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Robust adaptive stabilization of a class of systems under structured nonlinear perturbations with application to interconnected systems

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This paper presents a stabilization scheme for a class of multi-input/multi-output systems with nonlinear additive perturbations using high-gain adaptive controllers. The nominal system is assumed to satisfy some mild conditions required by standard adaptive control schemes, and the perturbations certain structural conditions. The controller is a dynamic output feedback containing a gain parameter, which is adjusted with a simple adaptation rule. The result is also applied to decentralized stabilization of a class of interconnected systems, where the interconnections are treated as perturbations to nominally decoupled subsystems.

1. Introduction

High-gain feedback control is a standard tool for robust stabilization in the presence of modelling uncertainties (see, for example, Zames and Bensoussan 1983 and Saberi and Sannuti 1990). In the case of a single-input/single-output system, design of such a controller requires that the system be minimum-phase and its relative degree, the sign of its high frequency gain and the bounds on the system parameters or perturbations be known. Similar information is needed for multi-input/multi-output (MIMO) systems. It has been shown by Byrnes and Willems (1984) and Byrnes and Isidori (1984) that for systems with relative degree one, robust stability can be achieved without the need to know the bounds of the system by tuning the gain parameter adaptively. A similar result has been obtained for systems with higher relative degree by Khalil and Saberi (1987), who used an adaptation mechanism to increment the gain parameter stepwise at discrete instants.

High-gain feedback finds a natural application in decentralized control of interconnected systems, where the essential uncertainty lies in the interconnections among the subsystems. Since the early work of Davison (1974), various high-gain decentralized stabilization schemes have been developed (see, for example, Ikeda *et al.* 1976, Sezer and Hüseyin 1978, 1980, Ikeda and Siljak 1980, Sezer and Siljak 1981, Hüseyin *et al.* 1982, Saberi and Khalil 1985, and Yu and Sezer 1992). As in high-gain control of a single system, adaptive techniques have been used (e.g. Gavel and Siljak 1989, Shi and Singh 1992) to eliminate the need for *a priori* information about the bounds of interconnections.

In this paper, we bring together several concepts and techniques of robust stabilization, adaptive control and decentralized control. We first present a high-gain adaptive robust stabilization scheme for a class of minimum-phase MIMO systems having uniform rank. The controller structure is determined

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completely offline by making use of the structural properties of the system. The feedback gain parameter is adjusted continuously using a simple adaptation rule by observing the closed-loop system output and the controller state. We then apply this result to decentralized stabilization of a class of interconnected systems. We show that a centralized adaptation rule for a gain parameter common to all local controllers provides closed-loop stability and boundedness of the gains for all interconnections having a certain structure.

2. A high-gain controller for a class of perturbed systems

Consider a system $\mathcal{S}$ described as

$$\left.\begin{aligned} \mathcal{S}\colon \dot{x}(t) &= Ax(t) + Bu(t) + I(t, x(t)) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{2.1}$$

where $x(t) \in \mathcal{R}^n$ is the state, $u(t)$, $y(t) \in \mathcal{R}^m$ are the input and output of $\mathcal{S}$, respectively, and A, B and C are constant matrices of appropriate dimensions. $I(t, x(t))$ in (2.1) stands for additive nonlinear perturbations to a linear, nominal system represented by the triple (C, A, B).

We make the following assumptions concerning the nominal system and the perturbations.

(a) (C, A, B) is controllable and observable.

(b) (C, A, B) is minimum-phase, that is, the set of transmission zeros defined as

$$\mathfrak{X} = \left\{ s \middle| \operatorname{rank} \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} < n + m,\ s \in \mathscr{C} \right\} \tag{2.2}$$

is included in the open left-half complex plane.

(c) (C, A, B) has uniform rank q (Sannuti 1983), that is

$$CA^iB = 0,\ i = 0, 1, \ldots, q - 2;\ M = CA^{q-1}B \text{ is non-singular} \tag{2.3}$$

(d) q and M in (2.3) are known.

(e) The perturbations are of the form $I(t, x) = Bg(t, x) + h(t, y)$, where g and h satisfy, for some constants c_g, $c_h > 0$

$$\left.\begin{aligned} \|g(t, x)\| &\le c_g \|x\| \\ \|h(t, y)\| &\le c_h \|y\| \end{aligned}\right\} \tag{2.4}$$

for all $t \in \mathcal{R}$, $x \in \mathcal{R}^n$.

As a first step toward the construction of a high-gain controller for $\mathcal{S}$, we recall the following result by Sannuti (1983).

Lemma 2.1: *Under the assumptions* (a)–(c), *there exists a non-singular matrix* T *such that*

$$\left.\begin{aligned} TAT^{-1} &= \begin{bmatrix} \bar{A}_0 & \bar{D}_{\mathrm{of}}\bar{C}_{\mathrm{f}} \\ \bar{B}_{\mathrm{f}}\bar{D}_{\mathrm{fo}} & \bar{A}_{\mathrm{f}} + \bar{B}_{\mathrm{f}}\bar{D}_{\mathrm{ff}} \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \bar{B}_{\mathrm{f}} \end{bmatrix} \\ CT^{-1} &= \begin{bmatrix} 0 & \bar{C}_{\mathrm{f}} \end{bmatrix} \end{aligned}\right\} \tag{2.5}$$

where $\bar{A}_0$ *is a stable matrix whose eigenvalues are the zeros of* (C, A, B) *defined* *in* (2.2)

$$\left.\begin{aligned}\bar{A}_{\mathrm{f}} &= \left[\begin{array}{cccc} 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_m \\ 0 & 0 & \cdots & 0 \end{array}\right]\Bigg\}\, q \text{ blocks}, \quad \bar{B}_{\mathrm{f}} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ M \end{array}\right] \\ \bar{C}_{\mathrm{f}} &= [I_m \quad 0 \quad \cdots \quad 0]\end{aligned}\right\} \tag{2.6}$$

and $\bar{D}_{\mathrm{of}}$, $\bar{D}_{\mathrm{fo}}$ *and* $\bar{D}_{\mathrm{ff}}$ *are constant matrices.*

We now consider a time-varying transformation

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_{\mathrm{o}}(t) \\ \bar{x}_{\mathrm{f}}(t) \end{bmatrix} = \begin{bmatrix} T_{\mathrm{o}} \\ R_{\mathrm{f}}(t)T_{\mathrm{f}} \end{bmatrix} x(t) \tag{2.7}$$

where $T = [T_{\mathrm{o}}^{\mathrm{T}} \quad T_{\mathrm{f}}^{\mathrm{T}}]^{\mathrm{T}}$ is as in Lemma 2.1, with $T^{-1} = [\bar{T}_{\mathrm{o}} \quad \bar{T}_{\mathrm{f}}]$, and

$$R_{\mathrm{f}}(t) = \operatorname{diag}\{\rho^{q-1}(t)I_m, \ldots, \rho(t)I_m, I_m\}$$

with $\rho(t) > 0$ being a time-varying parameter to be specified later. Noting that

$$\begin{aligned} R_{\mathrm{f}}(t)\bar{A}_{\mathrm{f}}R_{\mathrm{f}}^{-1}(t) &= \rho(t)\bar{A}_{\mathrm{f}} \\ R_{\mathrm{f}}(t)\bar{B}_{\mathrm{f}} &= \bar{B}_{\mathrm{f}} \\ \bar{C}_{\mathrm{f}}R_{\mathrm{f}}^{-1}(t) &= \rho^{1-q}(t)\bar{C}_{\mathrm{f}} \\ \dot{R}_{\mathrm{f}}(t) &= \rho^{-1}(t)\dot{\rho}(t)Q_{\mathrm{f}}R_{\mathrm{f}}(t) \end{aligned}$$

where $Q_{\mathrm{f}} = \operatorname{diag}\{(q-1)I_m, \ldots, I_m, 0\}$, the transformation in (2.7) brings the system $\mathscr{S}$ in (2.1) into

$$\left.\begin{aligned} \bar{\mathscr{S}}\colon \dot{\bar{x}}_{\mathrm{o}}(t) &= \bar{A}_{\mathrm{o}}\bar{x}_{\mathrm{o}}(t) + \bar{l}_{\mathrm{o}}(t, \bar{x}(t)) \\ \dot{\bar{x}}_{\mathrm{f}}(t) &= \rho(t)\bar{A}_{\mathrm{f}}\bar{x}_{\mathrm{f}}(t) + \bar{B}_{\mathrm{f}}u(t) + \bar{l}_{\mathrm{f}}(t, \bar{x}(t)) \\ y(t) &= \rho^{1-q}(t)\bar{C}_{\mathrm{f}}\bar{x}_{\mathrm{f}}(t) \end{aligned}\right\} \tag{2.8}$$

where

$$\left.\begin{aligned} \bar{l}_{\mathrm{o}}(t, \bar{x}) &= \bar{D}_{\mathrm{of}}y + T_{\mathrm{o}}h(t, y) \\ \bar{l}_{\mathrm{f}}(t, \bar{x}) &= \bar{B}_{\mathrm{f}}\{\bar{D}_{\mathrm{fo}}\bar{x}_{\mathrm{o}} + \bar{D}_{\mathrm{ff}}R_{\mathrm{f}}^{-1}(t)\bar{x}_{\mathrm{f}} + g[t, \bar{T}_{\mathrm{o}}\bar{x}_{\mathrm{o}} + \bar{T}_{\mathrm{f}}R_{\mathrm{f}}^{-1}(t)\bar{x}_{\mathrm{f}}]\} \\ &\quad + R_{\mathrm{f}}(t)T_{\mathrm{f}}h(t, y) + \rho^{-1}(t)\dot{\rho}(t)Q_{\mathrm{f}}\bar{x}_{\mathrm{f}} \end{aligned}\right\} \tag{2.9}$$

To the system $\bar{S}$ in (2.8) we apply a dynamic output feedback control as

$$\left.\begin{aligned} \mathscr{C}\colon \dot{z}(t) &= \rho(t)Fz(t) + \rho^{q}(t)Gy(t) \\ u(t) &= \rho(t)Hz(t) + \rho^{q}(t)Ky(t) \end{aligned}\right\} \tag{2.10}$$

where $z(t) \in \mathscr{R}^{(q-1)m}$, and the matrices F, G, H and K are chosen such that the matrix

$$\hat{A}_{\mathrm{f}} = \begin{bmatrix} \bar{A}_{\mathrm{f}} + \bar{B}_{\mathrm{f}}K\bar{C}_{\mathrm{f}} & \bar{B}_{\mathrm{f}}H \\ G\bar{C}_{\mathrm{f}} & F \end{bmatrix} \tag{2.11}$$

is stable.

Note that $\hat{A}_{\mathrm{f}}$ represents the A matrix of a hypothetical system consisting of a plant $(\bar{C}_{\mathrm{f}}, \bar{A}_{\mathrm{f}}, \bar{B}_{\mathrm{f}})$ and a dynamic output feedback compensator (H, F, G, K).

Since $(\bar{C}_f, \bar{A}_f, \bar{B}_f)$ is controllable and observable, a choice of (H, F, G, K) to result in an $\hat{A}_f$ with a desired spectrum is always possible (Brasch and Pearson 1970).

The closed-loop system consisting of the system $\bar{\mathscr{S}}$ in (2.8) and the controller $\mathscr{C}$ in (2.10) can now be described as

$$\left.\begin{aligned} \hat{\mathscr{S}}\colon \dot{\hat{x}}_o(t) &= \hat{A}_o\hat{x}_o(t) + \hat{l}_o(t, \hat{x}(t)) \\ \dot{\hat{x}}_f(t) &= \rho(t)\hat{A}_f\hat{x}_f(t) + \hat{l}_f(t, \hat{x}(t)) \end{aligned}\right\} \tag{2.12}$$

where $\hat{x}_o = \bar{x}_o$, $\hat{x}_f = [\bar{x}_f^T \quad z^T]^T$, $\hat{x} = [\hat{x}_o^T \quad \hat{x}_f^T]^T$, $\hat{A}_o = \bar{A}_o$, $\hat{A}_f$ is as given in (2.11), and

$$\left.\begin{aligned} \hat{l}_o(t, \hat{x}) &= \bar{l}_o(t, \bar{x}) \\ \hat{l}_f(t, \hat{x}) &= [\bar{l}_f^T(t, \bar{x}) \quad 0]^T \end{aligned}\right\} \tag{2.13}$$

From (2.4)–(2.9) and (2.13) it follows that for $0 \leqslant \dot{\rho}(t) \leqslant 1 \leqslant \rho(t)$

$$\left.\begin{aligned} \|\hat{l}_o(t, \hat{x})\| &\leqslant \alpha_{of}\|\hat{x}_f\| \\ \|\hat{l}_f(t, \hat{x})\| &\leqslant \alpha_{fo}\|\hat{x}_o\| + \alpha_{ff}\|\hat{x}_f\| \end{aligned}\right\} \tag{2.14}$$

for some constants α_{of}, α_{fo}, $\alpha_{ff} > 0$, which depend on the system parameters C, A, B and the perturbation bounds c_g, c_h in (2.4).

We are now ready to prove our first result.

Theorem 2.1: *There exists a parameter $\rho^* \geqslant 1$, such that the closed loop system $\hat{\mathscr{S}}$ in (2.12) is exponentially stable for all $\rho(t)$ satisfying $0 \leqslant \dot{\rho}(t) \leqslant 1 \leqslant \rho^* \leqslant \rho(t)$.*

Proof: Since $\hat{A}_o$ and $\hat{A}_f$ are stable matrices, there exist positive definite matrices P_o and P_f such that

$$\left.\begin{aligned} \hat{A}_o^T P_o + P_o\hat{A}_o &= -I_{n-qm} \\ \hat{A}_f^T P_f + P_f\hat{A}_f &= -I_{(2q-1)m} \end{aligned}\right\} \tag{2.15}$$

Consider

$$v(\hat{x}) = \hat{x}_o^T P_o \hat{x}_o + \hat{x}_f^T P_f \hat{x}_f \tag{2.16}$$

as a candidate for a Lyapunov function for the closed-loop system $\hat{\mathscr{S}}$ in (2.12). For $0 \leqslant \dot{\rho}(t) \leqslant 1 \leqslant \rho(t)$, the derivative of $v(\hat{x})$ along the solutions of $\hat{\mathscr{S}}$ can be evaluated and 'majorized' using (2.12)–(2.14) as

$$\left.\begin{aligned} \dot{v}(\hat{x}(t)) \leqslant{}& -\|\hat{x}_o(t)\|(\|\hat{x}_o(t)\| - 2\alpha_{of}\|P_o\|\,\|\hat{x}_f(t)\|) \\ & -\|\hat{x}_f(t)\|\{-2\alpha_{fo}\|P_f\|\,\|\hat{x}_o(t)\| + [\rho(t) - 2\alpha_{ff}\|P_f\|]\|\hat{x}_f(t)\|\} \\ ={}& -\xi_x^T(t)Q[\rho(t)]\xi_x(t) \end{aligned}\right\} \tag{2.17}$$

where $\xi_x(t) = [\|\hat{x}_o(t)\|\,\|\hat{x}_f(t)\|]^T$, and

$$Q(\rho) = \begin{bmatrix} 1 & -\alpha_{of}\|P_o\| - \alpha_{fo}\|P_f\| \\ -\alpha_{of}\|P_o\| - \alpha_{fo}\|P_f\| & \rho - 2\alpha_{ff}\|P_f\| \end{bmatrix} \tag{2.18}$$

From (2.18) it follows that for any $0 < \varepsilon < 1$, there exists a ρ^* such that $\lambda_{min}\{Q(\rho^*)\} \geqslant \varepsilon$. Then, for $\rho(t) \geqslant \rho^*$, $\dot{v}$ in (2.17) can be further 'majorized' as

$$\left.\begin{aligned}\dot{v}(\hat{x}(t)) &\leqslant -\xi_x^{\mathrm{T}}(t)Q(\rho^*)\xi_x(t) - [\rho(t) - \rho^*]\|\hat{x}_{\mathrm{f}}(t)\|^2 \\ &\leqslant -\varepsilon(\|\hat{x}_{\mathrm{o}}(t)\|^2 + \|\hat{x}_{\mathrm{f}}(t)\|^2) \\ &\leqslant -2\sigma v(\hat{x}(t))\end{aligned}\right\} \tag{2.19}$$

where $2\sigma = \varepsilon/\max\{\lambda_{\max}(P_{\mathrm{o}}), \lambda_{\max}(P_{\mathrm{f}})\}$, and is independent of $\rho(t)$. Equation (2.19), together with (2.7), implies that, provided $0 \leqslant \dot{\rho}(t) \leqslant 1 \leqslant \rho^* \leqslant \rho(t)$

$$\left.\begin{aligned}\|x(t)\| &\leqslant l_x \exp\{-\sigma(t - t_0)\} \\ \|y(t)\| &\leqslant l_y \exp\{-\sigma(t - t_0)\} \\ \|z(t)\| &\leqslant l_z \exp\{-\sigma(t - t_0)\}\end{aligned}\right\} \tag{2.20}$$

for some constants l_x, l_y, $l_z > 0$, which depend on the initial conditions. This completes the proof. □

Note that Theorem 2.1 brings a clarification to the choice of the controller as in (2.10). With the interconnection terms between $\bar{x}_{\mathrm{o}}$ and $\bar{x}_{\mathrm{f}}$ included in $\bar{I}_{\mathrm{o}}$ and $\bar{I}_{\mathrm{f}}$, the open-loop nominal system is treated as having a transfer function matrix

$$G(s) = \frac{d_{\mathrm{o}}(s)}{s^q d_{\mathrm{o}}(s)} M \tag{2.21}$$

where $d_{\mathrm{o}}(s) = \Pi_{z\in\mathfrak{Z}}(s - z)$. Then the controller parameters are designed to have a transfer function matrix

$$G_{\mathrm{c}}(s) = H(sI - F)^{-1}G + K = \frac{n_{\mathrm{c}}(s)}{d_{\mathrm{c}}(s)} M^{-1} \tag{2.22}$$

where $n_{\mathrm{c}}(s)$ and $d_{\mathrm{c}}(s)$ are chosen such that

$$\hat{d}_{\mathrm{f}}(s) = s^q d_{\mathrm{c}}(s) - n_{\mathrm{c}}(s) \tag{2.23}$$

is a stable polynomial. With the controller as above, and $\rho(t) = \rho$ (constant), the closed-loop system in (2.12) after neglecting the parameters $\hat{I}_{\mathrm{o}}(\hat{x})$ and $\hat{I}_{\mathrm{f}}(\hat{x})$ behaves as if having the transfer function matrix

$$\hat{G}(s) = \frac{d_{\mathrm{o}}(s) d_{\mathrm{c}}(\rho^{-1}s)}{d_{\mathrm{o}}(s)\hat{d}_{\mathrm{f}}(\rho^{-1}s)} \rho^{-q} M \tag{2.24}$$

Thus, choosing ρ sufficiently large, $\hat{G}(s)$ can be made stable with an arbitrarily small H^∞-norm. This is exactly what makes it possible to beat the destabilizing effect of the perturbations.

From the design procedure and the proof of the above theorem, we observe that the matrices F, G, H and K of the controller $\mathscr{C}$ in (2.10) are independent of the system parameters and perturbation bounds except q and M. However, the gain $\rho(t)$ should be larger than a critical value ρ^*, which is determined by the bounds of the system. To eliminate the need to know these bounds, we propose, in the next section, an adaptation mechanism, which increases the value of $\rho(t)$ slowly until it is large enough to stabilize the system.

3. Adaptation of the gain parameter

We employ a simple adaptation rule.

$$\mathscr{A}\colon \dot{\rho}(t) = \min\{1, \alpha_y\|y(t)\|^2 + \alpha_z\|z(t)\|^2\} \tag{3.1}$$

where α_y, $\alpha_z > 0$ are arbitrary numbers.

Let $(x, z, \rho)(t, t_0, x_0, z_0, \rho_0)$ denote the solution of the adaptive system consisting of the plant in (2.1), the controller in (2.10), and the adaptation rule in (3.1), starting from the initial conditions $x(t_0) = x_0$, $z(t_0) = z_0$, $\rho(t_0) = \rho_0$. We have the following.

Theorem 3.1: *For all* t_0, x_0, z_0, *and* $\rho_0 \geqslant 0$

$$\lim_{t\to\infty} x(t, t_0, x_0, z_0, \rho_0) = 0 \tag{3.2}$$

$$\lim_{t\to\infty} z(t, t_0, x_0, z_0, \rho_0) = 0 \tag{3.3}$$

$$\lim_{t\to\infty} \rho(t, t_0, x_0, z_0, \rho_0) = \rho_\infty < \infty \tag{3.3}$$

Proof: Two cases are possible.

Case 1. $\rho(t^*) \geqslant \rho^*$ for some $t^* \geqslant t_0$, where ρ^* is as in the statement of Theorem 2.1

In this case, (3.2) and (3.3) follow directly from (2.20). We also have, from (3.1)

$$\rho(t) \leqslant \rho(t^*) + \int_{t^*}^{t} (\alpha_y \|y(\tau)\|^2 + \alpha_z \|z(\tau)\|^2)\,\mathrm{d}\tau \tag{3.5}$$

$$\leqslant \rho(t^*) + \int_{t^*}^{t} (\alpha_y l_y^2 + \alpha_z l_z^2) \exp\{-2\sigma(\tau - t_0)\}\,\mathrm{d}\tau \tag{3.6}$$

$$\leqslant \rho(t^*) + \frac{\alpha_y l_y^2 + \alpha_z l_z^2}{2\sigma} \exp\{-2\sigma(t^* - t_0)\}$$

proving (3.4).

Case 2. $\rho(t) < \rho^*$ for all $t \geqslant t_0$

Since $\rho(t)$ is non-decreasing, $\lim_{t\to\infty} \rho(t) = \rho_\infty < \infty$ exists, proving (3.4). From (3.5) it follows that $\lim_{t\to\infty} z(t) = 0$, proving (3.3), and that $\lim_{t\to\infty} y(t) = 0$. Thus, from (2.10), $u(t)$ is bounded and $\lim_{t\to\infty} u(t) = 0$. Then (3.2) follows from the lemma in the Appendix. □

We illustrate the above discussions with an example.

Example 3.1: Consider the equation of a damped inverted pendulum

$$\ddot{\theta} + c_1\dot{\theta} - c_2 \sin\theta = u \tag{3.7}$$

where θ is the clockwise angular displacement from the vertical, u is the control torque, and the parameters c_1, $c_2 \geqslant 0$ are determined by the damping coefficient, mass and the length of the pendulum. With $x_1 = \theta$, $x_2 = \dot{\theta}$, we get the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ c_2 \sin x_1 - c_1 x_2 \end{bmatrix} \tag{3.8}$$
$$y = [1 \quad 0]x$$

The nominal system is minimum-phase and has relative degree two. Accordingly, we design a first-order high-gain controller as

$$\left.\begin{aligned} \dot{z} &= -4\rho z + 4\rho^2 y \\ u &= 5\rho z - 6\rho^2 y \end{aligned}\right\} \tag{3.9}$$

which would result in a stable nominal closed-loop system having the poles $\lambda_1 = \bar{\lambda}_2 = \rho(-1 + j)$ and $\lambda_3 = -2\rho$, if ρ were constant; and choose the adaptation rule quite arbitrarily as

$$\dot{\rho}(t) = \min\{1, 2(y^2 + z^2)\} \tag{3.10}$$

The simulation results corresponding to arbitrarily selected system parameters $c_1 = 2$, $c_2 = 0{\cdot}5$, and the initial conditions $x_1(0) = x_2(0) = 1$, $z(0) = \rho(0) = 0$ are shown in the Figure, and are obtained by numerical integration using the Euler method with a time-step of 0·01 s. It is observed that even though the pendulum is falling down with a positive initial angular velocity, and there is no initial control action, the adaptive controller stabilizes the system without the gains becoming excessively large. □

4. Adaptive robust stabilization of interconnected systems

Adaptive high-gain controller design of the previous sections finds a natural application in decentralized stabilization of interconnected systems, where the interconnections among the subsystems are treated as perturbations.

We consider an interconnected system consisting of N subsystems described as

$$\left.\begin{aligned} \mathcal{S}_i\colon \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + l_i(t, x(t)) \\ y_i(t) &= C_i x_i(t), \; i \in \mathcal{N} \end{aligned}\right\} \tag{4.1}$$

where $\mathcal{N} = \{1, 2, \ldots, N\}$; $x_i(t) \in \mathcal{R}^{n_i}$, $u_i(t) \in \mathcal{R}^{m_i}$ and $y_i(t) \in \mathcal{R}^{m_i}$ are the state, input and output of $\mathcal{S}_i$, respectively; A_i, B_i, C_i are constant matrices of appropriate dimensions; $x(t) = [x_1^{\mathrm{T}}(t)\; x_2^{\mathrm{T}}(t)\; \ldots\; x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$; and $l_i(t, x(t))$ represents the interconnections between $\mathcal{S}_i$ and the other subsystems.

As in § 2, we assume that

(a') (C_i, A_i, B_i) are controllable and observable,

(b') (C_i, A_i, B_i) are minimum-phase, that is, the sets of transmission zeros $\mathcal{Z}_i$ are stable,

(c') (C_i, A_i, B_i) have uniform rank q_i with $C_i A_i^{q_i - 1} B_i = M_i$,

(d') q_i and M_i are known,

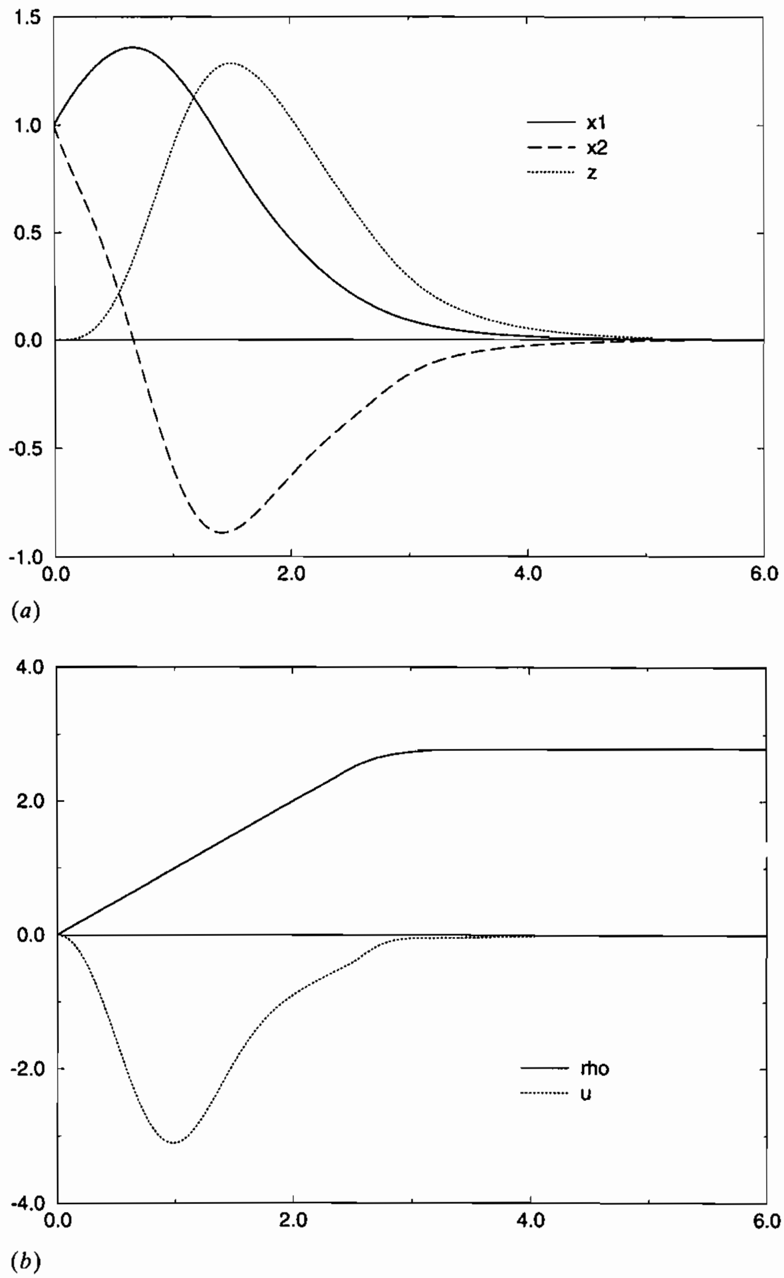
(e') $l_i(t, x) = B_i g_i(t, x) + h_i(t, y)$, where $y(t) = [y_1(t)\; y_2(t)\; \ldots\; y_N(t)]^{\mathrm{T}}$, and g_i and h_i satisfy, for some constants c_{ij}^g, $c_{ij}^h > 0$

$$\left.\begin{aligned} \|g_i(t, x)\| &\leqslant \textstyle\sum_{j=1}^N c_{ij}^g \|x_j\| \\ \|h_i(t, y)\| &\leqslant \textstyle\sum_{j=1}^N c_{ij}^h \|y_j\| \end{aligned}\right\} \tag{4.2}$$

for all $t \in \mathcal{R}$, $x_i \in \mathcal{R}^{n_i}$.

Imitating the design procedure of § 2, we transform the subsystems into the form of (2.8) with q replaced with q_i, and choose decentralized, high-gain dynamic controllers

$$\left.\begin{aligned} \mathcal{C}_i\colon \dot{z}_i(t) &= \rho_i(t) F_i z_i(t) + \rho_i^{q_i}(t) G_i y_i(t) \\ u_i(t) &= \rho_i(t) H_i z_i(t) + \rho_i^{q_i}(t) K_i y_i(t) \end{aligned}\right\} \tag{4.3}$$



Simulation results for the system in Example 3.1: (a) states (x_1: solid curve; x_2: dashed curve; z: dotted curve); (b) input and gain (u: dotted curve; ρ: solid curve).

so that the matrices

$$\hat{A}_{fi} = \begin{bmatrix} \bar{A}_{fi} + \bar{B}_{fi} K_i \bar{C}_{fi} & \bar{B}_{fi} H_i \\ G_i \bar{C}_{fi} & F_i \end{bmatrix} \tag{4.4}$$

are stable. The resulting closed-loop interconnected system is then described by

$$\left.\begin{aligned}\hat{\mathscr{S}}_i\colon \dot{\hat{x}}_{oi}(t) &= \hat{A}_{oi}\hat{x}_{oi}(t) + \hat{I}_{oi}(t, \hat{x}(t)) \\ \dot{\hat{x}}_{fi}(t) &= \rho_i(t)\hat{A}_{fi}\hat{x}_{fi}(t) + \hat{I}_{fi}(t, \hat{x}(t))\end{aligned}\right\} \tag{4.5}$$

where, for $\rho_i(t) \geq 1$,

$$\left.\begin{aligned}\|\hat{I}_{oi}(t, \hat{x})\| &\leq \sum_{j=1}^{N} \alpha_{ij}^{of}\|\hat{x}_{fj}\| \\ \|\hat{I}_{fi}(t, \hat{x})\| &\leq \rho_i^{-1}(t)|\dot{\rho}_i(t)|\beta_{ii}^{ff}\|\hat{x}_{fi}\| + \sum_{j=1}^{N}[\alpha_{ij}^{fo}\|\hat{x}_{oj}\| + (\alpha_{ij}^{ff} + \rho_i^{q_i-1}\rho_j^{1-q_i}\gamma_{ij}^{ff})\|\hat{x}_{fj}\|]\end{aligned}\right\} \tag{4.6}$$

for some constants α_{ij}^{pq}, β_{ij}^{ff}, γ_{ij}^{ff}, $p, q = \mathrm{o, f}$; $i, j \in \mathcal{N}$.

Similar to the choice of Saberi and Khalil (1985), we choose

$$\rho_i(t) = \rho^{\nu_i}$$

where

$$\nu_i = \begin{cases} 2\nu + 1, & q_i = 1 \\ \nu/(q_i - 1), & q_i \neq 1 \end{cases}$$

with $\nu = \prod_{q_i \neq 1}(q_i - 1)$, and state the following counterpart of Theorem 2.1.

Theorem 4.1: *There exist matrices F_i, G_i, H_i, K_i and a parameter $\rho^* \geq 1$, such that the closed loop system $\hat{\mathscr{S}}$ in* (4.5) *is exponentially stable for all $\rho(t)$ satisfying* $0 \leq \dot{\rho}(t) \leq 1 \leq \rho^* \leq \rho(t)$.

Proof (Outline): With

$$v(\hat{x}) = \sum_{i=1}^{N}(\hat{x}_{oi}^{T}P_{oi}\hat{x}_{oi} + \hat{x}_{fi}^{T}P_{fi}\hat{x}_{fi}) \tag{4.7}$$

where P_{oi} and P_{fi} are as in (2.15), we have

$$\dot{v}(\hat{x}(t)) \leq -\xi_x^{T}(t)Q[\rho(t)]\xi_x(t) \tag{4.8}$$

where $\xi_x = [\|\hat{x}_{1o}\| \; \ldots \; \|\hat{x}_{No}\| \; \|\hat{x}_{1f}\| \; \ldots \; \|\hat{x}_{Nf}\|]^{T}$, and

$$Q(\rho) = \begin{bmatrix} I_N & Q_{of} \\ Q_{of}^{T} & Q_{ff}(\rho) \end{bmatrix} \tag{4.9}$$

where $Q_{of} = (q_{ij}^{of})$, with $q_{ij}^{of} = -\alpha_{ij}^{of}\|P_{oi}\| - \alpha_{ji}^{fo}\|P_{fj}\|$, and $Q_{ff}(\rho) = [q_{ij}^{ff}(\rho)]$, with

$$q_{ij}^{ff}(\rho) = \begin{cases} \rho^{\nu_i} - \phi_{ii}, & j = i, \\ -\phi_{ij}, & j \neq i,\ q_i \neq 1 \neq q_j \text{ or } q_i = 1 = q_j \\ -\rho^{\nu}\theta_{ij} - \phi_{ij}, & \text{otherwise} \end{cases}$$

for some constants θ_{ij}, $\phi_{ij} > 0$.

The rest of the proof follows similar lines as the proof of Theorem 2.1. □

Now, we use the adaptation rule in (3.1) for the high-gain feedback parameter ρ, and state the following result, whose proof is exactly the same as the proof of Theorem 3.1.

Theorem 4.2: *The result of Theorem* 3.1 *is also valid for the interconnected system of* (4.1).

Note that although the controller in (4.3) is decentralized, the adaptation rule in (3.1) requires information about the outputs of all the subsystems as well as the state of each local controller.

5. Concluding remarks

We would like to discuss a few points about our results.

Assumptions (a)–(e) about the nominal systems and the perturbations guarantee the existence of a high-gain dynamic output-feedback controller that achieves robust stability. The assumptions that the system is square and has uniform rank are only made to provide simplicity in the presentation of the concept and the technique of the paper. They can easily be replaced with less restrictive assumptions, as has been done by Saberi and Sannuti (1990).

As explained in § 2, the controller is designed offline, and is quite arbitrary provided $\hat{d}_{\mathrm{f}}(s)$ in (2.23) is stable. It can easily be shown that if

$$\hat{d}_{\mathrm{f}}(s) = s^{2q-1} + p_1 s^{2q-2} + \cdots + p_{2q-1}$$

then the choice

$$F = \begin{bmatrix} 0 & \cdots & 0 & -p_{q-1}I_m \\ I_m & \cdots & 0 & -p_{q-2}I_m \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_m & -p_1 I_m \end{bmatrix}, \quad G = \begin{bmatrix} (p_{q-1}p_q - p_{2q-1})I_m \\ (p_{q-2}p_q - p_{2q-2})I_m \\ \vdots \\ (p_1 p_q - p_{q+1})I_m \end{bmatrix}$$

$$H = [0 \quad \cdots \quad 0 \quad M^{-1}], \quad K = -p_q M^{-1}$$

results in

$$\det(sI - \hat{A}_{\mathrm{f}}) = [\hat{d}_{\mathrm{f}}(s)]^m$$

Obviously, if $q = 1$, then the controller in (2.10) reduces to a simple constant output-feedback.

We would also like to point out that if the nominal system has no transmission zeros (which corresponds to maximal relative degree in the SISO case), then the closed-loop system can be made exponentially stable with an arbitrary degree of stability, as can be deduced from the proof of Theorem 2.1.

Increasing the gain $\rho(t)$ by the adaptation rule in (3.1) eliminates the need to know the perturbation bounds to achieve robust stability. From the proof of Theorem 2.1 it is observed that what is needed in reaching this result is a 'majorization' of $\hat{l}_{\mathrm{o}}$ and $\hat{l}_{\mathrm{f}}$ as in (2.14). The condition $\dot{\rho}(t) \leq 1$ is a simple one that satisfies (2.14), but can be replaced with less restrictive conditions. In fact, the result would be valid as long as $|\rho^{-1}(t)\dot{\rho}(t)|$ is bounded. Then the adaptation rule in (3.1) can be modified accordingly.

Finally, we would like to point out that high-gain decentralized controllers in (4.3) can also be used to stabilize interconnected systems having a more general interconnection structure, as considered by Yu and Sezer (1992).

Appendix

Lemma: *Under the assumptions (a)–(e), if*

$$\lim_{t\to\infty} u(t) = 0 \qquad \text{(A 1)}$$
$$\lim_{t\to\infty} y(t) = 0$$

for the system $\mathcal{S}$ *in* (2.1), *then*

$$\lim_{t\to\infty} x(t) = 0 \tag{A 2}$$

Proof: Without loss of generality, assume that $\mathcal{S}$ is already transformed into $\bar{\mathcal{S}}$ in (2.8)–(2.9), with $\rho(t)$ replaced with a constant parameter r, and consider the following observer for $\bar{\mathcal{S}}$.

$$\begin{aligned} \mathbb{O}\colon \dot{\zeta}_o(t) &= \bar{A}_o\zeta_o(t) \\ \dot{\zeta}_f(t) &= r(\bar{A}_f + \bar{G}_f\bar{C}_f)\zeta_f(t) + \bar{B}_f u(t) - r^q\bar{G}_f y(t) \end{aligned} \tag{A 3}$$

where $\bar{G}_f$ is such that $\bar{A}_f + \bar{G}_f\bar{C}_f$ is a stable matrix. Since $\mathbb{O}$ is stable, (A 1) implies

$$\lim_{t\to\infty} \zeta(t) = 0 \tag{A 4}$$

Let $e_o(t) = \bar{x}_o(t) - \zeta_o(t)$, and $e_f(t) = \bar{x}_f(t) - \zeta_f(t)$. Then, from (2.8) and (A 3) we have

$$\left.\begin{aligned} \dot{e}_o(t) &= \bar{A}_o e_o(t) + \bar{l}_o[t, e(t) + \zeta(t)] \\ \dot{e}_f(t) &= r(\bar{A}_f + \bar{G}_f\bar{C}_f)e_f(t) + \bar{l}_f[t, e(t) + \zeta(t)] \end{aligned}\right\} \tag{A 5}$$

(2.9), with $\rho(t)$ replaced with r, implies that for $r \geqslant 1$

$$\left.\begin{aligned} &\|\bar{l}_o[t, e(t) + \zeta(t)]\| \leqslant \gamma_o\|y(t)\| \\ &\|\bar{l}_f[t, e(t) + \zeta(t)]\| \leqslant \alpha_{fo}\|e_o(t)\| + \alpha_{ff}\|e_f(t)\| + \beta_{fo}\|\zeta_o(t)\| + \beta_{ff}\|\zeta_f(t)\| + \gamma_f\|y(t)\| \end{aligned}\right\} \tag{A 6}$$

Let $v(e) = e_o^T P_o e_o + e_f^T P_f e_f$, where P_o and P_f are the solutions of (2.15) with $\hat{A}_o$ and $\hat{A}_f$ replaced with $\bar{A}_o$ and $\bar{A}_f + \bar{G}_f\bar{C}_f$. Then, following the proof of Theorem 2.1, the derivative of $v(e)$ along the solutions of (A 5) can be 'majorized' as

$$\dot{v}(e(t)) \leqslant -\xi_e^T(t)Q(r)\xi_e(t) + \xi_e^T(t)\eta(t) \tag{A 7}$$

where $\xi_e = [\|e_o\| \ \|e_f\|]^T$, $Q(r)$ is as defined in (2.18) with $\alpha_{of} = 0$, and $\eta(t) = [\eta_o(t) \ \ \eta_f(t)]^T$ with

$$\left.\begin{aligned} \eta_o(t) &= 2\gamma_o\|P_o\|\,\|y(t)\|) \\ \eta_f(t) &= 2\|P_f\|(\beta_{fo}\|\zeta_o(t)\| + \beta_{ff}\|\zeta_f(t)\| + \gamma_f\|y(t)\|) \end{aligned}\right\} \tag{A 8}$$

By (A 1) and (A 4) $\lim_{t\to\infty}\eta(t) = 0$. Then, for sufficiently large r, (A 7) implies $\lim_{t\to\infty} e(t) = 0$, and the proof follows from (A 4).

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