Quantum stereographic projection and the homographic oscillator

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The quantum deformation created by the stereographic mapping from S_2 to $\mathbb{C}$ is studied. It is shown that the resulting algebra is locally isomorphic to su(2) and is an unconventional deformation of which the undeformed limit is a contraction onto the harmonic oscillator algebra. The deformation parameter is given naturally by the central invariant of the embedding su(2). The deformed algebra is identified as a member of a larger class of quartic q oscillators. We next study the deformations in the corresponding Jordan-Schwinger representation of two independent deformed oscillators and solve for the deforming transformation. The invertibility of this transformation guarantees an implicit coproduct law which is also discussed. Finally we discuss the analogy between Poincaré's geometric interpretation of the quantum Stokes parameters of polarization and the stereographic projection as an important physical application of the latter. [S1050-2947(96)00106-0]

I. INTRODUCTION

Since the discovery of the first deformed quantum algebra by Biedenharn and Macfarlane [1] a tremendous effort has been made to find physical realizations of these algebras. Much earlier, inspired by an operator representation for q-deformed dual resonance models [2], Baker, Coon, and Yu formulated the simplest q algebra which produced a suitably bounded spectrum determined by the parameter of deformation q. Apart from their profound mathematical significance mainly related to the solution of the quantum Yang-Baxter equation [3], and other problems [4], physical applications can be found in the deformed Jaynes-Cummings model [5], the ubiquitous quantum phase problem [6], the relativistic q oscillator [7], and recently, in reproducing deformed nuclear energy levels [8], the Morse oscillator [9], and the Kepler problem [10].

In particular, several possible realizations of deformed Lie algebras can be constructed by applying certain nonlinear invertible deforming transformations to the generators of the undeformed algebras [11]. In this context, some explicit cases have been examined by Curtright and Zachos [12], among several previously studied examples.

The quantum deformation of a physical symmetry should be identified by a deformation parameter which must be uniquely determined by a set of observables. In this work, we give a particular example of that by examining the stereographic projection of the su(2) generators on an extended complex plane and show that the resulting deformation is described by a deformation parameter which is directly connected with the central invariant of the embedding su(2). In Sec. II we define and derive the properties of the quantum stereographic projection. Section III is devoted to the properties of the central invariant. In Sec. IV the deformation induced by the homographic oscillator on su(2) is studied using Jordan-Schwinger representation. The proof of existence of the coproduct for the corresponding su(2) deformation is presented in Sec. V. In the last section, Sec. VI, we

discuss possible physical applications where quantized stereographic projection and the resulting homographic oscillator algebra become relevant.

II. STEREOGRAPHIC PROJECTION

Stereographic projection (SP) is a mapping from the Riemann sphere S_2 onto an extended complex plane $\mathbb{C}$. At the classical level, SP is defined by a mapping from S_2 in spherical J, θ, ϕ parametrization to that z, z^* on the complex plane given by

$$z = 2J\cot\theta/2\, e^{i\phi}, \quad \text{with} \quad S_\theta = \frac{2\omega}{1+\omega^2},$$

$$C_\theta = \frac{\omega^2-1}{\omega^2+1}, \quad \text{where} \quad \omega = \frac{\sqrt{z^*z}}{2J}, \tag{1}$$

where J is fixed θ and ϕ are real coordinates describing the polar and azimuthal angles, S_θ and C_θ describe the sine and cosine of θ, respectively, and z, z^* are defined on the projected plane as shown in Fig. 1. SP is an invertible mapping except for the *ideal point at infinity*. However, this does not violate the formal equivalence between the two representations; since, as will be shown later, when Eq. (1) is quantized, the ideal point at infinity is well represented by infinity in the discrete spectrum of the corresponding deformed algebra. We now proceed by defining an operator realization of Eq. (1) via *sine-cosine* operators $\hat{C}_\theta$ and $\hat{S}_\theta$ as

$$\hat{C}_\theta = \frac{\hat{\Omega}^2-1}{\hat{\Omega}^2+1}, \quad \hat{S}_\theta = \frac{2\hat{\Omega}}{1+\hat{\Omega}^2},$$

where

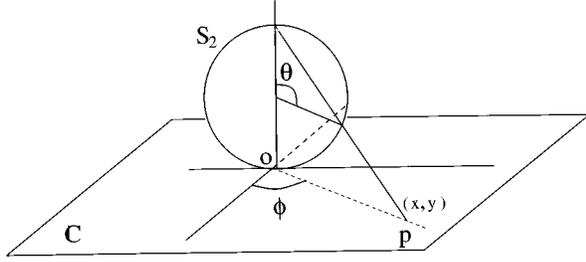$$\hat{\Omega} = \sqrt{\hat{Z}^\dagger\hat{Z}}, \quad \hat{Z} = \hat{E}_\phi\hat{\Omega}, \tag{2}$$

FIG. 1. Geometric interpretation of stereographic projection. Here $z = x + iy$ and z^* is conjugate to z.

and $\hat{\Omega}$ and $\hat{E}_\phi$ are operator counterparts of ω and $e^{i\phi}$, respectively. As in the case of angular decomposition of su(2), here we deal with *ideal* unitary polar operators (i.e., $\hat{C}_\theta^2 + \hat{S}_\theta^2 = 1$ and $[\hat{C}_\theta, \hat{S}_\theta] = 0$) whereas the azimuthal phase operator $\hat{E}_\phi$ has both a nonunitary as well as a unitary representation.

Throughout this work we assume that a quantum deformation is understood explicitly in the same sense as Refs. [11,12]. To be more precise, providing invertible nonlinear functionals of generators, the deformed algebraic structure and its representations are obtained by directly applying these nonlinear functionals to a particular representation of the undeformed mother algebra su(2) [11]. Depending on which level this substitution and following quantization take place, one naturally obtains different quantum deformations.

Using Eqs. (2) it can be found that

$$\hat{Z}^\dagger \hat{Z} = \frac{1 + \hat{C}_\theta}{1 - \hat{C}_\theta}, \quad \hat{Z}\hat{Z}^\dagger = \hat{E}_\phi \frac{1 + \hat{C}_\theta}{1 - \hat{C}_\theta} \hat{E}_\phi^\dagger. \tag{3}$$

In deriving Eq. (3) we did not assume the existence of any particular unitary representation for $\hat{E}_\phi$ (i.e., $\hat{E}_\phi^\dagger \neq \hat{E}_\phi^{-1}$). The algebraic structure of the $\hat{Z}, \hat{Z}^\dagger$ operators, however, as will be discussed later, is not influenced by any conflict between the unitary and nonunitary description of $\hat{E}_\phi$. The algebra defined by $\hat{Z}, \hat{Z}^\dagger$ can be found by using the well-known su(2) relation [13,14]

$$[\hat{C}_\theta, \hat{E}_\phi] = -\frac{1}{J} \hat{E}_\phi \tag{4}$$

in Eq. (3), which is expressed by the *generalized commutation relation*

$$\hat{a}\hat{a}^\dagger = \frac{p\hat{a}^\dagger\hat{a} + 1}{k\hat{a}^\dagger\hat{a} + 1}(1 + [\hat{E}_\phi, \hat{E}_\phi^\dagger]), \text{ with } \hat{a} = \frac{1}{\alpha}\hat{Z}, \quad \hat{a}^\dagger = \frac{1}{\alpha^*}\hat{Z}^\dagger, \tag{5}$$

where $\hat{a}(\hat{a}^\dagger)$ represent the normalized annihilation (creation) operators and the parameters q, k, α are given by

$$|\alpha|^2 = |2J - 1|^{-1}, \quad p = (2J+1)/(2J-1),$$
$$k = -(2J-1)^{-2}. \tag{6}$$

Equation (5) is not an example of prototype deformed algebras and has not been studied in the literature. The commutation $[\hat{E}_\phi, \hat{E}_\phi^\dagger]$ naturally arises in the derivation. It has been

recently shown that it is possible to find a manifestly unitary description of $\hat{E}_\phi$ without affecting any of its operator properties [6]. Here, $\hat{E}_\phi$ is identical to the azimuthal phase operator in the su(2) polar decomposition [6,15] and its unitary representation has the cyclic property

$$\hat{E}_\phi = \sum_{m=-j}^{j} |jm-1\rangle\langle jm| + \beta |j-j\rangle\langle jj|, \quad [\hat{E}_\phi, \hat{E}_\phi^\dagger] = 0 \tag{7}$$

in the finite dimensional Hilbert space spanned by the basis vectors $|jm\rangle$. Here $|\beta| = 1$ and is otherwise undetermined, referring to an arbitrary reference phase. On the other hand, we must mention as a side remark that, if one adopts the nonunitary description of $\hat{E}_\phi$ [i.e. $\beta = 0$ in (7)], the first deformed excitation energy of the algebra (5) is influenced by the nonunitarity of $\hat{E}_\phi$ in such a way that it produces a scale transformation on the operators $\hat{a}, \hat{a}^\dagger$. However, its effect can always be absorbed by a further trivial renormalization of these operators. We will not elaborate on the other implications of the nonunitary description of the $\hat{E}_\phi$ operator in this work.

Equation (5) is actually in the class of generalized quartic oscillators [16] whose properties cannot be simply obtained by taking the square of any generalized commutator. Because of the homographic dependence of $\hat{a}^\dagger\hat{a}$ on $\hat{a}\hat{a}^\dagger$ we term the algebra in Eq. (5) a homographic oscillator (HO). In the limit $J \to \infty$ we observe the limits $p \to 1$ and $k \to 0$, therefore HO contracts to the simple harmonic oscillator (SHO). This is reminiscent of the İnönü-Wigner contraction of su(2) onto the SHO [17]. The spectrum of HO can be solved exactly by considering a generalized Hermitian *number* operator $\hat{N}$ such that $[\hat{a}, \hat{N}] = \hat{a}; [\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger$. For the most general solution, we have $\hat{a}^\dagger\hat{a} = [\hat{N}]$ and $\hat{a}\hat{a}^\dagger = [\hat{N}+1]$ where $[\hat{N}]$ is the principal number operator, and $|n\rangle_J$ are the basis vectors such that

$$[n+1] = \frac{p[n]+1}{k[n]+1}, \tag{8}$$

where

$$\hat{a}|n\rangle_J = [n]^{1/2}|n-1\rangle_J, \quad \hat{a}^\dagger|n\rangle_J = [n+1]^{1/2}|n+1\rangle_J,$$
$$\hat{N}|n\rangle_J = n|n\rangle_J. \tag{9}$$

Enforcing the ground state $|0\rangle_J$ to be annihilated by $\hat{a}$, we have $[0] = 0$. Using this ground state in (8), the whole spectrum can be analytically iterated to yield

$$[n] = \frac{[[n]]}{[[n]] - [[n-1]]}, \quad [[n]] = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad (r_1 \neq r_2), \tag{10}$$

where $[0] = [[0]] = 0$ and $[1] = [[1]] = 1$, with

$$\frac{1}{r_1} + \frac{1}{r_2} = 1 + p, \quad \frac{1}{r_1 r_2} = p - k. \tag{11}$$

From Eqs. (6) and (11) we find $r_1 = r_2 = q$. The spectrum is thus given by the first derivative of $[[n]]$ with respect to q as

$[[n]] = nq^{n-1}$. Here we identify $q = (2J-1)/2J$ as the deformation parameter of the HO algebra.

It is known that the basic number $[[n]]$ arises in the solution of the generalized Fibonacci series [16]

$$[[n+2]] = \alpha[[n+1]] + \beta[[n]],$$

where

$$\alpha = r_1 + r_2, \quad \beta = -r_1 r_2. \quad (12)$$

Further, it can be shown that Eq. (12) defines a class of *generalized* (i.e., $r_1 \neq r_2^{-1}$) Biedenharn-Macfarlane (BM) (or Fibonacci) q oscillator. The commutation relation which yields (12) can be found if two new operators $\hat{b}, \hat{b}^\dagger$ are defined such that $\hat{b}^\dagger \hat{b} = [[\hat{N}]], \hat{b}\hat{b}^\dagger = [[\hat{N}+1]]$ as

$$\hat{b}\hat{b}^\dagger = r_2 \hat{b}^\dagger \hat{b} + r_1^{\hat{N}}. \quad (13)$$

In principle, Eqs. (10), (12), and (13) plausibly suggest that Eq. (5) can be effectively obtained from the generalized BM q oscillator by a second deformation. Although a direct transformation from one into the other has not been found, recently an attempt has been made to unify all quartic oscillators. In this scheme, (5) and (13) correspond to special cases such as $r_2 = r_1^{-1}$, $r_2 = r_1^2$, or $r_2 = r_1$ (see Ref. [18]). This can be shown by applying the nonlinear transformation

$$\hat{b} = \hat{\bar{b}}(\beta + \gamma \hat{\bar{b}}\hat{\bar{b}}^\dagger)^{1/2}, \quad (14)$$

where $[\hat{\bar{b}}, \hat{N}] = \hat{\bar{b}}$ in Eq. (13), we get the form

$$A\hat{\bar{b}}\hat{\bar{b}}^\dagger\hat{\bar{b}}\hat{\bar{b}}^\dagger + B\hat{\bar{b}}\hat{\bar{b}}^\dagger\hat{\bar{b}}^\dagger\hat{\bar{b}} + C\hat{\bar{b}}^\dagger\hat{\bar{b}}\hat{\bar{b}}^\dagger\hat{\bar{b}}$$

$$+ D\hat{\bar{b}}\hat{\bar{b}}^\dagger + E\hat{\bar{b}}^\dagger\hat{\bar{b}} + F = 0, \quad (15)$$

with the coefficients

$$A = \gamma, \quad B = 0, \quad C = -q\gamma,$$

$$D = \beta, \quad E = -q\beta, \quad F = -1, \quad (16)$$

whereas the HO corresponds to the special case of the generalized quartic oscillator in Eq. (15) with the coefficients

$$A = 0, \quad B = k, \quad C = 0, \quad D = 1, \quad E = -q, \quad F = -1. \quad (17)$$

Both Eqs. (16) and (17) have the property $AE = CD$ which is possessed by the quartic square root oscillator [16] as a special case of (15). In the limit $J \to \infty$ both homographic and Fibonacci oscillators contract to SHO.

III. CENTRAL INVARIANT

The HO algebra in (5) is isomorphic to its underlying su(2) algebra. The range of values which the quantum number n can take is limited by the total angular momentum J (i.e., $0 \leq n \leq 2J$). This can be seen easily by causing the diagonal operator $\hat{C}_\theta$ to act on the angular momentum $|Jm\rangle$ and, simultaneously, on the homographic oscillator $|n\rangle_J$ bases. The action of the $\hat{a}, \hat{a}^\dagger$ operators on the basis

vectors generates lower and upper bounds for its energy spectrum (i.e., $\hat{a}|0\rangle_J = \hat{a}^\dagger|2J\rangle_J = 0$). By direct substitution we find $[2J] = \infty$.

In order to find the central invariant $\hat{C}_q$ we first write the HO algebra in the conventional form

$$[\hat{a}, \hat{N}] = \hat{a}, \quad [\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger, \quad [\hat{a}, \hat{a}^\dagger] = G_q(\hat{N}),$$

where

$$G_q(\hat{N}) = \frac{Q(1+Q)}{(\hat{N}-Q)(\hat{N}-1-Q)}, \quad Q = q/(1-q). \quad (18)$$

$\hat{C}_q$ is then formulated as

$$\hat{C}_q = \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + F(\hat{N}),$$

where

$$F(\hat{N}) = \frac{S\hat{N}^2 + T\hat{N} + U}{(\hat{N}-Q)(\hat{N}-1-Q)} \quad (19)$$

with the coefficients $S = (c_q + 2Q)$, $T = [-c_q(1+2Q) - 2Q^2]$, and $U = (c_q - 1)(Q + Q^2)$. Here c_q is an undetermined eigenvalue of $\hat{C}_q$. It is easy to see also from Eqs. (18) and (19) that there are lower ($n = 0$) and upper ($n = 2J$) bounds in the spectrum such that

$$\hat{a}^\dagger\hat{a}|0\rangle_J = \hat{a}\hat{a}^\dagger|2J\rangle_J = 0$$

and

$$F(2J) - G_q(2J) = F(0) + G_q(0). \quad (20)$$

IV. THE HOMOGRAPHIC q-BOSON REALIZATION OF su(2) DEFORMATION

In the q-boson realization of su(2) deformation, the fundamental spinor realization is mapped onto a pair of commuting homographic oscillators as

$$\hat{I}_+ = \hat{a}_1^\dagger\hat{a}_2, \quad \hat{I}_- = \hat{a}_1\hat{a}_2^\dagger, \quad \hat{I}_z = \frac{1}{2}(\hat{N}_1 - \hat{N}_2), \quad (21)$$

where independent algebras for $\hat{a}_1$ and $\hat{a}_2$ are given by the analogs of (5). Using (5) and (6) the operators in (21) can be found to satisfy

$$[\hat{I}_\pm, \hat{I}_z] = \mp\hat{I}_\pm, \quad [\hat{I}_+, \hat{I}_-] = f_{\hat{I}}(\hat{I}_z) - f_{\hat{I}}(\hat{I}_z - 1), \quad (22)$$

where

$$f_{\hat{I}}(\hat{I}_z) = Q_1 Q_2 \frac{\hat{I} - \hat{I}_z}{\hat{I} - \hat{I}_z - 1 - Q_2} \frac{\hat{I} + \hat{I}_z + 1}{\hat{I} + \hat{I}_z - Q_1}. \quad (23)$$

Here $Q_i = q_i/(1-q_i)$ ($i = 1,2$) and q_i's are the deformation parameters. It is also easy to see that $\hat{I}^2 = 1/2(\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+) + \hat{I}_z^2$ is an invariant of the algebra with eigenvalue $i(i+1)$ where $i = 1/2(n_1 + n_2)$ and $i_z = 1/2(n_1 - n_2)$. Hence $|ii_z\rangle$ form the orthogonal basis vectors of the algebra (22).

As q_1 and q_2 independently approach unity in the zero deformation limit, the deformed algebra in (22) approaches su(2).

Now, our aim is to find the invertible nonlinear transformation between the generators $\hat{I}_{\pm}, \hat{I}_z$ and the generators $\hat{J}_{\pm}, \hat{J}_z$ of the limiting su(2). We seek an invertible operator function $\mathscr{G}(\hat{J}_z)$ such that [11]

$$\hat{I}_+ = \mathscr{G}(\hat{J}_z)\hat{J}_+, \quad \hat{I}_- = \hat{I}_+^\dagger, \quad \hat{I}_z = \hat{J}_z + \text{const}, \quad (24)$$

where the constant only depends on the central invariant. The type of deformation in (22) is not suitable for any Laplace (or Fourier) representation [11] in terms of the powers of $\hat{J}_z$. However, one can still find the central element $\hat{\mathscr{C}}$ of this algebra such that

$$\hat{I}_+\hat{I}_- = \mathscr{G}^2(\hat{J}_z)\hat{J}_+\hat{J}_- = \hat{\mathscr{C}} - \mathscr{F}(\hat{J}_z),$$

$$\hat{I}_-\hat{I}_+ = \hat{J}_- \mathscr{G}^2(\hat{J}_z)\hat{J}_+ = \hat{\mathscr{C}} - \mathscr{F}(\hat{J}_z + 1),$$

$$(25)$$

where $\mathscr{F}(\hat{J}_z)$ is to be found. Since $\hat{I}$ is the group invariant, $\hat{\mathscr{C}}$ is a function of $\hat{I}$ only. Therefore its eigenvalue $c_{Q_1 Q_2}$ only depends on $\hat{I}$ and the deformation parameters q_1, q_2. Furthermore, the existence of the lowermost and uppermost states on which the action of $\hat{I}_-$ and $\hat{I}_+$ yields zero, respectively, implies that $\hat{\mathscr{C}} = \mathscr{F}(-J) = \mathscr{F}(J+1)$. Using the condition $[\hat{I}_{\pm}, \hat{\mathscr{C}}] = 0$, the operator function $\mathscr{F}(\hat{J}_z)$ can be easily found as

$$\mathscr{F}(\hat{J}_z) = -f_{\hat{I}}(\hat{J}_z - 1). \quad (26)$$

From (24), (25), and the su(2) relation $\hat{J}_+\hat{J}_- = \frac{1}{2}(\hat{J}^2 - \hat{J}_z^2 + \hat{J}_z)$ we finally obtain

$$\mathscr{G}^2(\hat{J}_z) = 2\frac{f_{\hat{I}}(\hat{J}_z - 1) + \hat{\mathscr{C}}}{(\hat{J}^2 - \hat{J}_z^2 + \hat{J}_z)}. \quad (27)$$

Equations (23), (26), and (27) define the invertible deformation $\mathscr{G}(\hat{J}_z)$.

V. COPRODUCT

One implication of Sec. IV is that the existence of the simple su(2) coproduct

$$\Delta(\hat{J}_{\pm}) = \hat{J}_{\pm} \otimes \mathbb{I} + \mathbb{I} \otimes \hat{J}_{\pm},$$

$$\Delta(\hat{J}_z) = \hat{J}_z \otimes \mathbb{I} + \mathbb{I} \otimes \hat{J}_z, \quad (28)$$

and the invertibility of $\mathscr{G}(\hat{J}_z)$ guarantee a coproduct law [11] for $\hat{I}_{\pm}, \hat{I}_z$. The deforming map defined in Eqs. (24) and (27) is, however, not in the class of generalized prototype su(2)$_q$ deformations of Curtright and Zachos [11,12]. The nonpolynomial forms of $G_q(\hat{N})$ and $f_{\hat{I}}(\hat{I}_z)$ do not permit a closed analytic form for the coproduct $\Delta(\hat{I}_{\pm})$ and $\Delta(\hat{I}_z)$. However, one can get some hint from the interesting symmetry displayed by Eq. (22) in the limits of very large (i.e., $q_i \to -\infty$) and very small (i.e., $q_i \to 1$) deformations as

$$\lim_{q_i \to 1}[\hat{I}_+, \hat{I}_-] = 2\hat{I}_z,$$

$$\lim_{q_i \to -\infty}[\hat{I}_+, \hat{I}_-] = 4\epsilon\hat{I}_z / \hat{I},$$

where

$$Q_i = -(1 + \epsilon), \quad \epsilon \ll 1. \quad (29)$$

Hence in both limits the deformed algebra (22) behaves like a pure su(2). In principle, $\Delta(\hat{I}_{\pm})$ and $\Delta(\hat{I}_z)$ can be obtained by the application of the deforming invertible transformation found in Eqs. (24) and (27) (e.g., see Ref. [11]) as

$$\Delta(\hat{G}) = T[\mathbb{I} \otimes T^{-1}(\hat{G}) + T^{-1}(\hat{G}) \otimes \mathbb{I}], \quad (30)$$

where $\hat{g} = (\hat{J}_{\pm}, \hat{J}_z)$, $\hat{G} = (\hat{I}_{\pm}, \hat{I}_z)$, and $\hat{G} = T(\hat{g})$ is just a compact notation for the transformation in Eq. (24). Equation (30) implies

$$[\Delta(\hat{I}_{\pm}), \Delta(\hat{I}_z)] = \mp\Delta(\hat{I}_{\pm}), \quad [\Delta(\hat{I}_+), \Delta(\hat{I}_-)] = \Delta(\hat{I}_z). \quad (31)$$

Here we notice that $\Delta(\hat{G})$ should behave like $\Delta(\hat{g})$ in the symmetric limits (i.e., $q_i \to 1, q_i \to -\infty$). A possibly existing simple analytic form of $\Delta(\hat{G})$ might be connected to the closed form (30) by a unitary transformation [12]. However, no explicit and general form for such a transformation is known at the moment.

VI. DISCUSSION

The isomorphism between the homographic oscillator algebra (5) and su(2) has subtle implications in the angular momentum addition theorem and the coproduct law for $\hat{I}_{\pm}, \hat{I}_z$. From (24) and (25) it is easy to see that $[\hat{I}, \hat{J}_{\pm}] = [\hat{I}, \hat{J}_z] = 0$ and $[\hat{I}, \hat{J}^2] = 0$. These commutations further imply an isomorphism between $|ii_z\rangle$ and $|jj_z\rangle$. In other words, the two basis vectors are parallel to each other although they are raised and lowered in different scales on the z axis [19]. We also note that $\hat{I}$ is a function of $\hat{J}$ only. Let us now define i and i_z as quantum numbers corresponding to a basis set $|ii_z\rangle$ on which the generators in (22) apply. Then, using (27) and acting Eq. (24) on this basis, one obtains

$$j(j+1) = c_{Q_1 Q_2} + Q_1 Q_2 \frac{i(i+1)}{(i - Q_1)(i - Q_2 - 1)}, \quad (32)$$

which is the desired relation between $\hat{I}$ and $\hat{J}$. Here $c_{Q_1 Q_2}$ is the eigenvalue of $\hat{\mathscr{C}}$. In the undeformed limit (i.e., $Q_1, Q_2 \to \infty$) the equivalence of the two algebras requires $c_{Q_1 Q_2} \to 0$.

The arguments presented above guarantee the existence of an implicit coproduct, making it possible to consider $\hat{I}_+$ and $\hat{I}_z$ as elements of a quantum deformation of su(2). The deformation parameter is shown to be determined by the *total angular momentum J*. Our work is under progress to extend the arguments presented above to the most general case of quartic oscillator algebra. To the knowledge of the authors, such generalized cases have also been examined recently by Smith [20] as applied to the more general nonunitary case $\hat{I}_- \ne (\hat{I}_+)^\dagger$.

The stereographic projection is intimately connected with the polar construction of su(2) generators, the quantum phase problem, and, in particular, certain geometrical realizations [21] of quantum Stokes parameters of polarization [14,21,22]. Nevertheless, the nonunitarity of the azimuthal phase $\hat{E}_\phi$ and/or the unavoidable nonzero commutations between the azimuthal and polar phase operators [e.g., see Eq. (4) here] plague the polar operator construction of su(2). As briefly mentioned in Sec. II, the resolution was given by Ellinas [6] by adding a cyclic property to the matrix elements of $\hat{E}_\phi$ along the J_z axis with a periodicity of $(2J+1)$. This new term does not affect the original spectrum and further makes $\hat{E}_\phi$ manifestly unitary.

The homographic oscillator representation is physically relevant for its application, particularly in the weak intensity regime of quantum Stokes parameters [14,21]. The quantum phase problem has been studied in the context of an operational point of view by Noh, Fougéres, and Mandel using two coherent laser sources [23]. More recently this formalism has been applied to the case of polarization measurement of weak fields [14]. Using Poincaré's stereographic projection, the angular parameters of the polarization ellipse can be mapped conveniently on the Stokes parameters. This has a certain advantage from the operational point of view. The direct measurement of the quantum Stokes parameters might be more relevant in determining the orientation of the polarization ellipse both for experimental perspective and the suitable group properties that quantum Stokes parameters possess. This is where the authors believe that the homographic oscillator introduced here can be linked with the quantum Stokes parameters and polarization measurement. Another dimension of our work in progress is to exploit this physical application.

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