'_1 + \dots + D'_{g-1}$ is some positive divisor of degree $g-1$ (the latter may depend on $(\xi, \bar{\xi})$).

If divisor $\infty^{(1)} + D'$ is non-special, then the nominator of (15) remains non-vanishing:

$$\Theta(U(\infty^{(2)}) - U(D) + \frac{1}{4}n + b_W - K) \neq 0$$

and $\mathcal{E} = \infty$.

Generally, condition (18) is transcendental and it is difficult to use it efficiently. However, in the elliptic case $g = 1$ it is considerably simplified:

$$U(D) - \frac{1}{4} - b_W - U(\infty^{(1)}) \equiv 0.$$

In fact, this is impossible, because $U(D) - U(\infty^{(1)})$ and b_W are always imaginary together with the 1×1 matrix of b -periods B . We thus arrive at the following

Statement 1. For $g = 1$ all singularities of the Ernst potential (15) lie at the branch points $\xi = E, \bar{E}$, at the points of the divisor $\xi = D$ or at the poles of the integral $W(P)$.

Now we consider the surface of the stationary limit. From condition (6) (axiom **D** we see that

$$\det \Psi(P = \infty^{(1)}) = 2i \operatorname{Re} \mathcal{E}(\xi, \bar{\xi})$$

and, hence, if $\operatorname{Re} \mathcal{E} = 0$ then for the same ξ we have $\det \Psi(P = \infty^{(1)}) = 0$.

However, for every ξ we already know all poles of $\det \Psi(P)$ lying at D_j, D_j^* and the poles of $\exp(W(P))$. Hence, if there is an additional zero at $P = \infty^{(1)}$ then

$$\det \Psi(P) \equiv 0 \quad P \in \mathcal{L} \quad \xi \in S.$$

In particular, it means that

$$\det \Psi(P = \xi) = 0.$$

Therefore, $\varphi(P = \xi) = 0$ or $\psi(P = \xi) = 0$. Hence, a point ξ lies on the surface S if we can find a divisor $D'_1 + D'_2 + \dots + D'_{g-1}$ on $\mathcal{L}$ such that

$$U(D) \pm \frac{1}{4}n - b_W - (U(\xi) + U(D'_1) + \dots + U(D'_{g-1})) \equiv 0 \quad (19)$$

for any sign $\pm$ modulus of the arbitrary combination of the columns of the matrices I and B .

Note that if potential $\mathcal{E}$ is finite and non-vanishing at some point ξ lying on the surface S then point ξ belongs to the surface of the stationary limit of potential $\mathcal{E}^{-1}$ as well. For the points where $\mathcal{E} = 0$ or $\mathcal{E} = \infty$ this statement may in general be non-true. However, if surface S has no isolated points in the ξ plane (in this plane S is a curve), we can state that the points where $\mathcal{E} = \infty$ also lie on S .

For $g > 1$ condition (19) setting surface S is rather complicated as a singularity condition (18). However, for elliptic case we again have a considerable simplification. Namely, we can formulate the following

Statement 2. If $g = 1$, then the surface of the stationary limit S of solution (15) is defined by the condition

$$U(D) - U(\xi) - \frac{1}{4}n - b_W \equiv 0 \quad (\text{mod } 1, B). \quad (20)$$

We shall use condition (20) in sections 3 and 4 for the investigation of the infinite redshift surfaces of some special elliptic solutions.

As was shown in [1, 2], in some particular cases formula (15) includes the whole class of multi-Kerr-NUT solutions. However, here there is some subtlety. Note that a single Kerr-NUT solution can be obtained from (15) in two different ways. The first way is to take

$$g = 1 \quad W(P) = 2i\alpha W_{E_1 F_1}(P)$$

where α is an arbitrary real constant; notation $W_{QR}(P)$ is used for the Abelian integral of the third kind having logarithmic singularities at points Q and R with the residue $+1$ and -1 , respectively. Vector b_W related to the integral $W(P)$ is equal to $i\alpha$. Then as $E_1, F_1 \rightarrow \gamma \in \mathbb{C}$ one obtains

$$\Gamma \equiv \frac{1 - \varepsilon}{1 + \varepsilon} = \left[\frac{x_1}{-1 + ia} + \frac{ia y_1}{-1 + ia} \right]^{-1} \quad (21)$$

where $a = e^{-2\pi\alpha}$;

$$x_1 = \frac{f(\gamma) + f(D)}{D - \gamma} \quad y_1 = \frac{f(\gamma) - f(D)}{D - \gamma}$$

$$f(\lambda) \equiv \sqrt{(\lambda - \xi)(\lambda - \bar{\xi})}$$

are prolate ellipsoidal coordinates.

However, formula (21) gives only a three-dimensional subset of the general four-parametric Kerr-NUT family. To obtain the whole Kerr-NUT family by the degeneration of solutions (15) one should use the following alternative procedure.

Let $g = 2$ and

$$W = \sum_{i=1,2} (2i\alpha W_{E_i F_i} + \frac{1}{2}(W_{D_i E_i} + W_{D_i F_i})).$$

The addition of the integrals $W_{D_i E_i}$ and $W_{D_i F_i}$ into the static background is in fact equivalent to choosing $D_1 = E_1$ and $D_2 = E_2 = \bar{E}_1$.

Then formula (15) transforms into

$$\varepsilon = \frac{\Theta(U(\infty^{(2)}) - U(\xi) + \frac{1}{4}n + i\alpha - K)}{\Theta(U(\infty^{(1)}) - U(\xi) + \frac{1}{4}n + i\alpha - K)} \exp[2\pi i(U_1 |_{\xi}^{\infty^{(1)}} + U_2 |_{\xi}^{\infty^{(1)}})] \quad (22)$$

where $n_i = \pm 1, i = 1, 2$; $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$.

Formula (22) gives after the degeneration $E_i, F_i \rightarrow \gamma_i \in \mathbb{R}$ the whole Kerr-NUT family [1, 2]:

$$\Gamma = \left[\frac{i(a_1 - a_2)}{1 - a_1 a_2 + i(a_1 + a_2)} x_2 + \frac{1 + a_1 a_2}{1 - a_1 a_2 + i(a_1 + a_2)} y_2 \right]^{-1}$$

where $a_i = n_i e^{-2\pi i}$;

$$x_2 = \frac{f(\gamma_1) + f(\gamma_2)}{\gamma_1 - \gamma_2} \quad y_2 = \frac{f(\gamma_1) - f(\gamma_2)}{\gamma_1 - \gamma_2}$$

are the new prolate ellipsoidal coordinates.

Similarly, the overall N -Kerr-NUT solution may be obtained by the degeneration of the $g = 2N$ solution:

$$\varepsilon = \frac{\Theta(U(\infty^{(2)}) - U(\xi) + \frac{1}{4}n + i\alpha - K)}{\Theta(U(\infty^{(1)}) - U(\xi) + \frac{1}{4}n + i\alpha - K)} \exp \left[2\pi i \sum_{j=1}^{2N} U_j |_{\xi}^{\infty^{(1)}} \right] \quad (23)$$

where $n_j = \pm 1$, $j = 1, \dots, 2N$, $\alpha = (\alpha_1, \dots, \alpha_{2N}) \in \mathbb{R}^{2N}$; if we choose $P_0 = \xi$ then vector K may be represented in the form [16]:

$$K_j = \frac{1}{2} \sum_{k=1}^g B_{kj} + \frac{j}{2} \quad j = 1, \dots, g \quad (= 2N).$$

In the non-degenerated situation $E_i \neq F_i$, formula (23) gives the finite-gap generalization of the overall multi-Kerr-NUT solutions.

It is natural to start the investigation of the physical properties of non-degenerated solution (15) from the simplest (elliptic) case $g = 1$. In this case curve $\mathcal{L}$ is set by the equation

$$\omega^2 = (\lambda - \xi)(\lambda - \bar{\xi})(\lambda - E)(\lambda - \bar{E})$$

and the multidimensional theta-functions from (15) are the ordinary Jacobi theta-functions. The only holomorphic elliptic differential on $\mathcal{L}$ may be represented as

$$dU = \frac{C d\lambda}{\omega}$$

where C is a normalization constant defined by

$$C = \left[\oint_a \frac{d\lambda}{\omega} \right]^{-1}.$$

The matrix of b -periods is now a scalar imaginary function:

$$B = \left[\oint_a \frac{d\lambda}{\omega} \right]^{-1} \oint_b \frac{d\lambda}{\omega}.$$

The only Riemann constant is

$$K = \frac{B + 1}{2}.$$

However, in spite of a considerable simplification of all objects, even in this simplest case we have a very rich class of solutions due to the presence of integral $W(P)$ with an arbitrary set of constant poles and residues.

What should be the choice of this integral required to get the most elementary algebraic-geometric solution?

To answer this question one should recall that the singularities of the Ernst potential in the ξ -plane in the case $g = 1$ coincide with the spectrum of linear system (4), i.e. with the singularities of the function Ψ in the λ -plane. In order to minimize the number of singularities, it is natural to take integrals $W(P)$ which have no poles and zeros except $P = E, \bar{E}, D$ and ∞ . It turns out that $\mathcal{E} \rightarrow 1$ as $\xi \rightarrow \infty$, in a most physically reasonable way if one excludes $P = \infty$ from this list; otherwise (15) asymptotically coincides with the Kasner solution.

Turning to the other opportunities, consider first the simplest case $W(P) = 0$. An investigation of this solution began with [6]; in the next section we present a more complete analysis of this case.

Another way to choose $W(P)$ arises if we wish to 'delete' a pole of Ψ at $P = D$. For this purpose, taking into account the 'reality condition' (14), we have to choose

$$W = \frac{1}{2}(W_{DE} + W_{D\bar{E}})$$

Unfortunately, this choice leads to the trivial solution $\mathcal{E} = 1$. The simplest sophistication of the choice is

$$W = \frac{1}{2}(W_{DE} + W_{D\bar{E}}) + i\alpha W_{E\bar{E}}$$

where α is an arbitrary real constant. This leads to 'toron' solution which was already briefly considered in [2]. In section 3 we present a rigorous analysis of this solution.

The metric coefficient A can be found in terms of the function Ψ (see below). It turns out to be possible to avoid the contour integration when expressing A in terms of the Ernst potential (see (2)).

Statement 3. Let $\mathcal{E}$ be an arbitrary solution of the Ernst equation and $\Psi(P)$ be a solution of linear system (4) satisfying A-D. Then the metric coefficient $A(\xi, \bar{\xi})$ set by (2) may be expressed as

$$A = A_0 + 2\sqrt{2}(\Psi^{-1}\Psi_\delta(\delta=0))_{21}$$

where $\delta \equiv 1/\lambda$ is a local parameter in a neighbourhood of the infinite point $\lambda = \infty^+$ on the first sheet of $\mathcal{L}_0$ (where $\omega = \lambda + O(1)$; A_0 is an arbitrary constant).

Proof 1. It is not difficult to verify the following identity [17]:

$$(\Psi^{-1}\Psi_\delta)_\xi = \Psi^{-1}(\Psi_\xi\Psi^{-1})_\delta\Psi.$$

Now statement follows immediately from the form of matrix $\Psi_\xi\Psi^{-1}$ (4), normalization condition (6) and the expression of A in terms of the Ernst potential (2). $\square$

Using the basic ansatz (7), we can rewrite expression for $A(\xi, \bar{\xi})$ as

$$A = A_0 + 2\sqrt{2}\frac{(\varphi + \psi)_{1/\lambda}(P = \infty^{(1)})}{(\varphi + \psi)(P = \infty^{(2)})}$$

where subscript $1/\lambda$ means a partial derivative. The important open question is: is it possible to find some similar expression for the conformal factor $g_{\rho\rho}$ (i.e. for the function k in (2))?

3. Elliptic solution on the flat static background

Take $g = 1$ and $W(P) = 0$. Then (15) takes the form into

$$\mathcal{E}(\xi, \bar{\xi}) = \frac{\Theta(U(\infty^{(2)}) - U(D) + \frac{1}{4} - B/2)\Theta(U(\infty^{(1)}) - U(D) - \frac{1}{2}(B+1))}{\Theta(U(\infty^{(1)}) - U(D) + \frac{1}{4} - B/2)\Theta(U(\infty^{(2)}) - U(D) - \frac{1}{2}(B+1))}. \quad (24)$$

This solution was first investigated in [6] where we discussed the topology, behaviour on the symmetry axis, mass, NUT parameter, angular momentum and the infinite redshift surface (surface of stationary limit). In particular, we found that for $D = \text{Re } E$ the NUT parameter of (24) becomes zero. Without loss of generality we can take

$$D = 0^{(1)} \quad E = \bar{E} \quad (25)$$

Henceforth we restrict ourselves to this most interesting case. In [6] we have already established some properties of a solution defined by (24), (25). Since we fix the location of the object on the symmetry axis ($\text{Re } E = 0$), solution (24) includes only the parameter $|E|$.

To understand the nature of the dynamics of solution (24), note that in fact it is determined by the dynamics of the variables $B(\xi, \bar{\xi})$ and $(U(\infty^{(1,2)}) - U(D))(\xi, \bar{\xi})$.

In terms of non-normalized holomorphic differential

$$dU_0 = \frac{d\lambda}{\omega}$$

we have

$$B = \frac{B_0}{A_0}$$

$$U(\infty^{(1,2)}) - U(D) = \frac{1}{A_0}(U_0(\infty^{(1,2)}) - U_0(D))$$

$$A_0 \equiv \oint_a dU_0 \quad B_0 \equiv \oint_b dU_0.$$

It is easy to verify that differential dU_0 obeys the Laplace (Euler–Darboux) equation (the Ernst equation for the real Ernst potential (17)):

$$(\partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2)(dU_0) = 0$$

and, therefore, all the functions A_0 , B_0 and $U_0(\infty^{(1,2)}) - U_0(D)$ obey this linear equation too. (The same is true for g non-normalized holomorphic differentials in general genus g solution.) Combining then the solutions of the linear Euler–Darboux equation we obtain by means of (24) a solution of the non-linear Ernst equation (3). The Euler–Darboux equation arises here as a partial case of a general linear Picard–Fuchs system which describes the dependence of (hyper)elliptic integrals on the branch points.

In the limit $E \rightarrow 0$ curve $\mathcal{L}$ degenerates and then everything in (24) can be expressed in terms of elementary functions as follows [1, 2, 6]:

$$dU \rightarrow \frac{|\xi|d\lambda}{2\pi i \lambda \sqrt{(\lambda - \xi)(\lambda - \bar{\xi})}}$$

$$B \sim \frac{\log |E|}{\pi i} + O(1).$$

Taking into account the definition of theta-function (10), it is easy to verify [6] that $\mathcal{E}(\xi, \bar{\xi}) \rightarrow 1$ when $E \rightarrow 0$ i.e. in this limit the spacetime becomes flat. Similarly, for $\rho \rightarrow 0$ we have

$$dU \rightarrow \frac{\sqrt{(z - E)(z + E)}}{2\pi i (\lambda - z) \sqrt{(\lambda - E)(\lambda + E)}} \quad (26)$$

$$B \sim \frac{\log \rho}{\pi i} + O(1)$$

and after simple calculations the Ernst potential on the symmetry axis may be expressed in elementary functions as

$$\mathcal{E}(\rho = 0, z) = \pm \frac{z\sqrt{z^2 + |E|^2} + i|E|^2}{z^2 + z|E| + |E|^2}. \quad (27)$$

Due to the presence of two Killing vectors, the manifold $\mathcal{M}$ being the two-dimensional section $t = \text{constant}$, $\phi = \text{constant} + k\pi$, $k = 0, 1$ of the four-dimensional Einstein manifold $\hat{\mathcal{M}}$, may be considered as a covering of the plane with complex ξ -plane. So we can investigate $\mathcal{M}$ in terms of Riemann surfaces and represent the structure of this manifold by the Hurvitz diagram (which can be regarded as a ‘profile’ of the corresponding Riemann surface). In the sequel we shall not distinguish between $\mathcal{M}$ and $\hat{\mathcal{M}}$ when it is not essential.

Note also the difference between the Ernst manifold of some solution, i.e. the manifold with the value of Ernst potential fixed at every point, and the related Einstein manifold,

i.e. the manifold in which the metric tensor is fixed at every point. In the general case, due to the integral character of relation (2) between the metric coefficients and the Ernst potential, the Einstein manifold is a covering of the Ernst manifold. (If at some point on the ξ -plane the Ernst equation is not fulfilled, the metric coefficients may possess a monodromy property with respect to tracing around this point).

It has been found in [6] that the Ernst manifold of solution (24) (ξ being the coordinate) consists of two sheets $\mathcal{M}_1$ and $\mathcal{M}_2$ glued along the branch ring $\xi = E$. The Ernst potential has different signs on $\mathcal{M}_1$ and $\mathcal{M}_2$. (The latter can be easily verified: consider a deformation of $\mathcal{L}$ such that ξ traces around E . Then all a - and b cycles remain unchanged, and so $[D, \infty^{(1)}]$. The contour $[D, \infty^{(2)}]$ is modified by $2b$. Hence the expression (24) for the Ernst potential suffers the following simple modification:

$$U(\infty^{(2)}) - U(D) \rightarrow U(\infty^{(2)}) - U(D) \pm 2B.$$

By the periodicity property of theta-function, this immediately implies the change of the sign of $\mathcal{E}$.) The potential tends to its limit value $+1$ at infinity on $\mathcal{M}_1$ and to -1 at infinity of $\mathcal{M}_2$. Hence the sign in (27) should be chosen according to the asymptotic behaviour at the infinite points.

In our case the Einstein manifold coincides with the Ernst manifold (figure 3). To show this we have to check that the double tracing of ξ around E preserves the values of the metric coefficients $g_{\phi t} = -fA$ and $g_{\rho\rho} = f^{-1}e^{2k}$. For the coefficient A , the situation is especially simple because according to (2), expression for A_ξ has different signs on $\mathcal{M}_1$ and $\mathcal{M}_2$. The single-valuedness of $f^{-1}e^{2k}$ on the Ernst manifold (and, moreover, the coincidence of values of this function on $\mathcal{M}_1$ and $\mathcal{M}_2$) can be derived, for example, from formula (31) obtained below for the local behaviour of this metric coefficient near the branch ring $\xi = E$.

From (27) we can immediately find the mass and the angular momentum on $\mathcal{M}_1$:

$$M = \frac{1}{2}|E|$$

$$J = -\frac{1}{2}|E|^2$$

Sheet $\mathcal{M}_2$, in contrast to $\mathcal{M}_1$, is not stationary and axisymmetric (we shall later discuss the character of the symmetries on this sheet); it thus seems irrelevant to discuss the mass and angular momentum on it.

Figure 3. The Hurvitz diagram of the two-dimensional section $\phi = \text{constant}$, $t = \text{constant}$ of the Einstein manifold of the elliptic solution with flat background: points $\xi = E$ and $\xi = \bar{E}$ correspond to a singular 'branch ring'.

Now we will discuss the structure of the surface of stationary limit $\mathcal{S}$. In [6] we demonstrated that the relation $\text{Re } \mathcal{E} = 0$ is fulfilled on the disc $\text{Re } \xi = \text{Re } E$, $|\text{Im } \xi| \leq |\text{Im } E|$ on the both sheets of $\mathcal{M}$. Here we prove that this disc is the whole surface $\mathcal{S}$:

Statement 4. The surface of stationary limit of solution (24) consists of two discs $\text{Re } \xi = \text{Re } E$, $|\text{Im } \xi| \leq |\text{Im } E|$ lying on the different sheets of the related Einstein manifold $\mathcal{M}$.

Proof. According to statement 2, point ξ lies on $\mathcal{S}$ if and only if

$$U(D) - U(\xi) \equiv \frac{1}{4} \pmod{1, B}. \quad (28)$$

Consider the conformal (fractional-linear) transformation $\sigma(\lambda)$ mapping curve $\mathcal{L}$ (figure 4(a)) to the standard form: $\sigma(E) = 1$, $\sigma(\bar{E}) = -1$, $\sigma(\xi) = -\sigma(\bar{\xi}) < 1$ (figure 4(b)). Point D is mapped into some point $\sigma(D)$ ($\operatorname{Re} \sigma(D) = 0$); $\sigma(D) = 0$ if and only if $\operatorname{Re} \xi = \operatorname{Re} E$ and $|\operatorname{Im} \xi| < |\operatorname{Im} E|$. Since $\sigma(\lambda)$ is conformal, for the Abel map U^σ (see figure 4(b)) we have

Figure 4. The conformal map σ of the curve $\mathcal{L}$ (a) to a standard form (b). The involution $\sigma(l) \rightarrow -\sigma(l)$, invisible in figure 4(a), is seen in figure 4(b).

$$U^\sigma(\sigma(D)) - U^\sigma(\sigma(\xi)) \equiv \frac{1}{4} \pmod{1, B}. \quad (29)$$

Let $\sigma(D) \neq 0$. Then, using the symmetry $\sigma(\lambda) \rightarrow -\sigma(\lambda)$ (this symmetry is not visible in figure 4(a)) we see that

$$U^\sigma(-\sigma(D)) - U^\sigma(-\sigma(\xi)) \equiv -\frac{1}{4} \pmod{1, B}.$$

Besides that, we have

$$U^\sigma(\sigma(\xi)) - U^\sigma(-\sigma(\xi)) = U^\sigma(\sigma(\xi)) - U^\sigma(\sigma(\bar{\xi})) \equiv \frac{1}{2}.$$

Hence,

$$U^\sigma(-\sigma(D)) - U^\sigma(\sigma(\xi)) \equiv \frac{1}{4}.$$

Now, if $\sigma(D) \neq -\sigma(D)$, we arrive at a contradiction with (29), because an Abel map on an elliptic curve gives a one-to-one correspondence between the curve and its Jacobi torus. Thus to satisfy condition (28) we have to assume $\sigma(D) = 0$, then ξ belongs to the segment $[E, \bar{E}]$. (Note that the condition $\sigma(D) = -\sigma(D)$ is also fulfilled if $\sigma(D) = \infty$, but in this case $U^\sigma(\infty) - U^\sigma(\sigma(\xi)) \equiv \frac{1}{4} + \frac{B}{2}$.) $\square$

Using the Ernst equation, it is not difficult to verify that in spite of the presence of the singular factor f^{-1} in (1), the metric coefficients

$$\begin{aligned} g_{\rho\rho} &= g_{zz} \equiv \frac{e^{2k}}{\operatorname{Re} \mathcal{E}} \\ g_{\phi t} &\equiv -A \operatorname{Re} \mathcal{E} \\ g_{\phi\phi} &= -A^2 \operatorname{Re} \mathcal{E} + \frac{\rho^2}{\operatorname{Re} \mathcal{E}} \end{aligned}$$

remain finite and non-vanishing on $\mathcal{S}$ outside of the 'branch ring'. This is, of course, not surprising, because on the infinite redshift surface of Kerr-NUT or multi-Kerr-NUT solutions the metric coefficients are non-singular too. To provide continuous behaviour of the conformal factor $g_{\rho\rho}$ on $\mathcal{S}$ the function $2k$ should not be everywhere real: if it is real on

$\mathcal{M}_1$, then it will have imaginary part πi on $\mathcal{M}_2$ behind $\mathcal{S}$ (the same situation takes place in the case of a Kerr solution).

As a result, we see that all probable singularities of metric coefficients lie on the 'branch ring' $\xi = E$.

The natural local coordinates in a neighbourhood of this ring are

$$\tau = \sqrt{i(\xi + E)} \quad \bar{\tau} = \sqrt{-i(\xi - E)}$$

To find the asymptotical behaviour of $\mathcal{E}(\tau, \bar{\tau})$ in a neighbourhood of $\tau = 0$, it is convenient to introduce a new basis of cycles on $\mathcal{L}$: $(\tilde{a}, \tilde{b}) = (b, -a)$. Then a new 1×1 matrix of b -periods $\tilde{B}$ is connected with the former matrix B as follows:

$$\tilde{B} = -\frac{1}{B}$$

To obtain the expression for the Ernst potential (24) in the new basis, we will write first a new expression for $\varphi(P)$. Taking into account the analytic properties of $\varphi(P)$, we can represent it in the form

$$\varphi(P) = C_\varphi \frac{\Theta(\tilde{U}(P) - \tilde{U}(D) - \frac{\tilde{B}}{4} - \frac{1}{2}|\tilde{B}|)}{\Theta(\tilde{U}(P) - \tilde{U}(D) - \frac{\tilde{B}+1}{2}|\tilde{B}|)} \exp(\frac{1}{2}\pi i \tilde{U}(P))$$

where C_φ is a normalization constant; a tilde is a reminder that all objects relate to the basis $(\tilde{a}, \tilde{b})$.

Finally, we obtain in the new basis

$$\begin{aligned} \mathcal{E} = & \frac{\Theta(U(\infty^{(2)}) - U(D) - \frac{B}{4} - \frac{1}{2})\Theta(U(\infty^{(1)}) - U(D) - \frac{B+1}{2})}{\Theta(U(\infty^{(1)}) - U(D) - \frac{B}{4} - \frac{1}{2})\Theta(U(\infty^{(2)}) - U(D) - \frac{B+1}{2})} \\ & \times \exp[\frac{1}{2}\pi i(U(\infty^{(2)}) - U(\infty^{(1)}))] \end{aligned} \quad (30)$$

where the tilde is omitted.

Expression (30) is convenient to find the behaviour of $\mathcal{E}(\xi, \bar{\xi})$ on the 'branch ring' $\xi = E$; this asymptotics is obtained in the appendix (A15). Substituting (A15) into (2) we easily derive the asymptotics of all metric coefficients near the 'branch ring'. Namely, for the conformal factor $g_{\xi\bar{\xi}}$ we have:

$$g_{\xi\bar{\xi}} = \frac{2e^{2k}}{\text{Re } \mathcal{E}} = C(\tau\bar{\tau})^{-1/4} + o(|\tau|^{-1/2}) \quad (31)$$

where C is some integrability constant which is, in principle, fixed by the condition $k \rightarrow 0$ at $\xi \rightarrow \infty$.

We have thus seen that $g_{\xi\bar{\xi}}$ is singular at $\tau = 0$. However, in coordinates $\tau, \bar{\tau}$ this singularity disappears and

$$g_{\tau\bar{\tau}} = 4\tau\bar{\tau}g_{\xi\bar{\xi}} = 4C(\tau\bar{\tau})^{3/4} + o(|\tau|^{3/2}). \quad (32)$$

For other metric coefficients we have:

$$\begin{aligned} g_{\phi\phi} &= 2\sqrt{2}\tilde{C}|E| + (\tau + \bar{\tau})\left(\frac{\tilde{C}^2}{2\sqrt{2}|E|} - 2\sqrt{2}|E|^{3/2}\right) + o(|\tau|) \\ g_{\phi\tau} &= 2\sqrt{2}|E| + \frac{\tilde{C}(\tau + \bar{\tau})}{\sqrt{2}|E|} + \frac{(\tau - \bar{\tau})^2}{\sqrt{2}} + o(|\tau|^2) \end{aligned} \quad (33)$$

where $\tilde{C}$ is some integrability constant which is defined by the condition $A \rightarrow 0$ when $\xi \rightarrow \infty$. (Of course, working in the neighbourhood of the branch ring, we are unable to find constants C and $\tilde{C}$ explicitly).

The metric coefficients (32), (33) are thus non-singular at $\xi = E$ and everywhere on $\mathcal{M}$. However, (32) shows that the second derivatives of $g_{\tau\bar{\tau}}$ remain singular, and we cannot assert that the invariants of the Weyl tensor are non-singular at $\xi = E$. Indeed, one of two non-trivial invariants of the Weyl tensor turns out to be singular on the branch ring. Namely, according to [18], in the stationary axisymmetric situation the non-vanishing components of the Weyl tensor can be expressed in terms of the Ernst potential and of the function k as

$$\begin{aligned} C_{-2} &= \frac{e^{2k}}{4} \left[(2k)_{\xi} \mathcal{E}_{\xi} + \mathcal{E}_{\xi\xi} + \frac{\mathcal{E}_{\xi}^2}{\mathcal{E} + \bar{\mathcal{E}}} \right] \\ C_0 &= \frac{e^{2k}}{4} \left[\mathcal{E}_{\xi\bar{\xi}} - \frac{\mathcal{E}_{\xi} \mathcal{E}_{\bar{\xi}}}{\mathcal{E} + \bar{\mathcal{E}}} \right] \\ C_2 &= \frac{e^{2k}}{4} \left[(2k)_{\bar{\xi}} \mathcal{E}_{\bar{\xi}} + \mathcal{E}_{\bar{\xi}\bar{\xi}} + \frac{\mathcal{E}_{\bar{\xi}}^2}{\mathcal{E} + \bar{\mathcal{E}}} \right]. \end{aligned} \quad (34)$$

The invariants of the Weyl tensor may be expressed in terms of these components as follows:

$$I_1 = C_0 \quad I_2 = C_2 C_{-2} - 9C_0^2 \quad (35)$$

(components C_i do not coincide with the standard components Ψ_i). Substitution of expressions (A15) and (32) into (34), (35) shows that the components C_0 and C_2 are finite on the branch ring, but C_{-2} is singular. So the invariant I_1 is finite, but I_2 is divergent on this ring, and, therefore, the branch ring represents a geometrical singularity of the Einstein manifold corresponding to solution (24).

Figure 5. The manifold $\mathcal{M}$ in the coordinates $(\xi, \bar{\xi})$: the surface of stationary limit ($g_{tt} = 0$) is stretched on a singular 'branch ring' on both sides; the non-stationary sheet $\mathcal{M}_2$ lies behind $\mathcal{S}$. A section of $\mathcal{M}$ in the coordinates $(\gamma_1, \bar{\gamma}_1)$: $\mathcal{S}$ becomes a sphere with singular equator $\gamma_1 = E$; the sheet $\mathcal{M}_1$ lies outside and the sheet $\mathcal{M}_2$ lies inside $\mathcal{S}$.

Let us discuss in more detail the structure of the two-dimensional section $\mathcal{M}$ of this Einstein manifold. Regarding $\mathcal{M}$ as a covering of complex ξ -plane we see that it consists of two sheets: the stationary one ($\mathcal{M}_1$) and the non-stationary one ($\mathcal{M}_2$) glued at two points: $\xi = E$ and $\xi = \bar{E}$ (along the 'branch ring') and divided by the surface of stationary limit (on $\mathcal{M}_1$ it is depicted in figure 5(a)). Hence $\mathcal{M}$ has zero genus and can be mapped into a plane by some conformal transformation; also, we can define on $\mathcal{M}$ global complex coordinates which define this isomorphism:

$$\gamma_1 = \frac{1}{2} [\xi + \sqrt{\xi^2 + |E|^2}] \quad \text{and} \quad \bar{\gamma}_1. \quad (36)$$

Related real coordinates (ρ_1, z_1) defined by $\gamma_1 = (z_1 + i\rho_1)$ coincide on the infinity of $\mathcal{M}_1$ with the Weyl coordinates (ρ, z) ; a two-dimensional section of $\mathcal{M}$ looks in these coordinates as shown in figure 5(b). The surface of stationary limit $\mathcal{S}$ takes a spheroidal form $|\gamma_1| = |E|/2$; the singular ring coincides with its equator $\rho_1 = |E|/2$; the stationary sheet $\mathcal{M}_1$, where the square root in (36) is positive, lies outside. The non-stationary sheet $\mathcal{M}_2$ lies inside of $\mathcal{S}$; the infinity of $\mathcal{M}_2$ lies at the point $\gamma_1 = 0$.

Choosing the minus sign before the square root in (36), we obtain the alternative coordinates $(\gamma_2, \bar{\gamma}_2)$:

$$\gamma_2 = \frac{1}{2}[\xi - \sqrt{\xi^2 + |E|^2}] \equiv -\frac{|E|^2}{4\gamma_1} \quad \gamma_2 \equiv z_2 + i\rho_2$$

coinciding with the coordinates $(\xi, \bar{\xi})$ in the neighbourhood of the infinity of $\mathcal{M}_2$.

The symmetry axes $\rho = 0$ of $\mathcal{M}_{1,2}$ are defined in the coordinates (ρ_1, z_1) by the same condition $\rho_1 = 0$. The axis of $\mathcal{M}_1$ coincides with the segment $\rho_1 = 0, |z_1| \geq |E|/2$ and lies outside of $\mathcal{S}$ while the symmetry axis of $\mathcal{M}_2$ lies inside of $\mathcal{S}$. In coordinates (ρ_2, z_2) the situation is reverse.

The singular 'branch ring' $\xi = E$ in coordinates (ρ_1, z_1) has again the form of a ring (the same in γ_1 - and γ_2 -plane): $\gamma_1 = E/2$.

The interpretation of our solution in a neighbourhood of infinity on $\mathcal{M}_1$ is more or less clear: it describes the gravitational field of some massive rotating source which is generated by the singular 'hole' on the non-stationary sheet. This second sheet lies behind the surface of stationary limit; its interpretation is less clear. In [6] we named this sheet 'unphysical' because solution (24) is non-stationary and non-asymptotically-flat therein. In particular, as $\xi \rightarrow \infty$ on $\mathcal{M}_2$, we have $\mathcal{E} \rightarrow -1$, and the line element (1) has the form

$$ds^2 = d\rho^2 + dz^2 - \rho^2 d\phi^2 + (dt + A_0 d\phi)^2 \quad (37)$$

where $A_0 \in \mathcal{R}$ is some constant which provides the continuous behaviour of coefficient $g_{\phi t}$ on the surface $\mathcal{S}$. To write down interval (37) in Minkowski form we have to introduce on $\mathcal{M}_2$ new space-like coordinates x, y and new 'time' $\tilde{t}$ as follows:

$$x = t + A_0\phi \quad y = \rho \cosh \phi \quad \tilde{t} = \rho \sinh \phi.$$

Now (37) can be rewritten in the form

$$ds^2 = dx^2 + dy^2 + dz^2 - d\tilde{t}^2.$$

The metric on $\mathcal{M}_2$ becomes flat if z and (or) $\rho \equiv \sqrt{y^2 - \tilde{t}^2}$ tend to the infinity. The metric coefficients A, k and f depend on z and $y^2 - \tilde{t}^2$. In contrast to the 'time' and 'axial' symmetries on $\mathcal{M}_1$, the sheet $\mathcal{M}_2$ is non-stationary and possesses translational invariance in x -direction and boost symmetry in $(y, \tilde{t})$ -plane. The parameter ϕ which plays on $\mathcal{M}_1$ the role of 'angle' can be interpreted on $\mathcal{M}_2$ as a parameter of the Lorentz transform.

Hence sheet $\mathcal{M}_2$ is again 'asymptotically flat' in (z, ρ) -plane, but the ordinary asymptotical flatness is certainly absent due to the x -invariance of the whole configuration.

The physical interpretation of spacetime with boost symmetry has been thoroughly considered in [19]. $\mathcal{M}_2$ can probably be interpreted in a similar way.

4. 'Toron' solution

Here we consider the second class of the simplest elliptic solutions, namely, the solution (15) with static background given by (16). Simple calculation shows that the Ernst potential

is given by the expression

$$\mathcal{E} = \frac{\Theta(U(\infty^{(1)}) - U(\xi) - i\alpha)}{\Theta(U(\infty^{(1)}) - U(\xi) + i\alpha)} \quad (38)$$

where $\alpha \in \mathbf{R}$ is an arbitrary constant; the contour $l^0 \equiv [\xi, \infty^{(2)}]$, the branch cuts and basic cycles which should be chosen to evaluate (38), are shown in figure 6(a).

Figure 6. The branch cuts, basic cycles and the contour l^0 on the curve $\mathcal{L}$ by the tracing of ξ around E in the positive direction. Basic cycles remain unchanged; the contour l^0 transforms into $l^1 = l^0 + b$.

The properties of solution (38) were already briefly discussed in [7]; here we consider them in more detail. As was shown in section 2, the Ernst potential (38) may have branch points and other singularities of the related Einstein manifold $\mathcal{M}$ only at $\xi = E$ (in $\mathbf{R}^3$, this condition defines a ring). To verify that $\xi = E$ is a branch point on $\mathcal{M}$ we will continuously deform curve $\mathcal{L}$ so that ξ travels around E in the positive direction. We obtain curve $\mathcal{L}^1$ (figure 6(b)). The branch cuts and basic cycles (a, b) on $\mathcal{L}^1$ coincide with those on $\mathcal{L}$. The contour l^0 transforms into a new contour l^1 (all objects related to curve $\mathcal{L}^1$ we shall mark by the superscript 1). These contours are related by

$$l^1 = l^0 + b$$

therefore

$$[U(\infty^{(1)}) - U(\xi)]^1 = U(\infty^{(1)}) - U(\xi) + B.$$

Substituting this relation into (38) and using the periodicity property of the theta-function we obtain

$$\mathcal{E}^1 = e^{4\pi\alpha} \mathcal{E}.$$

So $\xi = E$ is indeed a branch point on the Einstein manifold; after j turns about E in the positive direction (if $j < 0$, the direction of a turn is in fact negative) we obtain the following new value of Ernst potential:

$$\mathcal{E}^j(\xi, \bar{\xi}) = e^{4\pi\alpha j} \mathcal{E}(\xi, \bar{\xi}) \quad j = \pm 1, \pm 2, \dots \quad (\mathcal{E}^0 \equiv \mathcal{E}). \quad (39)$$

Hence the points $\xi = E$ and $\bar{\xi} = F$ are the branch points of the infinite degree on the two-dimensional manifold $\mathcal{M}$ and the Hurvitz diagram of $\mathcal{M}$ can be represented in the form shown in figure 7.

Obviously, the fundamental group of this manifold is trivial; hence the metric coefficients (1) are single-valued on $\mathcal{M}$. The values of the Ernst potential differ on different sheets by a positive constant factor. Hence all sheets $\mathcal{M}_j$, $j \in \mathbf{Z}$ are similar to the initial sheet $\mathcal{M}_0$. This situation is different in comparison with that considered in section 2 where

Figure 7. The Hurvitz diagram of a *toron* infinite-sheet manifold $\mathcal{M}$. All sheets $\mathcal{M}_j$ are glued along the 'branch ring' $\xi = E$; the Ernst potential differs on different sheets by a positive constant factor.

the Ernst potential had different signs on different sheets. As a result, the physical properties of the sheets were radically different. However, the branch ring $\xi = E$ remains unchanged.

The behaviour of solution (38) on the branch ring is more complicated in comparison with the solution considered in the previous section. Once again we encounter a curvature singularity. In the limit $\alpha \rightarrow 0$, the 'branch ring' loses its monodromy property and the Einstein manifold turns to be an infinite set of flat spacetimes.

Following the scheme of [6], we can prove that on the 'zero' sheet of the Einstein manifold

$$\mathcal{E}(\xi, \bar{\xi}) \rightarrow 1 \quad \text{as} \quad |\xi| \rightarrow \infty$$

In order to find the mass and NUT parameter of our solution we can explicitly obtain the Ernst potential $\mathcal{E}_0(z, \rho)$ on the symmetry axis of the 'zero' sheet $\mathcal{M}_0$. Consider the degenerate curve $\mathcal{L}^0 \equiv \mathcal{L}$ being the limit as $\rho \rightarrow 0$. From formulae (26) describing the behaviour of the differential $dU(P)$ in a neighbourhood of the symmetry axis we have

$$U(\infty^{(1)}) - U(\xi) = U(\infty^{(1)}) - U(E) + \frac{B}{2} \sim -\frac{1}{2\pi i} \log \frac{z + \sqrt{z^2 + |E|^2}}{E} + \frac{B}{2}$$

$$\Theta(U(\infty^{(1)}) - U(\xi) \pm i\alpha) = 1 + \frac{E e^{\pm 2\pi\alpha}}{z + \sqrt{z^2 + |E|^2}}$$

Finally,

$$\mathcal{E}(z, \rho = 0) = \frac{z + \sqrt{z^2 + |E|^2} + E e^{-2\pi\alpha}}{z + \sqrt{z^2 + |E|^2} + E e^{2\pi\alpha}}$$

Hence, as $z \rightarrow i$, we obtain

$$\mathcal{E}(\rho = 0, z) = 1 - \frac{i|E| \sinh(2\pi\alpha)}{z} + O(z^{-2})$$

We see that the ordinary mass of our solution on $\mathcal{M}_0$ is zero. Instead, it has a NUT parameter, or 'magnetic mass'

$$L = -|E| \sinh 2\pi\alpha. \quad (40)$$

(Note that in [6] we understood the NUT parameter as an arctan of the ratio of the 'magnetic mass' to the ordinary mass.) The interpretation of the NUT parameter as a 'magnetic mass' was discussed in [20].

Consider an additional term $m = \frac{1}{2}$ instead of $m = 0$ in the argument of the theta-function in (38). Then the NUT-parameter changes its sign: $L \rightarrow -L$.

Hence solution (38) is not asymptotically flat due to the non-zero NUT-parameter; to provide the asymptotical flatness, one should consider the interaction of at least two objects of this kind. This is an obvious defect of solution (38) from the physical point of view.

We will now discuss the structure of the surface of stationary limit $\mathcal{S}$ setting by the condition $\operatorname{Re} \mathcal{E} = 0$ which is the most exotic feature of solution (38). Condition (20) setting the surface $\mathcal{S}$ takes for (38) the following form:

$$i\alpha = M + (N - \tfrac{1}{2})B$$

for some $M, N \in \mathbb{Z}$. Taking into account that function $B(\xi, \bar{\xi})$ is pure imaginary, we see that $M = 0$ and this condition becomes very simple:

$$i\alpha = (N - \tfrac{1}{2})B \quad (41)$$

for any $N \in \mathbb{Z}$. This means that $\mathcal{S}$ has a component for every N . Since $\operatorname{Im} B > 0$, if we choose $\alpha > 0$, condition (41) can be satisfied only for $N = 1, 2, \dots$. If $\alpha < 0$, the situation is reversed. Henceforth we take $\alpha > 0$.

If $N = 1$, then condition $B = 2i\alpha$ defines a family of surfaces $\mathcal{L}$ with fixed matrices of b -periods. Certainly, considering this condition as an equation in the (ρ, z) -plane, we cannot resolve it explicitly. However, to find the form of the one-dimensional submanifold of (ρ, z) plane satisfying the condition $\oint_{\mathcal{L}} dU = 2i\alpha$ we can proceed as follows. Consider a conformal map β

$$\beta(\lambda) = \frac{a\lambda + b}{c\lambda + d}$$

such that $\beta(E) = 0$, $\beta(-E) = \infty$ and $\beta(\xi) = 1$. Then

$$\beta(\bar{\xi}) \equiv p(\xi, \bar{\xi}) = \frac{(\bar{\xi} - E)(\xi + E)}{(\xi - E)(\bar{\xi} + E)} > 0. \quad (42)$$

In terms of the spectral parameter $\beta(\lambda)$, the curve $\mathcal{L}$ has fixed branch points at 0, 1, ∞ and only one moving real branch point at $p(\xi, \bar{\xi}) > 0$ defined by (42). Since the conformal map does not change the matrix of b -periods, the module of $\mathcal{L}$ depends only on the combination (42) of ξ and $\bar{\xi}$: $B = B(p(\xi, \bar{\xi}))$. Thus the condition $B(p) = 2i\alpha$ is equivalent to the condition

$$p(\xi, \bar{\xi}) = p_1 > 0 \quad (43)$$

where p_1 is a positive constant which is entirely defined by α : $p_1 \equiv B^{-1}(2i\alpha)$. Although we do not know the explicit form of this correspondence, the qualitative analysis can be carried out if we fix some value of p_1 and find the curve in the (ρ, z) -plane defined by (43).

Simple calculations show that (43) defines two circumferences, $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$; the symmetry transformation $\rho \rightarrow -\rho$ maps $\mathcal{S}_1^+$ to $\mathcal{S}_1^-$. The equation for $\mathcal{S}^\pm$ is

$$z^2 + (\rho \pm |E| \frac{|p_1 + 1|}{|p_1 - 1|})^2 = |E|^2 \frac{4p_1}{(p_1 - 1)^2}. \quad (44)$$

The centres lie at

$$\left(z = 0, \rho = \pm |E| \frac{p_1 + 1}{p_1 - 1} \right)$$

and the radius equals

$$R_1 = |E| \frac{2\sqrt{p_1}}{|1 - p_1|}.$$

It is easy to verify that the points $\xi = E$ and $\xi = -E$ lie inside S_1^- and S_1^+ , respectively (figure 8).

Figure 8. The infinite redshift surface $\mathcal{S}$ ($g_{tt} = 0$) of a *toron* that is an infinite family of tori enumerated by $N = 1, 2, \dots$, containing one another and a singular 'branch ring'.

Hence the component of $\mathcal{S}$ corresponding to $N = 1$ in (41) is a torus S_1 ; the singular ring $\xi = E$ lies inside this torus. From (39) we see that this torus lies on all sheets of $\mathcal{M}$.

Choosing $N = 2$ in (41) we obtain the second torus S_2 which solves the equation

$$p(\xi, \bar{\xi}) = p_2$$

and envelops the ring $\xi = E$. The only question is which of the two tori, S_1 and S_2 , is closer to the 'branch ring'. Our choice of the basic cycles (a, b) on $\mathcal{L}$ implies that $B \rightarrow 0$ as $\xi \rightarrow E$, and, therefore, S_2 lies inside of S_1 . Similarly, every next torus S_N lies inside $S_1, \dots, S_{N-1}$. Condition (39) provides the same form of $\mathcal{S}$ on all sheets of $\mathcal{M}$; thus in figure 8 we draw the complete infinite redshift surface on an arbitrary sheet of $\mathcal{M}$.

Notice that parameter p_1 defining the external torus by (44) may be chosen arbitrarily due to the arbitrary choice of α . Therefore the parameter α defines (see (43)) the size of the external torus; all the inner tori are entirely fixed by the largest one.

Taking into account the structure of the infinite redshift surface of (38), we named this solution 'toron'.

We can now discuss the behaviour of the metric coefficients (1) on symmetry axis. This behaviour is typical for the localized solutions having non-zero NUT parameter and zero mass:

$$\mathcal{E} = 1 + \frac{iL}{|\xi|} + O(|\xi|^{-2}) \quad |\xi| \rightarrow \infty. \quad (45)$$

Namely, from (1) it is obvious that at the any point of the symmetry axis we have $k_{\xi} = A_{\xi} = 0$; hence, the coefficients k and A are constant on the every connected component of the symmetry axis of $\mathcal{M}$. Each such component lies on two sheets: $\mathcal{M}_n$ and $\mathcal{M}_{n+1}$. Suppose that on the $-\infty$ of the symmetry axis of $\mathcal{M}_0$ the behaviour of the metric coefficients k and A is regular, i.e.

$$k(\rho = 0, z) = 0 \quad A(\rho = 0, z) = 0.$$

Integrating now the (1) with the Ernst potential defined by (45) along a large half-circle, we see that at the opposite 'end' of the symmetry axis of $\mathcal{M}_0$ the coefficients A and k take the values

$$k(\rho = 0, z) = 0 \quad A(\rho = 0, z) = -2L.$$

Hence the conformal factor $g_{\rho\rho}$ takes its 'physical' value at the opposite end of the symmetry axis of $\mathcal{M}_0$, but the coefficient $g_{\phi t}$ takes an 'unphysical' non-zero value $-2L$. Taking into account the topological structure of the manifold $\mathcal{M}$, we see that the conformal factor is regular on all symmetry axes of $\mathcal{M}$ (i.e. there are no conical points on any of the sheets). The behaviour of the coefficient $g_{\phi t}$ turns out to be irregular on all symmetry axes except the initial one. However, locally this defect may be compensated for by the linear transformation of the coordinate t . Namely, if $A(z, \rho = 0) = A_0 \neq 0$, we can define $\tilde{t} = t + A_0\phi$; then the new coefficient $g_{\tilde{t}\phi}$ will tend to zero as $\rho \rightarrow 0$.

This situation is typical for an arbitrary solution having non-zero NUT parameter: it is impossible to define the asymptotically a Minkowski coordinate system covering the whole asymptotical region.

Some analogies between the magnetic monopole in electrodynamics and the NUT-solution were established in [20]. The NUT-solution was interpreted as a 'gravitational magnetic monopole' and the NUT-parameter—as a 'magnetic mass'. Then the irregular behaviour of the metric coefficient $A(\rho, z)$ on one 'half' of the symmetry axis can be considered as an analogue of the Dirac string; in both cases it can be removed by an appropriate choice of the new coordinate system.

Finally, we can summarize the properties of *toron*. Topological structure of the Einstein manifold is shown in figure 7. In contrast to the solution considered in the previous section, it consists of an infinite number of stationary sheets. All sheets are glued along the 'branch ring' $z = \text{Re } E = 0$, $\rho = \text{Im } E$. The Ernst equation fails on this ring only. The Ernst potential on different sheets differs by the positive constant factor, so all the sheets are similar. As the radius of the 'branch ring' tends to zero, *toron* transforms into a flat spacetime.

The mass of a *toron* is zero; instead, it has a non-zero NUT parameter, or 'magnetic mass' (40).

The most unusual feature of the (38) is the structure of the infinite redshift surface (the surface of stationary limit), being an infinite family of tori enveloping one another and the 'branch ring'. Taking into account this property we call this solution a *toron*.

5. Conclusions and perspectives

We can summarize the results of this paper as follows. First, we present a version, simplified in comparison with [1, 2], of the finite-gap construction of solutions of the Ernst equation. Of course, the final expression for the solutions of Ernst equation in terms of multidimensional theta-functions remains the same. Besides that, we obtained a formula for metric coefficient $g_{\phi t}$ corresponding to an arbitrary solution of the Ernst equation in terms of the derivative

of related function Ψ with respect to the spectral parameter. This allows us to obtain $g_{\phi t}$ avoiding the contour integration; in particular, in the case of finite-gap solutions it can be expressed in terms of derivatives of theta-functions.

Second, we analyse two particular examples of the simplest elliptic (genus 1) solutions which are expressed in terms of the ordinary Jacobi theta-functions and elliptic integrals. The properties of these solutions described in the previous two sections differ essentially from the properties of the multisoliton (multi-Kerr-NUT) solutions constituting the subclass of degenerate finite-gap solutions.

A solution presented in section 3 describes the interaction of a stationary axisymmetric asymptotically-flat sheet (where mass and angular momentum are real and non-zero) with a non-stationary sheet (where the symmetry becomes translational and boost). The sheets are separated by a spheroidal surface of stationary limit and are glued along a singular branch ring lying on the equator of this surface (this is the only singularity of the Weyl tensor invariants on the Einstein manifold).

A solution presented in section 4 (*a toron*) has a more exotic structure: its Einstein manifold consists of an infinite number of sheets glued along a singular 'branch ring'. All sheets are similar one to another; the Ernst potential differs on different sheets by a positive constant factor. The mass of *toron* is zero; instead it has a non-zero NUT-parameter. The surface of stationary limit is an infinite family of tori enveloping one another and the singular ring.

These two simplest examples begin the investigation of the properties of general finite-gap solutions. Even in the elliptic case a broad freedom in the choice of static background remains; besides that, both cases correspond to only one of the two possible types of the basic real algebraic curve.

The straightforward reformulation of the construction presented in section 2 allows one to obtain new solutions describing the interaction of the vacuum impulse or smooth gravitational waves; the application to this subject should be developed in the future.

Note a more difficult (but also more promising) problem of investigating finite-gap solutions of the Einstein-Maxwell system obtained in [1, 2]. The basic algebraic curve has in this case a more sophisticated structure than the simple hyperelliptic 'vacuum' curve. It is natural to expect that it will be possible to find new interesting examples even among the degenerate solutions which can be expressed in terms of elementary functions.

Finally, there are some important unsolved problems of the algebraic-geometric construction itself. First, it would be desirable to express not only the metric coefficient $g_{\phi t}$ but also the 'conformal factor' $g_{\rho\rho}$ (or, equivalently, the function k) in terms of theta-functions without contour integration. The existence of such an expression in the multisoliton case [12] makes us believe in the feasibility of its generalization to finite-gap solutions. Besides that, it should be possible even to express coefficient $g_{\phi t}$ not in terms of the derivatives of theta-function, but via theta-functions themselves. For this purpose it is necessary to establish explicitly the relation between the algebraic-geometric solutions of the Ernst equation and the solutions of the Einstein equation written in terms of metric coefficients which were obtained in [2] using the zero curvature representation of [9, 12].

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style of the present paper.

Appendix A. Asymptotics of elliptic solution on flat background near the 'branch ring'

Here we find the asymptotics of the solution (30) near the 'branch ring' $\xi = E$. For this purpose let us prove several lemmas:

Lemma 1. The Ernst potential (30) has the following local behaviour as $\xi \rightarrow E$:

$$\mathcal{E}(\mu, \bar{\mu}) = 2(\mu - \mu^2 \bar{\mu} + \bar{\mu}^3) + O(|\mu|^5) \quad (\text{A1})$$

where

$$\mu = \exp \left[\frac{1}{2} \pi i (U(\infty^{(2)}) - U(D) - \frac{1}{2}) \right] \rightarrow 0 \quad \xi \rightarrow E \quad (\text{A2})$$

is a local parameter in a neighbourhood of $\xi = E$.

Proof. Introducing new variables

$$\begin{aligned} p &= \exp[2\pi i (U(\infty^{(1)}) - U(D))] \\ q &= \exp[2\pi i (U(\infty^{(2)}) - U(D) - \frac{1}{2}B)] \\ r &= \exp[\pi i B/2]. \end{aligned}$$

Taking into account the following behaviour of the b -period

$$B \sim \frac{2}{\pi i} \log |\xi - E| + O(1) \quad \tau \equiv \sqrt{i(\bar{\xi} + E)} \rightarrow 0$$

and the definition of theta-function (10), we have

$$\begin{aligned} \Theta(U(\infty^{(2)}) - U(D) - \frac{B}{4} + \frac{1}{2}) &= \sum_m \exp\{\pi i B m(m + \frac{1}{2}) + 2\pi i m(U(\infty^{(2)}) - U(D) \\ &\quad - \frac{1}{2}B + \frac{1}{2})\} = -\frac{1}{4}r + 1 + O(r^2) \end{aligned}$$

Likewise,

$$\begin{aligned} \Theta(U(\infty^{(1)}) - U(D) - \frac{1}{4}B + \frac{1}{2}) &= -pr + 1 + O(r^2) \\ \Theta(U(\infty^{(2)}) - U(D) - \frac{1}{2}B + \frac{1}{2}) &= 1 + O(r^2) \\ \Theta(U(\infty^{(1)}) - U(D) - \frac{1}{2}B + \frac{1}{2}) &= 1 - p + O(r^4) \end{aligned}$$

and

$$\exp\left(\frac{\pi i}{2}(U(\infty^{(2)}) - U(\infty^{(1)}))\right) = \sqrt{r} \sqrt[4]{q/p}$$

As a result, we obtain

$$\mathcal{E} = (1 - p) \sqrt[4]{\frac{q}{p}} \left[\sqrt{r} + \left(p - \frac{1}{q}\right) \sqrt{r^3} \right] + O(|r|^{5/2})$$

that gives (A1) if we choose the local parameter μ according to (A2). $\square$

This proves formula (A1). However, to find the asymptotic representation of $\mathcal{E}(\tau, \bar{\tau})$ in terms of $\tau = \sqrt{i(\bar{\xi} + E)}$ and $\bar{\tau}$ we have to express μ in terms of τ and $\bar{\tau}$. The next lemma establishes the relations between τ , $\bar{\tau}$ and $|\mu|$.

Lemma 2. The following is true:

$$\operatorname{Re} \log \mu \equiv \frac{1}{4} \pi i B \equiv \frac{1}{2} \log \frac{\tau \bar{\tau}}{8|E|} - \frac{\tau^2 + \bar{\tau}^2}{8|E|} + o(|\tau|^2). \quad (\text{A3})$$

Proof. Let us transform $\mathcal{L}$ to the standard form by an appropriate conformal map; it will allow us to use the asymptotic representation of full elliptic integrals [21]. Assume

$$\sigma(E) = 1 \quad \sigma(-E) = -1 \quad \sigma(\xi) = -\sigma(-\xi) \equiv 1 - \epsilon$$

where σ is a fraction-linear transformation of the complex plane. Then

$$\sigma(\lambda) = \frac{\lambda + \beta E}{\beta \lambda + E}$$

where

$$\beta(\tau, \bar{\tau}) = \frac{-(\tau^2 \bar{\tau}^2 + |E|(\tau^2 + \bar{\tau}^2)) \pm \tau \bar{\tau} \sqrt{(\tau^2 + 2|E|)(\bar{\tau}^2 + 2|E|)}}{|E|(\tau^2 - \bar{\tau}^2)}. \quad (\text{A4})$$

Hence

$$\epsilon = (\beta - 1) \frac{\xi - E}{c\xi + E} = \frac{1}{|E|} \tau \bar{\tau} - \frac{1}{4|E|^2} \tau \bar{\tau} (\tau + \bar{\tau})^2 + O(|\tau|^6).$$

In terms of ϵ the asymptotics of b -period B can be written in the standard form [21]:

$$\frac{\pi i B}{4} = \frac{1}{2} \log \frac{\epsilon}{8} + \log \epsilon (b_1 \epsilon + b_2 \epsilon^2 + \dots) + a_1 \epsilon + a_2 \epsilon^2 + \dots$$

Therefore,

$$\frac{\pi i B}{4} = \frac{1}{2} \log \frac{\tau \bar{\tau}}{8|E|} + \frac{(\tau + \bar{\tau})^2}{8|E|} + \frac{\tau \bar{\tau}}{2|E|} a_1 + o(|\tau|^2) \quad (\text{A5})$$

where a_1 is a constant.

To find this constant, let us consider the differential of the second kind $dU_{\xi}(P)$ on $\mathcal{L}$ with a pole at $\lambda = \xi$. From the normalization condition $\oint_a dU(P) = 1$ we conclude that

$$\oint_a dU_{\xi}(P) = 0. \quad (\text{A6})$$

Writing the standard expression for $dU(P)$,

$$dU = \frac{C d\lambda}{\omega} \equiv \frac{C d\lambda}{\sqrt{(\lambda - \xi)(\lambda - \bar{\xi})(\lambda - E)(\lambda - \bar{E})}}$$

we obtain

$$(dU)_{\xi} = \frac{d\lambda}{\omega} \left[C_{\xi} - \frac{C}{2(\lambda - \bar{\xi})} \right]. \quad (\text{A7})$$

Taking into account (A6), we see that the residue of (A7) at $\lambda = E$ at $\lambda = \bar{E}$ should vanish as $\xi \rightarrow E$. Hence

$$C_{\xi} - \frac{C}{2(E - \bar{E})} \rightarrow 0. \quad (\text{A8})$$

Likewise,

$$C_{\xi} - \frac{C}{2(\bar{E} - E)} \rightarrow 0, \quad \xi \rightarrow E. \quad (\text{A9})$$

Consider the differential

$$(dU)_{\xi \bar{\xi}} = \frac{d\lambda}{\omega} \left[C_{\xi \bar{\xi}} - \frac{C_{\xi}}{2(\lambda - \bar{\xi})} - \frac{C_{\bar{\xi}}}{2(\lambda - \xi)} + \frac{C}{4(\lambda - \xi)(\lambda - \bar{\xi})} \right].$$

Conditions (A8), (A9) imply that in the limit $\xi \rightarrow E$ this differential has no poles at $\lambda = E$ and $\lambda = \bar{E}$. Therefore,

$$\oint_b (dU)_{\xi\bar{\xi}} \rightarrow \text{constant}, \quad \xi \rightarrow E$$

and

$$B_{\tau\bar{\tau}} \equiv \left(\oint_b dU \right)_{\tau\bar{\tau}} = 4\tau\bar{\tau} \oint_b (dU)_{\xi\bar{\xi}} \rightarrow 0, \quad \tau \rightarrow 0.$$

Hence, the term proportional to $\tau\bar{\tau}$ in (A5) vanishes and (A5) gives (A3). $\square$

It remains for us to find the asymptotical representation of $\text{Im} \log \mu \equiv \text{Im}(\frac{1}{2}\pi i(U(\infty^{(2)}) - U(D) - \frac{1}{2}))$ in terms of τ and $\bar{\tau}$.

Lemma 3. Let $\phi \equiv \arg \tau$. Then

$$\text{Im} \left\{ \frac{1}{2}\pi i(U(\infty^{(2)}) - U(D) - \frac{1}{2}) \right\} = i\phi + O(|\tau|^4). \quad (\text{A10})$$

Proof. It is convenient to rewrite the LHS of (A10) in the form

$$\begin{aligned} \text{Im} \left\{ \frac{1}{2}\pi i(U(\infty^{(2)}) - U(D) - \frac{1}{2}) \right\} &\equiv \frac{1}{2}\pi i(U(\infty^{(2)}) - U(D) - \frac{1}{2}) - \frac{1}{4}\pi iB \\ &= \frac{1}{2}\pi i(U(\infty^{(2)}) - U(P_0^*) + U(P_0) - U(D) - \frac{1}{2}) \end{aligned} \quad (\text{A11})$$

where the real point $P_0(\tau, \bar{\tau})$ satisfies $\int_{P_0}^{\infty} dU = B/2$. This point can be found from the condition $\sigma(P_0) = 0$ where σ is a conformal transformation (see Lemma 2) that maps $\mathcal{L}$ to its the standard form. Therefore,

$$P_0 = -\beta E = \frac{(\tau + \bar{\tau})(2|E| - \tau\bar{\tau})}{2i(\tau - \bar{\tau})} + O(|\tau|^4)$$

where β is set by (A4). In terms of $|\tau|$ and ϕ we have:

$$P_0 = \frac{1}{2} \tan \phi (2|E| - |\tau|^2 + O(|\tau|^4)). \quad (\text{A12})$$

To use (A11) we have to find the asymptotical behaviour of $dU(P)$ as $|\tau| \rightarrow 0$ up to second-order terms. After simple calculations, we obtain on the first sheet of $\mathcal{L}$:

$$dU(\lambda) = \frac{d\lambda}{2\pi i} \left[\frac{2E}{(\lambda - E)(\lambda + E)} + \frac{i|\tau|^2 e^{2i\phi}}{2(\lambda + E)^2} + \frac{i|\tau|^2 e^{-2i\phi}}{2(\lambda - E)^2} \right] + O(|\tau|^4). \quad (\text{A13})$$

On the second sheet, $dU(\lambda)$ has the opposite sign.

Substituting (A12) and (A13) into (A11), we obtain

$$\frac{\pi i}{2} (U(\infty^{(2)}) - U(P_0^*) + U(P_0) - U(D) - \frac{1}{2}) = i\phi + O(|\tau|^4)$$

which implies (A10). $\square$

Relations (A3) and (A10) can be summarized as follows:

$$\mu = \frac{\tau}{2\sqrt{2|E|}} \left[1 - \frac{\tau^2 + \bar{\tau}^2}{8|E|} \right] + o(|\tau|^2). \quad (\text{A14})$$

Substituting (A14) into (A1) we obtain the following

Theorem A. 1. In a neighbourhood of the point $\xi = E$ the Ernst potential (24) has the following asymptotical behaviour:

$$\mathcal{E}(\tau, \bar{\tau}) = \frac{1}{\sqrt{2|E|}} \left[\tau + \frac{1}{8|E|} (-\tau^3 + \tau\bar{\tau}^2 - \tau^2\bar{\tau} + \bar{\tau}^3) \right] + o(\tau^3) \quad (\text{A15})$$

where $\tau = \sqrt{i(\xi + E)}$.

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