

Exact solutions of topologically massive gravity with a cosmological constant

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Abstract. Exact solutions of the Deser–Jackiw–Templeton field equations including a cosmological constant are presented. They generalize the finite-action homogeneous, anisotropic vacuum solutions of topologically massive gravity. We find that only Vuorio-type solutions, where any two of the constant scale factors coincide, admit a cosmological constant. One of these solutions can be dressed up to yield a two parameter black hole solution of topologically massive gravity.

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Deser, Jackiw and Templeton [1], hereafter referred to as DJT, have proposed a very interesting theory of topologically massive gravity in $2 + 1$ dimensions. The DJT field equations possess the elegance and consistency of the Einstein field equations and yet they are qualitatively different. This is a dynamical theory of gravity, unique to three dimensions and the geometry of its exact solutions is non-trivial. We shall be interested in the DJT field equations including a cosmological constant. They are given by [2]

$$D^i_k = G^i_k - \Lambda \delta^i_k + (1/\mu)C^i_k = 0 \quad (1)$$

where we have the Einstein tensor

$$G^i_k = R^i_k - \frac{1}{2}\delta^i_k R \quad (2)$$

and

$$C^i_k = \frac{1}{\sqrt{|g|}}\epsilon^{imn}(R_{km} - \frac{1}{4}g_{km}R)_{;n} \quad (3)$$

is Cotton's conformal tensor for three-dimensional manifolds. The constants μ and Λ are the DJT coupling and cosmological constants, respectively.

We shall first consider the finite-action exact solutions of the vacuum DJT field equations [3]. These are generalizations of the Vuorio solution [4]. Subsequently they were republished in [5]. It will be of interest to see if any of these solutions can be modified to admit a cosmological constant. For this purpose we shall consider an orthonormal frame with the metric

$$ds^2 = \eta_{ik}\omega^i \otimes \omega^k \quad (4)$$

where the co-frame

$$\omega^i = \lambda_i \sigma^i \quad (5)$$

is proportional to the left-invariant 1-forms σ^i of either Bianchi type VIII or IX depending on the choice of Lorentz or Euclidean signature for η_{ik} respectively. In equation (5) the λ_i are constants and there is no summation convention over the label i . We refer to [3] for a representation of σ^i in terms of Euler angles. Cartan's equations of structure are

$$d\sigma^i = \frac{1}{2} c_j^i{}_k \sigma^j \wedge \sigma^k \quad (6)$$

where c_{jk}^i are the structure constants of Bianchi type VIII or IX.

With this choice of frame and assuming Euclidean signature for η_{ik} , we find that the field equations (1) reduce to

$$D_{00} = \frac{1}{4\lambda_0^3\lambda_1^3\lambda_2^3} \{2\lambda_0^4(2\lambda_0^2 - \lambda_1^2 - \lambda_2^2) - 2(\lambda_1^2 + \lambda_2^2)(\lambda_1^2 - \lambda_2^2)^2 + \mu\lambda_0\lambda_1\lambda_2[2\lambda_0^2(2\lambda_0^2 - \lambda_1^2 - \lambda_2^2) - (\lambda_1^2 - \lambda_2^2)^2 - \lambda_0^4 - 4\lambda_0^2\lambda_1^2\lambda_2^2\Lambda]\} = 0 \quad (7)$$

$$D_{11} = \frac{1}{4\lambda_0^3\lambda_1^3\lambda_2^3} \{2\lambda_1^4(2\lambda_1^2 - \lambda_0^2 - \lambda_2^2) - 2(\lambda_0^2 + \lambda_2^2)(\lambda_0^2 - \lambda_2^2)^2 + \mu\lambda_0\lambda_1\lambda_2[2\lambda_1^2(2\lambda_1^2 - \lambda_0^2 - \lambda_2^2) - (\lambda_0^2 - \lambda_2^2)^2 - \lambda_1^4 - 4\lambda_0^2\lambda_1^2\lambda_2^2\Lambda]\} = 0 \quad (8)$$

$$D_{22} = \frac{1}{4\lambda_0^3\lambda_1^3\lambda_2^3} \{2\lambda_2^4(2\lambda_2^2 - \lambda_1^2 - \lambda_0^2) - 2(\lambda_0^2 + \lambda_1^2)(\lambda_0^2 - \lambda_1^2)^2 + \mu\lambda_0\lambda_1\lambda_2[2\lambda_2^2(2\lambda_2^2 - \lambda_1^2 - \lambda_0^2) - (\lambda_0^2 - \lambda_1^2)^2 - \lambda_2^4 - 4\lambda_0^2\lambda_1^2\lambda_2^2\Lambda]\} = 0 \quad (9)$$

and the Ricci scalar is given by

$$R = \frac{1}{2\lambda_0^2\lambda_1^2\lambda_2^2} (\lambda_0 + \lambda_1 + \lambda_2)(\lambda_0 + \lambda_1 - \lambda_2)(\lambda_0 - \lambda_1 + \lambda_2)(-\lambda_0 + \lambda_1 + \lambda_2) = -6\Lambda. \quad (10)$$

For Lorentz signature these equations are modified by the formal replacement of

$$\lambda_0 \rightarrow -\lambda_0 \quad \Lambda \rightarrow -\Lambda \quad (11)$$

while the others remain unchanged. This simple rule is useful for discussing both cases of signature. The details of the computation of curvature for the frame (5) can be found in Ozsváth [6] who also showed that the criterion for the vanishing of the scalar of curvature is necessary and sufficient for the imbeddability of the resulting solution in an anti-de Sitter universe in four dimensions.

There is a simple criterion for equations (7)–(9) to admit a solution. We can eliminate μ and Λ between any two of these equations and obtain a constraint involving only λ_i from the remaining equation. This constraint is a polynomial in λ_i which can be readily factorized. We find that its roots are given by either

$$\lambda_i \pm \lambda_j \pm \lambda_k = 0 \quad (12)$$

or

$$\lambda_i \pm \lambda_k = 0 \quad (13)$$

for i, j, k all different. Among these, solutions (12) must be rejected as they give rise to vacuum solutions [3] which is clear from equation (10). Only the spheroidal-type solutions (13) allow the introduction of a cosmological constant. These solutions are given by

$$\lambda_0 = -\frac{6e\mu}{\mu^2 - 27e\Lambda} \quad \lambda_1 = \lambda_2 = \frac{3}{\sqrt{\mu^2 - 27e\Lambda}} \quad (14)$$

where e is the sign of η_{00} . Clearly any permutation of the labels 0, 1, 2 above will also yield a solution. In the limit $\Lambda \rightarrow 0$ equations (14) reduce to Vuorio's result for vacuum.

The interesting feature of equations (14) lies in the apparent singularity of the metric when the DJT coupling constant and the cosmological constant are related by

$$\mu^2 - 27e\Lambda = 0. \quad (15)$$

On the other hand, solution (14) consists of a homogeneous space and all the curvature invariants are regular at this critical value. Hence this singularity can be removed, as we shall see below.

For stationary solutions the introduction of a cosmological constant into the DJT field equations typically leads to relation (15) between the DJT coupling constant and the cosmological constant. We note, however, that for propagating wave solutions of DJT no such behaviour is expected. In particular, we find that with the inclusion of the cosmological constant the DJT wave solution [7] is given by

$$ds^2 = \Omega^{-2}(2 du dv - dx^2 + 2H du^2) \quad (16)$$

where

$$\Omega = h - \sqrt{-\Lambda}x \quad H = -\frac{1}{\sqrt{-\Lambda}}h_{uu} + c\Omega^{1-\mu/\sqrt{-\Lambda}} \quad (17)$$

and h is an arbitrary function of u while c is an arbitrary constant. This form of the solution is not conducive for considering the limit $\Lambda \rightarrow 0$, but it is interesting because it does not display singular behaviour at the critical value (15).

We shall conclude with an examination of the solution (14) at the critical value (15). The metric for one of these solutions can be written in the form [3]

$$ds^2 = -\left(\frac{2\mu}{3\epsilon^2}\right)^2 (d\psi + \sinh \theta d\phi)^2 + \frac{1}{\epsilon^2} (d\theta^2 + \cosh^2 \theta d\phi^2) \quad (18)$$

where we have introduced the critical parameter

$$\epsilon = \frac{1}{3}\sqrt{\mu^2 + 27\Lambda} \quad (19)$$

and we shall be interested in the limit $\epsilon \rightarrow 0$. For this purpose we first rescale and relabel the Euler angles according to

$$\theta = \epsilon x \quad \phi = \epsilon y \quad \psi = \epsilon^2 T \quad (20)$$

which brings the metric (18) to a form suitable for passing to the limit. At the critical value (15) we find that the metric is given by

$$ds^2 = -(\frac{2}{3}\mu)^2(dT + y dx)^2 + dx^2 + dy^2 \quad (21)$$

and it is possible to put this solution into a more interesting form by transforming the coordinates again. We find that the change

$$T = \frac{3}{2\mu}t - \frac{1}{4}r^2 \sin 2\theta \quad x = r \cos \theta \quad y = r \sin \theta \quad (22)$$

results in

$$ds^2 = -(dt - \frac{1}{3}\mu r^2 d\theta)^2 + dr^2 + r^2 d\theta^2 \quad (23)$$

and the restriction (15) can now be removed to yield

$$ds^2 = -(dt - \frac{1}{3}\mu r^2 d\theta)^2 + \frac{dr^2}{\frac{1}{36}(\mu^2 + 27\Lambda)r^2 + 1} + r^2[\frac{1}{36}(\mu^2 + 27\Lambda)r^2 + 1] d\theta^2 \quad (24)$$

which is a limiting form of the DJT black hole solution.

The metric (24) can be dressed up with two parameters to result in a black hole solution

$$ds^2 = -\frac{1}{6}(2J - M) dt^2 + 2\left(\frac{1}{3}\mu r^2 - \frac{1}{2\mu}J\right) dt d\theta \\ + \frac{dr^2}{\frac{1}{36}(\mu^2 + 27\Lambda)r^2 - \frac{1}{6}M + (J^2/4\mu^2 r^2)} + r^2\left(1 - \frac{1}{2}\frac{\mu^2 - 9\Lambda}{2J - M}r^2\right) d\theta^2 \quad (25)$$

which can be interpreted as the field of an isolated spinning mass in topologically massive gravity. Here M, J are constants which need to be restricted for physical reasons such as the one given by equation (26) below.

In this coordinate system the metric is regular at the origin of polar coordinates and has the proper asymptotic behaviour expected from a solution with a cosmological constant. The vacuum solution obtained by setting $\Lambda = 0$ in equation (25) is not asymptotically flat. We note that an extra parameter can always be introduced into this solution through the transformation $t \rightarrow t + \alpha\theta$ for α a constant, but this will either spoil the interpretation of θ as an angle altogether, or regularity at the origin.

For this solution the triad, i.e. the physical components of the Riemann tensor and therefore all the curvature scalars, are constants which depend only on μ and Λ . In fact the triad components of the Riemann tensor for the metrics (24) and (25) are indistinguishable. This situation is quite similar to the relationship between the Bafados-Teitelboim-Zanelli (BTZ) black hole solution [8] of the three-dimensional Einstein equations with a cosmological constant and the de Sitter solution. The full analysis of the global structure of the solution (25) is too involved as there are many possible choices for various constants. We shall only remark that depending on the sign of M and the value of Λ there are cases where this solution can have an inner and outer horizon, a single horizon, or none at all. For vacuum we require both M, J to be positive and

$$1 > J/M > \frac{1}{2} \quad (26)$$

is the physically interesting range of the parameters. These restrictions come from the requirement of Minkowskian signature for the metric and

$$r_{\pm}^2 = \frac{3}{\mu^2} [M \pm (M^2 - J^2)^{1/2}] \quad (27)$$

which determines the location of the horizons. The infinite redshift surface for this solution takes place at

$$r_0^2 = \frac{2}{\mu^2} (2J - M) \quad (28)$$

which coincides with the radius at which the circumference of circles shrinks back to zero.

Historical note

Originally I derived the black hole solution (25) subject to the restriction given by equation (15). M Gürses has pointed out to me that this restriction could be removed simply by including a term proportional to r^2 in the denominator of g_{rr} as in the passage from equation (23) to equation (24) above. I thank M Gürses for this remark. M Gürses has not otherwise contributed towards the derivation of the metric (25).

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