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Second-order second-degree Painleve equations related to´ Painleve IV,V,VI equations´

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Abstract. The algorithmic method introduced by Fokas and Ablowitz to investigate the transformation properties of Painleve equations is used to obtain one-to-one correspondence´ between the Painleve IV, V and VI equations and the second-order second-degree equations of´ Painleve type.´

1. Introduction

Painleve and his school addressed a question raised by Picard concerning second-order´ first-degree ordinary differential equations of the form

$$y'' = F(z,y,y') \tag{1.1}$$

where F is rational in y', algebraic in y and locally analytic in z, which have the property that singularities other than poles of any of the solutions are fixed [22, 15, 17]. This property is known as the Painleve property. Within the M´ obius transformation, Painlev¨ e´ and his colleagues found that there are 50 canonical equations of the form (1.1). Among these equations six are irreducible and define classical Painleve transendents PI–PVI. These´ may be regarded as nonlinear counterparts of some of the classical special functions. For example, PIII has solutions which have similar properties to the Bessel functions. The solutions of the 11 equations of the remaining 44 equations can be expressed in terms of the solutions of the Painleve equations PI, PII, or PIV and 33 equations are solvable in´ terms of the linear equations of order two or three, or are solvable in terms of the elliptic functions.

Although the Painleve´ equations were discovered from strictly mathematical considerations they have appeared in many physical problems. Besides their physical importance, the Painleve equations possess a rich internal structure. Some of these properties´ can be summarized as follows. (i) For a certain choice of parameters PII–PVI admit oneparameter families of solutions which are either rational or expressible in terms of the classical transcendental functions. For example, PVI admits a one-parameter family of solutions expressible in terms of hypergeometric functions [9]. (ii) There are transformations (Backlund or Schlesinger) associated with PII–PVI, these transformations map the solution¨ of a given Painleve equation to the solution of the same equation but with different values´ of parameters [11, 19, 20]. (iii) PI–PV can be obtained from PVI by the process of contraction [17]. It is possible to obtain the associated transformations for PII–PV from

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the transformation for PVI. (iv) They can be obtained as the similarity reduction of the nonlinear partial differential equations solvable by the inverse scattering transform (IST). (v) PI–PVI can be considered as the isomonodromic conditions of suitable linear system of ordinary differential equations with rational coefficients possessing both regular and irregular singularities [18]. Moreover, the initial value problem of PI–PVI can be studied by using the inverse monodromy transform (IMT) [12, 13, 21].

The Riccati equation is the only example for the first-order first-degree equation which has the Painleve property.´ Before the work of Painleve and his school, Fuchs [14, 17]´ considered equations of the form

$$F(z,y,y^0) = 0 \tag{1.2}$$

where F is polynomial in y and y^0, and locally analytic in z, such that the movable branch points are absent, that is, the generalization of the Riccati equation. Briot and Bouquet [17] considered a subcase of (1.2), that is, first-order binomial equations of degree $m \in Z_+$:

$$(y_0)^m + F(z,y) = 0 \tag{1.3}$$

where F(z,y) is a polynomial of degree at most 2m in y. It was found that there are six types of equation of the form (1.3). However, all these equations are either reducible to a linear equation or solvable by means of elliptic functions [17]. Second-order binomial-type equations of degree m > 3

$$(y_{00})_m + F(z,y,y_0) = 0 \tag{1.4}$$

where F is polynomial in y and y^0 and locally analytic in z, were considered by Cosgrove [4]. It was found that there are nine classes. Only two of these classes have arbitrary degree m and the others have degree three, four and six. As in the case of first-order binomialtype equations, all these nine classes are solvable in terms of the first, second and fourth Painleve transcendents, elliptic functions or by quadratures. Chazy [3], Garnier [16] and´ Bureau [1] considered third-order differential equations possessing the Painleve property of˝ the following form

$$y^{000} = F(z,y,y^0,y^{00}) \tag{1.5}$$

where F is assumed to be rational in y,y^0,y^{00} and locally analytic in z. However, in [1] the special form of $F(z,y,y^0,y^{00})$

$$F(z,y,y^0,y^{00}) = f_1(z,y)y^{00} + f_2(z,y)(y^0)^2 + f_3(z,y)y^0 + f_4(z,y) \tag{1.6}$$

where $f_k(z,y)$ are polynomials in y of degree k with analytic coefficients in z was considered. In this class no new Painleve transcendents were discovered and all of them´ are solvable either in terms of the known functions or one of six Painleve transcendents.´ Second-order second-degree Painleve-type equations of the following form´

$$(y^{00})^2 = E(z,y,y^0)y^{00} + F(z,y,y^0) \tag{1.7}$$

where E and F are assumed to rational in y and y^0 and locally analytic in z were the subject of [2, 8]. In [2] the special case of (1.7)

$$y^{00} = M(z,y,y^0) + \sqrt{N(z,y,y^0)} \tag{1.8}$$

was considered, where M and N are polynomials of degree 2 and 4 respectively in y^0, rational in y and locally analytic in z. Also, in this classification, no new Painleve´ transcendents were found. In [8], the special form, E = 0 and hence F polynomial in y and y^0 of (1.7) was considered and six distinct classes of equations were obtained by using the so-called α-method. These classes were denoted by SD-I,...,SD-VI and all are solvable in terms of the classical Painleve transcendents (PI,´ ...,PVI), elliptic functions or solutions of the linear equations.

Second-order second-degree equations of Painleve type appear in physics [5–7].´ Moreover, second-degree equations are also important in determining transformation properties of the Painleve equations [10, 11]. In [11], the aim was to develop an algorithmic´ method to investigate the transformation properties of the Painleve equations. However,´ certain new second-degree equations of Painleve type related to PIII and PVI were also´ discussed. By using the same notation, the algorithm introduced in [11] can be summarized as follows: let v(z) be a solution of any of the 50 Painleve equations, as listed by Gambier´ [15] and Ince [17], each of which takes the form

$$v^{00} = P_1(v^0)^2 + P_2v^0 + P_3 \tag{1.9}$$

where P_1,P_2,P_3 are functions of v,z and a set of parameters α. The transformation i.e. Lie-point discrete symmetry which preserves the Painleve property of (1.9) of the form´ $u(z;\hat{\alpha}) = F(v(z;\alpha),z)$ is the Mobius transformation

$$u(z;\hat{\alpha}) = \frac{a_1(z)v + a_2}{a_3(z)v + a_4(z)} \quad (z) \tag{1.10}$$

where $v(z,\alpha)$ solves (1.9) with the set of parameters α, and $u(z;\hat{\alpha})$ solves (1.9) with the set of parameters $\hat{\alpha}$. Lie-point discrete symmetry (1.10) can be generalized by involving the $v^0(z;\alpha)$, i.e. the transformation of the form $u(z;\hat{\alpha}) = F(v^0(z;\alpha),v(z,\alpha),z)$. The only transformation which contains v^0 linearly is the one involving the Riccati equation, i.e.

$$u(z,\hat{\alpha}) = \frac{v' + av^2 + bv + c}{dv^2 + ev + f} \tag{1.11}$$

where a,b,c,d,e,f are functions of z only. The aim is to find a,b,c,d,e,f such that (1.11) define a one-to-one invertible map between solutions v of (1.9) and solutions u of some second-order equation of the Painleve type. Let´

$$J = dv^2 + ev + f \qquad Y = av^2 + bv + c \tag{1.12}$$

then differentiating (1.11) and using (1.9) to replace v^{00} and (1.11) to replace v^0, one obtains,

$$Ju' = [P_1J^2 - 2dJv - eJ]u^2 + [-2P_1JY + P_2J + 2avJ + bJ + 2dvY + eY - (d'v^2 + e'v + f')]u + [P_1Y^2 - P_2Y + P_3 - 2avY - bY + a'v^2 + b'v + c']. \qquad (1.13)$$

There are two distinct cases.

(1) Find a,...,f such that (1.13) reduce to a linear equation for v,

$$A(u^0,u,z)v + B(u^0,u,z) = 0. \qquad (1.14)$$

Having determined a,...,f upon substitution of $v = -B/A$ in (1.11) one can obtain the equation for u, which will be one of the 50 Painleve equations.´

(2) Find a,...,f such that (1.13) reduces to a quadratic equation for v,

$$A(u_0,u,z)v_2 + B(u_0,u,z)v + C(u_0,u,z) = 0. \qquad (1.15)$$

Then (1.11) yields an equation for u which is quadratic in the second derivative. As mentioned before in [11] the aim is to obtain the transformation properties of PII–PVI. Hence, case 1 for PII–PV, and case 2 for PVI was investigated.

In this article, we investigate the transformation of type 2 to obtain the one-to-one correspondence between PIV, PV, PVI and the second-order second-degree Painleve-type´ equations. Similar work has been carried out for PI, PII, PIII in [23]. Some of the seconddegree equations related to PIV–PVI has been obtained in [2, 8] without giving the relation between the Painleve equations but many of them have not been considered in the literature.´ Instead of having the transformation of the form (1.11) which is linear in v^0, one may use the appropriate transformations related to

$$(v^0)^m + \sum_{j=1}^{m} P_j(z,v)(v^0)^{m-j} \qquad m > 1 \qquad (1.16)$$

where $P_j(z,v)$ is a polynomial in v, which satisfies the Fuchs theorem concerning the absence of movable critical points [14, 17]. This type of transformation yields the relation between the Painleve equations PI–PVI and higher-order higher-degree Painlev´ e-´ type equations. Throughout this article ^{0} denotes the derivative with respect to z and $\cdot$ denotes the derivative with respect to x.

2. Painleve IV´

Let v(z) be a solution of the PIV equation,

$$v'' = \frac{1}{2v}(v')^2 + \frac{3}{2}v^3 + 4zv^2 + 2(z^2 - \alpha)v + \frac{\beta}{v}. \qquad (2.1)$$

Then, for PIV the equation (1.13) takes the form

$$A_4v_4 + A_3v_3 + A_2v_2 + A_1v + A_0 = 0 \qquad (2.2)$$

where

$A_4 = 3[d^2u^2 - 2adu + a^2 - 1]$

$A_3 = 2[du^0 + 2deu^2 + (d^0 - 2ae - 2bd)u - (a^0 - 2ab + 4z)]$

(2.3)

$A_2 = 2eu^0 + (2df + e^2)u^2 + 2(e^0 - af - be - cd)u - (2b^0 - b^2 - 2ac + 4z^2 - 4\alpha)$ $A_1 = 2(fu^0 +$

$f\,{}^0u - c^0)$ $A_0 = -(f^2u^2 - 2cfu + c^2 + 2\beta)$.

Now, the aim is to choose a,b,...,f so that (2.2) becomes a quadratic equation for v. There are three cases: (1) $A_4 = A_3 = 0$, (2) $A_4 = 0, A_3$ 6= 0 and (3) A_4 6= 0.

Case 1. In this case the only possibility is $e = d = 0, a^2 = 1$ and $b = 2az$. One can always absorb c and f in u by a proper Mobius transformation, and hence, without loss of¨ generality, one can set $c = 0$, and $f = 1$. Then equation (2.2) takes the following form,

$$2(au + 2a - 2\alpha)v^2 - 2u^0v + u^2 + 2\beta = 0. \qquad (2.4)$$

Following the procedure discussed in the introduction yields the following second-order second-degree Painleve-type equation for´ u(z)

$$[u^{00} - 3au^2 - 4(a - \alpha)u - 2a\beta]^2 = 4z^2[u^{02} - 2(au + 2a - 2\alpha)(u^2 + 2\beta)] \qquad (2.5)$$

and there exits the following one-to-one correspondence between solutions v(z) of PVI and solutions u(z) of the equation (2.5)

$$u = v' + av^2 + 2azv \qquad v = -\frac{u'' - 2azu' - 3au^2 - 4(a-\alpha)u - 2a\beta}{4az(au + 2a - 2\alpha)}. \qquad (2.6)$$

The change of variable $u(z) = 4ay(x)$, $x = a\sqrt{2}\,z$ transforms equation (2.5) into the equation

$$\left(\ddot{y} - \frac{1}{2}\frac{\partial Q_3(y)}{\partial y}\right)^2 = x^2[\dot{y}^2 - Q_3(y)]$$
$$Q_3(y) := 4y^3 + 2(a-\alpha)y^2 + \tfrac{1}{2}\beta y + \tfrac{1}{4}\beta(a-\alpha). \qquad (2.7)$$

Equation (2.7) was first obtained by Bureau [2, equation 19.2, p 207] without giving the relation to PV and the special case, $\alpha = a$, of (2.7) was solved in terms of the first Painleve´ transcendent. However, we have not been able to recover this relation.

Case 2. In this case $d = 0$, $a^2 = 1$ and to reduce the equation (2.2) to a quadratic equation for v one should take e 6= 0. Since one can always absorb b and e in u by a proper Mobious¨ transformation, without loss of generality one may take $b = 0$, $e = 1$. Now equation (2.2) can be written as

$$(v + f)(Av^2 + Bv + C) = 0 \qquad (2.8)$$

where

$$A = 4(au + 2z) \quad B = -[2u' + u^2 - 2afu + 2(ac + 4zf - 2z^2 + 2\alpha)] \quad C = fu^2 - 2(f' - af^2)u + 2c' + 2f(ac + 4zf - 2z^2 + 2\alpha) \tag{2.9}$$

and c,f satisfy the following equations

$$f(f' - af^2 - c) = 0 \qquad 2f[c' + f(ac + 4zf - 2z^2 + 2\alpha)] = c^2 + 2\beta. \tag{2.10}$$

If $f \neq 0$, then equations (2.10) give

$$c = f' - af^2 \qquad f'' = \frac{1}{2f}(f')^2 + \frac{3}{2}f^3 - 4zf^2 + 2(z^2 - \alpha)f + \frac{\beta}{f}. \tag{2.11}$$

Let $\mu = (-2\beta)^{1/2}$, and $u(z) = \xi(z)y(x) + \eta(z)$, $x = \zeta(z)$, where

$$\begin{aligned}
\eta(z) &= \frac{1}{f}(f' - af^2 + \mu)\\
\xi(z) &= \exp\left[-\int^z \{\eta(\hat z) + af(\hat z)\}\,\mathrm{d}\hat z\right]\\
\zeta(z) &= -\tfrac{1}{2}\int^z \xi(\hat z)\mathrm{d}\hat z.
\end{aligned} \tag{2.12}$$

Then one obtains the following second-order second-degree equation of Painlevé type for $y(x)$

$$\begin{aligned}
&[4(y + a_0)(\ddot y - 2y\dot y) - (\dot y - y^2)^2 - P_2(y)(\dot y - y^2) - Q_3(y)]^2\\
&\quad = [\dot y - y^2 - R_2(y)][(\dot y - y^2)^2 - S_3(y)]
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
P_2(y) &= 2p^{-2}[3p^2y^2 + 2(p^2a_0 - 2ah - 2pq)y - p^2a_0^2\\
&\qquad - a_0(7ah + 8pq) + 2\alpha - a\mu - 2a]\\
Q_3(y) &= 16p^{-2}(y + a_0)[hy^2 + (h + 2\mu)y + \mu]\\
R_2(y) &= 2p^{-2}[5p^2y^2 + 2p(3pa_0 + 2q)y + p^2a_0^2\\
&\qquad + (4pq - ah)a_0 - 2\alpha + a\mu + 2a]\\
S_3(y) &= 16ap^{-2}y(y + a_0)(hy + 2\mu).
\end{aligned} \tag{2.14}$$

The coefficients in equations (2.13) and (2.14) are given as follows

$$\begin{aligned}
&h = 2\mu x + \nu \qquad |\mu| + |\nu| \neq 0 \qquad \nu = \text{constant}\\
&p(x) := \xi(z) \qquad q(x) := \eta(z) \qquad a_0 := \frac{1}{\xi(z)}[\eta(z) + 2az].
\end{aligned} \tag{2.15}$$

The functions $p(x)$, $q(x)$, and $a_0(x)$ satisfy the following equations

$$\begin{aligned}
&p\dot p - 2(pq + ah) = 0 \qquad p\dot q + p^2a_0^2 - 2(pq + 2h)a_0 - 2(a\mu - 2\alpha) = 0\\
&p^2(\dot a_0 + a_0^2) - aha_0 - 2\alpha + a\mu + 2a = 0.
\end{aligned} \tag{2.16}$$

The one-to-one correspondence between solutions v(z) of the PIV and solutions y(x) of the equation (2.13) is given as follows

$$y = \frac{v' + av^2 - \eta v - \mu}{\xi(v+f)} \qquad 4a(y + a_0)v^2 + p(\dot{y} - y^2)v + y(hy + 2\mu) = 0. \qquad (2.17)$$

If f = 0, then equation (2.10) gives $c = (-2\beta)^{1/2}$. Thus C = 0 and equation (2.8) gives a linear equation for v. Then one obtains the following transformation for the PIV equation [11]

$$\begin{aligned} \bar{v} &= \frac{1}{2v}[v' - v^2 - 2zv - (-2\beta)^{1/2}] \\ \bar{\alpha} &= \tfrac{1}{4}[2 - 2\alpha + 3(-2\beta)^{1/2}] \qquad \bar{\beta} = -\tfrac{1}{2}[1 + \alpha + \tfrac{1}{2}(-2\beta)^{1/2}]^2 \end{aligned} \qquad (2.18)$$

Case 3. In this case equation (2.2) can be written as follows

$$(v^2 + \tilde{g}v + \tilde{h})(\tilde{B}_2 v^2 + B_1 v + B_0) = 0 \qquad (2.19)$$

where B_j, j = 1,2,3 are functions of u^0, u, z and $\tilde{g}$, $\tilde{h}$ are functions of z only. One may consider the two subcases (3.1) d = 0, and (3.2) d ≠ 0 separately.

Case 3.1. If d = 0, then one must take $f = \tilde{g} = \tilde{h} = 0$ and $c = \mu, \mu := (-2\beta)^{1/2}$. Since e ≠ 0 then without loss of generality one can take b = 0, e = 1. The quadratic equation for v becomes

$$3(a^2 - 1)v^2 - 2(au + a^0 + 4z)v + 2u^0 + u^2 + 2(\mu a - 2z^2 + 2\alpha) = 0. \qquad (2.20)$$

One may distinguish between the two cases $a^2 + 3 = 0$ and $a^2 + 3 \neq 0$.

Case 3.1.1. If $a^2 + 3 = 0$, then by using the change of variable $u = 2(\nu y + \frac{1}{a\nu}x)$, $x = \nu z$, where ν is a nonzero constant, one obtains the following second-order second-degree equation of Painleve´ type for y(x)

$$[\ddot{y} - \lambda x\dot{y} + (\lambda - 2\lambda^2x^2)y - 2\kappa\lambda x]^2 = -4(y^2 + \sigma)^2(\dot{y} + \lambda xy + \kappa) \qquad (2.21)$$

where

$$\lambda = \frac{-2}{a\nu^2} \quad \kappa = \frac{1}{2\nu^2}\left(2\alpha + a\mu + \frac{2}{a}\right) \quad \sigma = \frac{1}{2\nu^2}\left(2\alpha - a\mu - \frac{2}{a}\right). \qquad (2.22)$$

The one-to-one correspondence between solutions v(z) of the PIV and solution y(x) of the equation (2.21) is given by

$$\begin{aligned} y &= \frac{av' - 3v^2 - 2zv + a\mu}{2a\nu v} \\ & 3v^2 + 2a\nu(y - \lambda x)v - \nu^2(\dot{y} + y^2 - \lambda xy + \lambda^2x^2 + \kappa) = 0 \end{aligned} \qquad (2.23)$$

Equation (2.21) was first obtained by Cosgrove [8], and was labelled as SD-IV^0.A.

Case 3.1.2. If $a^2 + 3$ 6= 0, let z = r(x), a(z) := s(x), and u(z) = p(x)y(x) + q(x), where p(x),q(x) and r(x) satisfy the following equations

$$p(s^2+3)\dot{r} = 6(s^2-1) \qquad (s^2+3)(\dot{p}-2q) = 4s\left(\frac{\dot{s}}{\dot{r}}+4r\right)$$
$$(s^2+3)(p\dot{q}-q\dot{p}+q^2) = \left(\frac{\dot{s}}{\dot{r}}+4r\right)^2 - 6(s^2-1)(\mu s - 2r^2 + 2\alpha). \tag{2.24}$$

such that $(s^2 - 1)(s^2 + 3)$ 6= 0. Then the equations

$$py + q = \frac{v' + av^2 + \mu}{v} \qquad Av^2 + Bv + C = 0 \tag{2.25}$$

where

$$A = 3(s^2-1) \qquad B = -2\left(2spy + 2sq + \frac{\dot{s}}{\dot{r}} + 4r\right)$$
$$C = \frac{1}{3(s^2-1)}\left[p^2(s^2+3)\dot{y} + 3p^2(s^2-1)y^2 + 4ps\left(2sq + \frac{\dot{s}}{\dot{r}} + 4r\right)y + \left(2sq + \frac{\dot{s}}{\dot{r}}\right)^2\right] \tag{2.26}$$

give a one-to-one correspondence between solutions v(z) of the PIV and solutions y(x) of the following second-order second-degree equation of Painleve type´

$$[\ddot{y} + 2(y+a_0)\dot{y} - 4y^2(y+a_0)]^2 = \frac{-16}{(s^2+3)^2}[s(s^2-9)y^2 + c_1 y + c_0]^2(\dot{y} - y^2). \tag{2.27}$$

The functions a_0, c_1, c_0 are given in terms of the functions p(x), q(x), r(x) and s(x) as follows

$$a_0 = \frac{1}{p(s^2+3)}\left[2q(s^2+3) + \frac{s(s^2-9)}{6(s^2-1)}p\dot{s} + 16rs\right]$$
$$c_1 = \frac{2}{p}[sq(s^2-9) - 2r(5s^2+3)] + \frac{(s^2+3)^2}{6(s^2-1)}\dot{s}$$
$$c_0 = -\frac{(s^2+3)^2}{12(s^2-1)}\ddot{s} + \frac{s(s^2+3)(s^2+15)}{18(s^2-1)^2}(\dot{s})^2 - \frac{1}{p^2}[sq^2(s^2-9) - 4rq(5s^2+3) - 32sr^2] + \frac{3}{p^2}(s^2-1)[\mu(s^2-3) + 4(sr^2 - \alpha s + 1)] \tag{2.28}$$

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Case 3.2. If d 6= 0, then without lose of generality one can take a = 0, d = 1. Equation (2.2) can be written as follows

$$(v^2 + g\tilde{v} + f)(B_2v^2 + B_1v + B_0) = 0 \tag{2.29}$$

where

$B_2 = 3(u^2 - 1)$ $\qquad$ $B_1 = 2u^0 + (4e - 3g)\tilde{u}^{\,2} - 4bu + 3\tilde{g} - 8z$

$$B_0 = 2(e - g)\tilde{u}^0 + (e^2 - \tilde{g}^2 - f)u^2 + 2(e^0 + 2b\tilde{g} - be - c)u \quad (2.30)$$

$$-(2b^0 - b^2 + 4z^2 - 4\alpha) - g(\tilde{} 3\tilde{g} - 8z) + 3f$$

and $b,c,e,f,\tilde{g}$ satisfy the following equations:

$$g(\tilde{e} - g)\tilde{} = 0 \; f(e - g)\tilde{} = 0 \; f(e^0 + be_2 - 2c)_2 = 0 \; f + 2bf_2 = g(\tilde{e}^0 + be_2 - c) \quad (2.31)$$

$$f(2b^0 - b + 3e - 8ze - 3f + 4z - 4\alpha) = c + 2\beta \; g(\tilde{} 2b^0 - b^2 + 3e^2 - 8ze + 4z^2 -$$

$$4\alpha) = 2c^0 + 2f(3e - 4z).$$

The following three subcases (3.2.1) $f = \tilde{g} = 0$, (3.2.2) $f = 0, \tilde{g} 6= 0$, (3.2.3) $f 6= 0$ may be considered separately.

Case 3.2.1. If $f = \tilde{g} = 0$, then equation (2.31) gives $c = \mu, \mu^2 + 2\beta = 0$. Let $z = r(x)$, $u(z) = p(x)y(x)+q(x)$ where $p(x),q(x)$ and $r(x)$ are solutions of the following equations

$$p\dot{r} = 1 \qquad \dot{p} = 2(t + sq) \qquad p\dot{q} = sq^2 + 2tq - 3s + 4r \quad (2.32)$$

and $t(x)$ and $s(x)$ are arbitrary functions. Moreover, defining $b(z) := t(x)$, $e(z) := s(x)$ then the quadratic equation for v can be written as

$$3(y^2 + 2a_1y + a_0)v^2 + 2(\dot{y} - sy^2)v + (2c_1y + c_0) = 0 \quad (2.33)$$

where

$$a_1 = p_{-1}q \quad a_0 = p_{-2}(q_2 - 1) \quad c_1 = \dot{s} + p_{-1}(ts - \mu) \quad (2.34) \; c_0 = p^{-2}[2pqc_1 -$$

$$(2p\dot{t} - t^2 - 8rs + 3s^2 + 4r^2 - 4\alpha)].$$

Assume that c_1 and c_0 are not both zero, then equations (2.33) and

$$py + q = \frac{v' + bv + c}{v^2 + ev} \quad (2.35)$$

give one-to-one correspondence between solutions $v(z)$ of the PIV and solutions $y(x)$ of the following second-order second-degree Painleve-type equation`

$$[P_3(y)(\ddot{y} - 2sy\dot{y} - \dot{s}y^2) - 2Q_2(y)(\dot{y} - sy^2)^2 - R_4(y)(\dot{y} - sy^2) + P_3(y)F_2(y)]^2$$
$$= [2F_2(y)(\dot{y} - sy^2) + G_4(y)]^2[(\dot{y} - sy^2)^2 - P_3(y) \quad]$$

(2.36) where

$$P_3(y) = 6(c_1y^3 + g_2y^2 + g_1y + g_0)$$
$$Q_2(y) = 5c_1y^2 + 2(g_2 + 2a_1c_1)y + 3g_1 - 2a_1c_0$$
$$R_4(y) = 3[3sc_1y^4 + (\dot{c}_1 + 2sg_2)y^3 + (\dot{g}_2 + sg_1)y^2 + \dot{g}_1y + \dot{g}_0]$$
$$F_2(y) = c_1y^2 - 2(g_2 - 4a_1c_1)y + 2g_1 - 3a_1c_0$$
$$\begin{aligned} G_4(y) = 3\bigg[&sc_1y^4 + \left(\dot{c}_1 + \frac{\dot{p}}{p}c_1 + 4sa_1c_1\right)y^3 \\ &+ \left(\dot{g}_2 + \frac{\dot{p}}{p}g_2 - 4\dot{a}_1c_1 + 3sg_1 - 2sa_1c_0\right)y^2 \\ &+ \left(\dot{g}_1 + \frac{\dot{p}}{p}g_1 - 2\dot{a}_0c_1 - 2\dot{a}_1c_0 + 2sg_0\right)y + \dot{g}_0 + \frac{\dot{p}}{p}g_0 - \dot{a}_0c_0\bigg] \end{aligned} \tag{2.37}$$

and

$$g_2 = \tfrac{1}{2}(c_0 + 4a_1c_1) \qquad g_1 = a_0c_1 + a_1c_0 \qquad g_0 = \tfrac{1}{2}a_0c_0. \tag{2.38}$$

When $c_1 = c_0 = 0$, one obtains

$$e^0 + be - c = 0 \qquad 2b^0 - b^2 + 3e^2 - 8ze + 4(z^2 - \alpha) = 0 \tag{2.39}$$

and equation (2.33) reduces to a linear equation for v. In this case $w = \frac{u+1}{u-1}$ solves PXLII in [17, p 341].

Case 3.2.2. If f = 0, g 6= 0, then equation (2.31) gives g = e, and

$$c = (-2\beta)^{1/2} \qquad e^0 + be - (-2\beta)^{1/2} = 0 \qquad 2b^0 - b^2 + 3e^2 - 8ze + 4(z^2 - \alpha) = 0. \tag{2.40}$$

In this case, $B_0 = 0$ in equation (2.30) and equation (2.29) is linear in v, $B_2v + B_1 = 0$. This case is the same as case 3.2.1 when $c_1 = c_0 = 0$.

Case 3.2.3. If f 6= 0, then equation (2.31) gives g = e and

$$\begin{aligned} &c = \tfrac{1}{2}(e' + be) \qquad f' + 2bf - ce = 0 \qquad f[2c' + f(3e - 8z)] = e(c^2 + 2\beta) \\ &f(2b' - b^2 + 3e^2 - 8ze - 3f + 4z^2 - 4\alpha) = c^2 + 2\beta. \end{aligned} \tag{2.41}$$

Note that if e = 0, then equations (2.41) imply c = 0,f = 0 which contradicts the assumption f 6= 0, thus one has to take e 6= 0. Let $\mu = (-2\beta)^{1\,2}$ and let η(z), x = ζ(z), $u(z) = \xi(z)y(x) +$ where

$$\begin{aligned} \xi(z) &= \exp\left[-\int^z \frac{f'(\hat{z}) + \mu e(\hat{z})}{f(\hat{z})}\,\mathrm{d}\hat{z}\right] \\ \zeta(z) &= -\frac{1}{2}\int^z \xi(\hat{z})e(\hat{z})\,\mathrm{d}\hat{z} \\ \eta(z) &= \frac{1}{2f}(e' + be + 2\mu). \end{aligned} \tag{2.42}$$

Then the equation (2.29) can be written as

$$3[py^2 + 2qy + p^{-1}(q^2 - 1)]v^2 - s(\dot{y} - y^2)v - y(hy + 2\mu) = 0 \tag{2.43}$$

where

$$h(x) = 2\mu x + \nu \qquad |\mu| + |\nu| \neq 0 \qquad \nu = \text{constant}$$
$$p(x) := \xi(z) \qquad q(x) := \eta(z) \qquad s(x) := \xi(z)e(z). \tag{2.44}$$

Equation (2.43) and

$$y = \frac{v' - \eta v^2 - (e\eta - b)v - \mu}{\xi(v^2 + ev + f)} \tag{2.45}$$

give one-to-one correspondence between solutions v(z) of the PIV and solutions y(x) of the following second-order second-degree Painleve equation´

$$\begin{aligned}[F_2(y)(\ddot{y} - 2y\dot{y}) - 2(py + q)(\dot{y} - y^2)^2 - P_3(y)(\dot{y} - y^2) + F_2(y)Q_3(y)]^2 \\ = [2(py + q)(\dot{y} - y^2) - R_3(y)]^2[(\dot{y} - y^2)^2 + S_4(y)]\end{aligned} \tag{2.46}$$

where

$$\begin{aligned}
F_2(y) &= 3[py^2 + 2qy + p^{-1}(q^2 - 1)]\\
P_3(y) &= 3\left[2py^3 + \left(\frac{1}{2}\dot{p} - p\frac{\dot{s}}{s} + 3q\right)y^2 + \left(\dot{q} - 2q\frac{\dot{s}}{s} + \frac{q^2 - 1}{p}\right)y + \frac{1}{p}q\dot{q}\right.\\
&\qquad \left. - \frac{1}{p}(q^2 - 1)\left(\frac{\dot{p}}{2p} + \frac{\dot{s}}{s}\right)\right]\\
Q_3(y) &= 4s^{-2}\{4hpy^3 + (7hq + 5\mu p)y^2 + [8\mu q + 3hp^{-1}(q^2 - 1)]y\\
&\qquad + 3\mu p^{-1}(q^2 - 1)\}\\
R_3(y) &= 3\left[2py^3 + 6qy^2 + \frac{1}{p}(q\dot{p} - p\dot{q} + 5q^2 - 3)y + \frac{1}{p^2}(q^2 - 1)(\dot{p} + q) - \frac{1}{p}q\dot{q}\right]\\
S_4(y) &= 12s^{-2}y(hy + 2\mu)[py^2 + 2qy + p^{-1}(q^2 - 1)].
\end{aligned} \tag{2.47}$$

3. Painleve V´

Let v(z) be a solution of the fifth Painleve equation, PV,´

$$v'' = \frac{3v - 1}{2v(v - 1)}(v')^2 - \frac{1}{z}v' + \frac{\alpha}{z^2}v(v - 1)^2 + \frac{\beta(v - 1)^2}{z^2 v} + \frac{\gamma}{z}v + \frac{\delta v(v + 1)}{v - 1}. \tag{3.1}$$

Equation (1.13) becomes a fifth-order polynomial in v as follows

$$A_5v^5 + A_4v^4 + A_3v^3 + A_2v^2 + A_1v + A_0 = 0 \tag{3.2}$$

where

$$
\begin{aligned}
A_5 &= (du-a)^2 - \frac{2\alpha}{z^2}\\
A_4 &= 2(du'+d'u-a') - 3(du-a)^2 + \frac{2}{z}(du-a) + \frac{6\alpha}{z^2}\\
A_3 &= 2(eu'+e'u-b') - 2(du'+d'u-a') - (eu-b)^2 + \frac{2}{z}(eu-b)\\
&\quad -2(du-a)\left[2(eu-b)+(fu-c)+\frac{1}{z}\right] - \frac{6\alpha}{z^2} - \frac{2\beta}{z^2} - \frac{2\gamma}{z} - 2\delta\\
A_2 &= 2(fu'+f'u-c') - 2(eu'+e'u-b') - (eu-b)^2 - \frac{2}{z}(eu-b)\\
&\quad -2(fu-c)\left[2(eu-b)+(du-a)-\frac{1}{z}\right] + \frac{2\alpha}{z^2} + \frac{6\beta}{z^2} + \frac{2\gamma}{z} - 2\delta\\
A_1 &= -\left[2(fu'+f'u-c') + 3(fu-c)^2 + \frac{2}{z}(fu-c) + \frac{6\beta}{z^2}\right]\\
A_0 &= (fu-c)^2 + \frac{2\beta}{z^2}.
\end{aligned}
\tag{3.3}
$$

Note that if $A_5 = 0$, then $A_4 = 0$. Therefore in order to reduce (3.2) to a quadratic equation for v one must consider the following three cases: (1) $A_5 = A_3 = 0$, (2) $A_5 = 0, A_3 \neq 0$ and (3) $A_5 \neq 0$.

Case 1. If $A_5 = A_3 = 0$, then $d = e = 0$. Thus one should take $f \neq 0$ and hence without loss of generality $c = 0$, $f = 1$. This gives $a = \alpha = 0$ and

$$2b' + b^2 + \frac{2}{z}b + \frac{2\beta}{z^2} + \frac{2\gamma}{z} + 2\delta = 0. \tag{3.4}$$

Let $zu(z) = p(x)y(x) + \mu$, $z = r(x) = \exp[-2\int^x \frac{1}{p(\hat{x})}\,\mathrm{d}\hat{x}]$, where $p(x) = 2\mu x + \nu$, $\mu = (-2\beta)^{1/2}$, and ν is a constant such that $|\mu|+|\nu| \neq 0$. Then the quadratic equation for v can be written as

$$y(y+2\mu p^{-1})\left(\frac{1}{v}-1\right)^2 + (\dot{y}-y^2)\left(\frac{1}{v}-1\right) - 2(y^2+2c_1y+c_0) = 0 \tag{3.5}$$

where

$$c_0 = \frac{1}{p^2}(p^2c_1^2 + 2\delta r^2) \tag{3.6}$$

and $c_1(x) = \frac{1}{p}[\mu - zb(z)]$ satisfies the Riccati equation

$$\dot{c}_1 + c_1^2 + 2(\gamma+\delta r)rp^{-2} = 0. \tag{3.7}$$

Equations (3.5) and

$$y = p^{-1}[z(v^0 + bv) - \mu] \tag{3.8}$$

give one-to-one correspondence between a solution v(z) of PV and a solution y(x) of the following second-order second-degree equation:

$$[2(y^2+2c_1y+c_0)(\ddot{y}-2y\dot{y})-(y+c_1)(\dot{y}-y^2)^2-P_3(y)(\dot{y}-y^2)+Q_5(y)]^2$$
$$=[(y+c_1)(\dot{y}-y^2)^2-R_3(y)]^2[(\dot{y}-y^2)^2+8y(y+2\mu p^{-1})(y^2+2c_1y+c_0)] \tag{3.9}$$

where

$$P_3(y)=4y^3+2(3c_1-\mu p^{-1})y^2-4p^{-2}(\mu pc_1+\gamma r)y$$
$$-2p^{-3}[p^3c_1^3+\mu p^2c_1^2+2(\gamma+\delta r)rpc_1+2\delta(3\mu+2)r^2]$$

$$Q_5(y)=8(y^2+2c_1y+c_0)[3y^3+(5c_1+4\mu p^{-1})y^2+2(3\mu p^{-1}c_1+c_0)y$$
$$+2\mu p^{-1}c_0] \tag{3.10}$$

$$R_3(y)=8y^3+2(11c_1+\mu p^{-1})y^2+2p^{-2}[10p^2c_1^2+2\mu pc_1+2(\gamma+6\delta r)r]y+2p^{-3}[3p^3c_1^3+\mu p^2c_1^2+$$
$$2(\gamma+3\delta r)rpc_1+2\delta(3\mu+2)r^2].$$

Case 2. If $A_5=0, A_3\neq 0$, then $d=0, a=\frac{(2\alpha)^{1/2}}{z}$. If $e=0$, then (3.2) cannot be reduced to a quadratic equation in v. Let $e\neq 0$ and without loss of generality let $b=0, e=1$. Then equation (3.2) can be reduced to the following quadratic equation for v:

$$A_3v^2+(A_2-fA_3)v+A_1-fA_2+f^2A_3=0 \tag{3.11}$$

where f and c satisfy the following equations

$$f(f+1)(f'-af^2-c)=0$$
$$(3f+1)(c^2+f^2a^2)+2f^2(f-1)(ac+\delta)$$
$$=(f+1)\left[2f\left(c'+\frac{c}{z}\right)-\frac{2\gamma}{z}f^2-\frac{2\beta}{z^2}(f+1)^2\right]. \tag{3.12}$$

One has to consider the following three cases $f=0, f=-1$ and $f(f+1)\neq 0$ separately.

Case 2.1. If $f=0$, then equation (3.12) gives $z^2c^2+2\beta=0$. Thus $A_0=A_1=0$ and equation (3.2) reduce to a linear equation $A_3v+A_2=0$ for v. Therefore one obtains the following transformation for the PV [11]

$$\bar{v}=1-\frac{2(-2\delta)^{1/2}zv}{zv'-(2\alpha)^{1/2}v^2+[(2\alpha)^{1/2}-(-2\beta)^{1/2})+(-2\delta)^{1/2}z]v+(-2\beta)^{1/2}}$$
$$\bar{\alpha}=-\frac{1}{16\delta}\{\gamma+(-2\delta)^{1/2}[1-(2\alpha)^{1/2}-(-2\beta)^{1/2}]\}^2$$
$$\bar{\beta}=\frac{1}{16\delta}\{\gamma-(-2\delta)^{1/2}[1-(2\alpha)^{1/2}-(-2\beta)^{1/2}]\}^2$$
$$\bar{\gamma}=(-2\delta)^{1/2}[(-2\beta)^{1/2}-2\alpha)^{1/2}] \qquad \bar{\delta}=\delta. \tag{3.13}$$

Case 2.2. If f = −1, then equation (3.12) gives $(a + c)^2 + 2\delta = 0$. Assume that γ and δ are not both zero, and let $\mu = (2\alpha)^{1/2} + \frac{1}{2}$, $\nu = (-2\delta)^{1/2}$ $y(x) = z(u - \nu) - \frac{1}{2}(4\mu - 1)$, $x = \frac{\sqrt{z}}{\kappa}$, where

$$\kappa = \begin{cases} (2\nu)^{-1/2} & \text{and when } \nu \neq 0 \\ \frac{1}{2} & \text{when } \nu = 0. \end{cases} \qquad (3.14)$$

Then the quadratic equation for v can be written as follows

$$(y^2 + 2\mu y + \mu^2 + 2\beta)\left(\frac{1}{v} - 1\right)^2 + x\dot{y}\left(\frac{1}{v} - 1\right) + \kappa^2 x^2(2\nu y - \nu + 2\gamma) = 0. \qquad (3.15)$$

The equations (3.15) and (3.16)

$$y = \frac{2zv' + (2\mu - 1)v^2 - (2\nu z + 4\mu - 1)v + 2\mu}{2(v - 1)}$$

give one-to-one correspondence between solutions v(z) of PV and solutions y(x) of the following second-order second-degree Painlevé-type equation

$$\left[\ddot{y} - \frac{1}{2}\frac{\partial Q_3(y)}{\partial y}\right]^2 = \left[\frac{2}{x}y - x\right]^2 [\dot{y}^2 - Q_3(y)]$$
$$Q_3(y) := 4y^3 + \frac{2}{\nu}[\nu(4\mu - 1) + 2\gamma]y^2 + \frac{4}{\nu}[\nu(\mu^2 - \mu + 2\beta) + 2\mu\gamma]y + \frac{2}{\nu}(\mu^2 + 2\beta)(2\gamma - \nu) \qquad (3.17)$$

when $\nu \neq 0$, and

$$\left[\ddot{y} - \frac{1}{2}\frac{\partial Q_2(y)}{\partial y}\right]^2 = \frac{4}{x^2}y^2[\dot{y}^2 - Q_2(y)]$$
$$Q_2(y) := 2\gamma(y^2 + 2\mu y + \mu^2 + 2\beta) \qquad (3.18)$$

when ν = 0. The equations (3.17) and (3.18) were obtained by Bureau [2, equations (18.6), (20.5), p 206, 209 resp.] Note that if γ = δ = 0, then

$$w = \frac{2zv' + (8\alpha)^{1/2}v^2 - 2[(8\alpha)^{1/2} + 1]v + (8\alpha)^{1/2} + 2}{v - 1} \qquad (3.19)$$

is a solution of the following equation

$$zw'' = ww' \qquad (3.20)$$

which has the first integral $2zw' = w^2 + 2w + K$, where K is the integration constant.

Case 2.3. If $f(f + 1) \neq 0$, then equation (3.12) gives

$$c = f' - \frac{(2\alpha)^{1/2}}{z}f^2$$
$$f'' = \frac{3f + 1}{2f(f + 1)}(f')^2 - \frac{1}{z}f' + \frac{\alpha}{z^2}f(f + 1)^2 + \frac{\beta(f + 1)^2}{z^2 f} + \frac{\gamma}{z}f + \frac{\delta f(f - 1)}{f + 1}. \qquad (3.21)$$

By using the linear transformation $u(z) = \xi(z)y(x) + \eta(z)$ and the change of variable $x = \zeta(z)$,

$$\mu = (-2\beta)^{1/2} \qquad \xi(z) = \exp\left[-\int^z \frac{\hat{z}f'(\hat{z}) + (\mu+1)f(\hat{z}) + \mu}{\hat{z}f(\hat{z})}\,\mathrm{d}\hat{z}\right] \quad \text{where}$$

$$= -\frac{1}{2}\int^z \xi(\hat{z})[f(\hat{z})+1]\,\mathrm{d}\hat{z} \qquad \eta(z) = \frac{1}{zf}(zf' - (2\alpha)^{1/2}f^2 + \mu) \tag{3.22}$$

$\zeta(z)$ one can write the quadratic equation for v as

follows

$$\frac{\phi}{(\phi+1)}y(y+2a_1)\left(\frac{1}{v}-1\right)^2 + (\dot{y}-y^2)\left(\frac{1}{v}-1\right) - 2(y^2+2c_1y+c_0) = 0 \tag{3.23}$$

where

$$h(x) = 2\mu x + \nu \qquad a_1 = \frac{\mu}{h} \qquad |\mu| + |\nu| \neq 0$$

$$:= f(z) \qquad c_1 = \frac{1}{\xi(f+1)}[\eta(f+1) - a - c]$$

$$c_0 = c_1^2 + \frac{2\delta r^2\phi^2}{h^2(\phi+1)^2} \qquad r(x) = \exp\left[-2\int \frac{\phi(x)}{h(\phi+1)}\,\mathrm{d}x\right]_{\varphi(x)} \tag{3.24}$$

and

$$\dot{\phi} + 2\phi c_1 + \frac{2(2\alpha)^{1/2}}{h}\phi^2 - \frac{2\mu}{h}\phi = 0$$

$$\dot{c_1} + c_1^2 + \frac{2\gamma r\phi^2}{h^2(\phi+1)^2} + \frac{2\delta r^2\phi^2(\phi-1)}{h^2(\phi+1)^3} = 0 \tag{3.25}$$

.

Equations (3.23) and

$$y = \frac{1}{z\xi(v+f)}(zv' + (2\alpha)^{1/2}v^2 - z\eta v - \mu) \tag{3.26}$$

give a one-to-one correspondence between solutions $v(z)$ of the PV and solution $y(x)$ of the following equation:

$$[2(y^2+2c_1y+c_0)(\ddot{y}-2y\dot{y}) - (y+c_1)(\dot{y}-y^2)^2 - P_3(y)(\dot{y}-y^2) + Q_5(y)]^2$$
$$= [(y+c_1)(\dot{y}-y^2) - R_3(y)]^2[(\dot{y}-y^2)^2 + S_4(y) \quad] \tag{3.27}$$

where

$$
\begin{aligned}
P_3(y) &= 4y^3 + 2\left(3c_1 + \frac{\dot{\phi}}{2\phi(\phi+1)} - a_1\right)y^2 \\
&\quad +2\left[\dot{c_1} + c_0 + c_1\left(\frac{\dot{\phi}}{2\phi(\phi+1)} - a_1\right)\right]y \\
&\quad +\dot{c_0} + 2c_0\left(\frac{\dot{\phi}}{2\phi(\phi+1)} - a_1\right) \\
Q_5(y) &= \frac{8\phi}{\phi+1}(y^2+2c_1y+c_0)[3y^3+(5c_1+4a_1)y^2+2(3a_1c_1+c_0)y+2a_1c_0] \\
R_3(y) &= \frac{4(2\phi+1)}{(\phi+1)}y^3 + 2\left[a_1 - \frac{\dot{\phi}}{2\phi(\phi+1)} + \frac{(11\phi+5)}{(\phi+1)}c_1\right]y^2 \\
&\quad +2\left[\frac{4\phi}{(\phi+1)}c_1^2 + \frac{(5\phi+3)}{(\phi+1)}c_0 - 2c_1\left(\frac{\dot{\phi}}{2\phi(\phi+1)} - a_1\right) - \dot{c_1}\right]y \\
&\quad -2c_0\left(\frac{\dot{\phi}}{2\phi(\phi+1)} - a_1\right) - \dot{c_0} + \frac{4\phi c_0c_1}{\phi+1} \\
S_4(y) &= \frac{8\phi}{\phi+1}y(y+2a_1)(y^2+2c_1y+c_0).
\end{aligned}
\tag{3.28}
$$

Case 3. If $A_5 \neq 0$, then equation (3.2) can be written as

$$(v^3 + \tilde{g}v^2 + \tilde{h}v + \tilde{k})(\tilde{B}_2v^2 + B_1v + B_0) = 0 \tag{3.29}$$

where $B_j, j = 0,1,2$ are functions of u', u, z and $\tilde{g}, \tilde{h}, \tilde{k}$ are functions of z only. If $\tilde{k} = 0$, then one obtains $A_0 = 0$, and hence $f = 0, z^2c^2 + 2\beta = 0$. This implies that $A_1 = 0$, and (3.2) takes the form

$$A_5v^3 + A_4v^2 + A_3v + A_2 = 0 \tag{3.30}$$

and equation (1.11) becomes

$$u = \frac{v' + bv + c}{v^2 + ev}. \tag{3.31}$$

The discrete Lie-point symmetry of PV [11]

$$\bar{v} = \frac{1}{v} \qquad \bar{\alpha} = -\beta \qquad \bar{\beta} = -\alpha, \bar{\gamma} = -\gamma \qquad \bar{\delta} = \delta \tag{3.32}$$

transform this case to case 2.

When $\tilde{k} \neq 0$, one should take $\gamma = 0, d \neq 0$, and without loss of generality $a = 0$, $d = 1$. The functions $e, \tilde{g}, \tilde{h}, \tilde{k}$ should satisfy $e = -(f+1), \tilde{g} = -(f+2), \tilde{h} = 2f+1, \tilde{k} = -f$, where b, c, f are solutions of the following equations

$$(b+c)(f-1)=0 \qquad (b+c)\left(c-\frac{1}{z}\right)=0 \tag{3.33}$$

$$(b+c)^2+2\delta=0 \qquad f'+b(f-1)=0$$

$$2f\left[b'+\frac{b}{z}-bc+\frac{\alpha}{z^2}f(f-1)\right]-(f+1)c^2+\frac{2\beta}{z^2}(f-1)=0$$

The equations (1.11) and (1.15) become

$$u=\frac{v'+bv+c}{(v-1)(v-f)} \qquad B_2v^2+B_1v+B_0=0 \tag{3.34}$$

where

$$B_2=u^2-\frac{2\alpha}{z^2} \qquad B_1=2u'+(f-1)u^2+\frac{2}{z}u-\frac{2\alpha}{z^2}(f-1)$$

$$B_0=\frac{-1}{f}\left[f^2u^2-2cfu+c^2+\frac{2\beta}{z^2}\right]. \tag{3.35}$$

The following two subcases (3.1) f = 1 and (3.2) f 6= 1 should be considered separately.

Case 3.1. If f = 1, then equation (3.33) gives $b=\frac{2\mu}{z}+\nu, c=-\frac{2\mu}{z}$, where $\nu=(-2\delta)^{1/2}$, and μ is a constant such that ν(2μ + 1) = 0. Let y(x) = −i(zu + μ),x = lnz. Then the equations

$$y=\frac{zv'+\mu v^2+\nu zv-\mu}{\mathrm{i}(v-1)^2}$$

$$(y^2+2\mathrm{i}\mu y-\mu^2+2\alpha)v^2-2\mathrm{i}\dot{y}v-(y^2-2\mathrm{i}\mu y-\mu^2-2\beta)=0 \tag{3.36}$$

give one-to-one correspondence between solutions v(z) of PV and y(x) of the following second-order second-degree Painleve-type equation´

$$\left[\ddot{y}-\frac{1}{2}\frac{\partial Q_4(y)}{\partial y}\right]^2=-[2y-\mathrm{i}\nu\exp(x)]^2[\dot{y}-Q_4(y)]$$

$$Q_4(y):=y^4+2(\mu^2+\alpha-\beta)y^2-4\mathrm{i}\mu(\alpha+\beta)y+(\mu^2-2\alpha)(\mu^2+2\beta). \tag{3.37}$$

When ν = −2i equation (3.37) becomes

$$\left[\ddot{y}-\frac{1}{2}\frac{\partial Q_4(y)}{\partial y}\right]^2=-4[y-\exp(x)]^2[\dot{y}-Q_4(y)]$$

$$Q_4(y):=y^4+2(\mu^2+\alpha-\beta)y^2-4\mathrm{i}\mu(\alpha+\beta)y+(\mu^2-2\alpha)(\mu^2+2\beta). \tag{3.38}$$

Equation (3.38) was also obtained by Bureau [2, equation (16.13), p 203] but the relation between (3.38) and PV has not been mentioned.

Case 3.2. If f 6= 1, then (3.33) gives $\delta=0, b=-c, c=\frac{f'}{f-1}$, and f satisfies PV with $\delta=\gamma=0$

$$f''=\frac{(3f-1)}{2f(f-1)}(f')^2-\frac{1}{z}f'+\frac{\alpha}{z}f(f-1)^2+\frac{\beta(f-1)^2}{z^2f}. \tag{3.39}$$

Let u(z) = ξ(z)y(x) + η(z),x = ζ(z), where

(3.40)

$$\mu = (-2\beta)^{1/2} \qquad \xi(z) = \exp\left[-\int^z \frac{\hat{z}f'(\hat{z}) + (\mu+1)f(\hat{z}) - \mu}{\hat{z}f(\hat{z})}\,\mathrm{d}\hat{z}\right]$$

$$= -\frac{1}{2}\int^z \xi(\hat{z})[f(\hat{z}) - 1]\,\mathrm{d}\hat{z} \qquad \eta(z) = \frac{1}{zf(f-1)}[zf' + \mu(f-1)]$$

ζ(z) and let φ(x) := f(z). Then the equations

$$y = \frac{zv' - z\eta v^2 + (z\eta+\mu)v - \mu}{z\xi(v-1)(v-f)}$$
$$(y^2 + 2a_1 y + a_0)v^2 - (\phi - 1)(\dot{y} - y^2)v - \phi y(y + 2c_1) = 0 \tag{3.41}$$

where

$$h(x) = 2\mu x + \nu \qquad c_1 = \frac{\mu}{h(x)} \qquad |\mu| + |\nu| \neq 0$$
$$a_1(x) = c_1 - \frac{\dot{\phi}}{2\phi} \qquad a_0 = a_1^2 - \frac{2\alpha\phi^2}{h^2} \tag{3.42}$$

give one-to-one correspondence between solutions v(z) of PV and y(x) of the following second-degree equation

$$\left[\ddot{y} - 2y\dot{y} - 2(y - d_1)(\dot{y} - y^2) + \frac{1}{2}\frac{\partial Q_4(y)}{\partial y}\right]^2 = (d_2 y - d_3)^2[(\dot{y} - y^2)^2 + Q_4(y)]. \tag{3.43}$$

where

$$d_1 = \frac{1}{2\phi(\phi-1)}\dot{\phi} + \frac{\mu}{h} \qquad d_2 = \frac{2(\phi+1)}{(\phi-1)}$$
$$d_3 = \frac{1}{\phi(\phi-1)}\dot{\phi} - \frac{2\mu(\phi+1)}{h(\phi-1)}$$
$$Q_4(y) = \frac{4\phi y}{(\phi-1)^2}[y^3 + 2(a_1+c_1)y^2 + (a_0 + 4a_1c_1)y + 2a_0c_1] \tag{3.44}$$

and the function φ(x) satisfies the following equation

$$\ddot{\phi} = \frac{3}{2\phi}(\dot{\phi})^2 - \frac{\dot{h}}{h}\dot{\phi} + \frac{4\alpha}{h^2}\phi^3 - \frac{2\mu^2}{h^2}\phi. \tag{3.45}$$

4. **Painleve VI´** Let

v(z) be a solution of PVI

$$v'' = \frac{1}{2}\left(\frac{1}{v} + \frac{1}{v-1} + \frac{1}{v-z}\right)(v')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{v-z}\right)v'$$
$$+ \frac{v(v-1)(v-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{v^2} + \frac{\gamma(z-1)}{(v-1)^2} + \frac{\delta z(z-1)}{(v-z)^2}\right) \tag{4.1}$$

then, for PVI the equation (1.13) takes the form

$$A_6v^6 + A_5v^5 + A_4v^4 + A_3v^3 + A_2v^2 + A_1v + A_0 = 0 \tag{4.2}$$

where

$$A_6 = (du - a)^2 - \frac{2\alpha}{z^2(z-1)^2}$$

$$A_5 = 2(du' + d'u - a') - 2(z+1)(du - a)^2 + \frac{2(2z-1)}{z(z-1)}(du - a) + \frac{4\alpha(z+1)}{z^2(z-1)^2}$$

$$\begin{aligned} A_4 = {} & 2(eu' + e'u - b') - 2(z+1)(du' + d'u - a') - (eu - b)^2 + \frac{2(2z-1)}{z(z-1)}(eu - b) \\ & - \frac{2}{z^2(z-1)^2}[\alpha(z^2 + 4z + 1) + \beta z + (\gamma + \delta z)(z-1)] - (du - a) \\ & \times \left[2(z+1)(eu - b) + 2(fu - c) - 3z(du - a) + \frac{2(z^2 + 2z - 1)}{z(z-1)}\right] \end{aligned}$$

$$\begin{aligned} A_3 = {} & 2(fu' + f'u - c') - 2(z+1)(eu' + e'u - b') + 2z(du' + d'u - a') \\ & + \frac{4}{z(z-1)^2}[(\alpha + \beta)(z+1) + (\gamma + \delta)(z-1)] \\ & - 2(eu - b)\left[2(fu - c) - 2z(du - a) + \frac{(z^2 + 2z - 1)}{z(z-1)}\right] \\ & + \frac{2z}{z-1}(du - a) + \frac{2(2z-1)}{z(z-1)}(fu - c) \end{aligned}$$

$$\begin{aligned} A_2 = {} & 2z(eu' + e'u - b') - 2(z+1)(fu' + f'u - c') + z(eu - b)^2 + \frac{2z}{z-1}(eu - b) \\ & - \frac{2}{z(z-1)^2}[\alpha z + \beta(z^2 + 4z + 1) + (\gamma z + \delta)(z-1)] + (fu - c) \\ & \left[2(z+1)(eu - b) - 3(fu - c) + 2z(du - a) - \frac{2(z^2 + 2z - 1)}{z(z-1)}\right] \end{aligned}$$

$$A_1 = 2z(fu' + f'u - c') + 2(z+1)(fu - c)^2 + \frac{2z}{z-1}(fu - c) + \frac{4\beta(z+1)}{(z-1)^2}$$

$$A_0 = -z\left[(fu - c)^2 + \frac{2\beta}{(z-1)^2}\right]. \tag{4.3}$$

Note that if $A_6 = 0$ then $A_5 = 0$. Moreover, one cannot find the functions a, b, c, d, e, f such that $A_6 = A_5 = A_4 = 0$. Thus, to reduce (4.2) to a quadratic equation for v one may consider the following two cases (1) $A_6 = 0$, $A_4 \neq 0$ and (2) $A_6 \neq 0$.

Case 1. If $A_6 = 0$, then one obtains. $d = 0, a = \frac{(2\alpha)^{1/2}}{z(z-1)}$ Note that when e = 0 equation (4.2) cannot be reduced to a quadratic equation for v. Therefore one should take $e \neq 0$, and hence without loss of generality b = 0,e = 1. In this case equation (4.2) can be written as

$$(v + f)^2(B_2v^2 + B_1v + B_0) = 0 \tag{4.4}$$

where f and c are solutions of the following equations

$f(f + 1)(f + z) = 0$

$$[z(z-1)(af^2+c)+f(f+1)]^2+2\beta z(f+1)(f+z)+2\gamma(z-1)f(f+z)+(2\delta-1)z(z-1)f(f+1)=0. \quad (4.5)$$

The following three subcases should be considered separately.

Case 1.1. If $f = 0$, then equation (4.5) implies $(z-1)^2c^2 + 2\beta = 0$. By using the change of variable

$$y(x) = zu - \frac{(\mu+1)z}{2(z-1)} - \frac{\mu-1}{2(z-1)} \qquad x = \arcsin\frac{z+1}{z-1} \qquad \mu = (2\alpha)^{1/2} + (-2\beta)^{1/2} \quad (4.6)$$

one may write the quadratic equation for v as follows

$$A\left(\frac{z-1}{v-1}-1\right)^2 + B\left(\frac{z-1}{v-1}-1\right) + C = 0 \quad (4.7)$$

where

$$\begin{aligned} A &= y^2 + 2\lambda y + \lambda^2 - 2\gamma \\ B &= -2\sqrt{\frac{1+\sin x}{1-\sin x}}\,\dot{y} \\ C &= \frac{\sin x + 1}{\sin x - 1}(y^2 - 2\lambda y + \lambda^2 + 2\delta - 1) \end{aligned} \quad (4.8)$$

and $\lambda = \frac{1}{2}[(2\alpha)^{1/2} - (-2\beta)^{1/2} + 1]$. The second-degree equation for y(x) is

$$\left[\ddot{y} - \frac{1}{2}\frac{\partial Q_4(y)}{\partial y}\right]^2 = 4\tan^2 x\left[y - \frac{K_2}{\sin x}\right]^2[\dot{y} - Q_4(y)]$$

$$\begin{aligned} Q_4(y) &:= y^4 + (2\delta - 2\gamma - 2\lambda^2 - 1)y^2 + 2\lambda(2\gamma + 2\delta - 1)y \\ &\quad + (\lambda^2 - 2\gamma)(\lambda^2 + 2\delta - 1) \end{aligned} \quad (4.9)$$

where $K_2 = \frac{-\mu}{2}$. The equation (4.9) was obtained by Bureau [2, equation (16.12), p 202] and also by Fokas and Ablowitz [11].

Case 1.2. If $f = -1$, then equation (4.5) gives $z^2(a + c)^2 = 2\gamma$. The quadratic equation for v becomes

$$A\left(\frac{1}{v}-\frac{1}{z}\right)^2 + B\left(\frac{1}{v}-\frac{1}{z}\right) + C = 0 \quad (4.10)$$

where

$$A = z[u^2 + 2cu + c^2 + \frac{2\beta}{(z-1)^2}] \qquad B = \frac{2}{z-1}[z(z-1)u' + zu + za + c]$$

$$C = \frac{1}{z(z-1)}[(z-1)^2u^2 - 2(z-1)(z^2a + c + 1)u + (z^2a + c + 1)^2 + 2\delta - 1]. \quad (4.11)$$

The Lie-point symmetry of PVI [11]

$$\begin{aligned} &\bar{v}(\bar{z};\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}) = 1 - v(z;\alpha,\beta,\gamma,\delta) \\ &\bar{z} = 1 - z \qquad \bar{\alpha} = \alpha \qquad \bar{\beta} = -\gamma \quad \bar{\gamma} = -\beta \qquad \bar{\delta} = \delta \end{aligned} \quad (4.12)$$

transforms this case to the case 1.1

Case 1.3. If $f = -z$, then equation (4.5) gives $(za + c + 1)^2 = 1 - 2\delta$. The quadratic equation for v now may be written as

$$A\left(\frac{1}{v}-1\right)^2 + B\left(\frac{1}{v}-1\right) + C = 0 \tag{4.13}$$

where

$$A = \frac{1}{z}\left[z^2u^2 + 2zcu + c^2 + \frac{2\beta}{(z-1)^2}\right] \qquad B = 2\left[u' + \frac{(2z-1)}{z(z-1)}u + \frac{(za+c)}{z(z-1)}\right]$$
$$C = -\left[(z-1)u^2 + 2(a+c)u + \frac{(a+c)^2}{z-1} - \frac{2\gamma}{z^2(z-1)}\right]. \tag{4.14}$$

The Lie-point symmetry of PVI [11]

$$\bar{v}(\bar{z};\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}) = 1 - (1-\bar{z})v(z;\alpha,\beta,\gamma,\delta)$$
$$z = \frac{1}{1-\bar{z}} \qquad \alpha = \bar{\alpha} \qquad \beta = -\bar{\gamma} \qquad \gamma = -\bar{\delta} + \frac{1}{2} \qquad \delta = \bar{\beta} + \frac{1}{2} \tag{4.15}$$

transforms this case to the case 1.1.

Case 2. If $A_6 \neq 0$, then to reduce equation (4.2) to a quadratic equation for v one should take $d \neq 0$ and hence without loss of generality one may take $a = 0, d = 1$. Then equation (4.2) can be written as

$$(v^4 + \tilde{g}v^3 + \tilde{h}v^2 + \tilde{k}v + \tilde{l})(\tilde{B}_2v^2 + B_1v + B_0) = 0 \tag{4.16}$$

where $B_j, j = 1,2,3$ are functions of u^0, u, z and $b, c, e, f, \tilde{g}, \tilde{h}, \tilde{k}, \tilde{l}$ may be chosen such that one of the following two cases are satisfied.

(i) $e = -(z+1) \qquad f = z \qquad \tilde{g} = -2(z+1) \qquad \tilde{h} = z^2 + 4z + 1$
$\tilde{k} = -2z(z+1) \qquad \tilde{l} = z^2$

$$b = \frac{-1}{2z(z-1)}[(\mu+\nu)z + \mu - \nu] \qquad c = \frac{\mu}{z-1} \tag{4.17}$$

where μ and ν are constants such that

$$\mu + \nu - \mu\nu = 2(\gamma + \delta) \qquad \mu^2 + \nu^2 = 2(\mu + \nu) + 4(\gamma - \delta) \tag{4.18}$$

(ii) $e = -z, \tilde{g} = -2z, \tilde{h} = z^2, f = \tilde{k} = \tilde{l} = 0$

$$(z-1)^2c^2 + 2\beta = 0 \qquad (zb + c + 1)^2 + 2\delta - 1 = 0. \tag{4.19}$$

Case 2.1. With the choices (4.17), (4.18) equations (1.11) and (1.13) take the following form

$$u = \frac{v' + bv + c}{(v-1)(v-z)} \qquad Av^2 + Bv + C = 0 \tag{4.20}$$

respectively, where

$$A = u^2 - \frac{2\alpha}{z^2(z-1)^2} \qquad B = 2\left[u' + \frac{(2z-1)}{z(z-1)}u\right]$$
$$C = -\left[zu^2 - 2cu + \frac{1}{z}c^2 + \frac{2\beta}{z^2(z-1)^2}\right]. \tag{4.21}$$

Let $y(x) = z(z-1)u - \frac{1}{2}\mu$, $x = \arcsin\frac{z+1}{z-1}$, then y(x) is a solution of (4.9) with $K_2 = \frac{\nu-1}{2}$, and

$$Q_4(y) := y^4 + (2\beta - 2\alpha - \tfrac{1}{2}\mu^2)y^2 + 2\mu(\alpha+\beta)y + (\tfrac{1}{4}\mu^2 - 2\alpha)(\tfrac{1}{4}\mu^2 + 2\beta). \tag{4.22}$$

Using the fact that both subcases 1.1 and 2.1 give the same second-degree equation one obtains the following new Lie-point symmetry for PVI:

$$\bar{v} = \frac{v-z}{v-1} \qquad \bar{\alpha} = \gamma \qquad \bar{\beta} = \delta - \tfrac{1}{2} \qquad \bar{\gamma} = \alpha \qquad \bar{\delta} = \beta + \tfrac{1}{2}. \tag{4.23}$$

Case 2.2. In this case equations (1.11) and (1.13) respectively become

$$u = \frac{v' + bv + c}{v(v-z)} \qquad A(v-z)^2 + B(v-z) + C = 0 \tag{4.24}$$

where

$$A = u^2 - \frac{2\alpha}{z^2(z-1)^2} \qquad B = 2\left[u' + \frac{(2z-1)}{z(z-1)}u\right]$$
$$C = (z-1)u^2 + 2(b+c)u + \frac{(b+c)^2}{z-1} - \frac{2\gamma}{z^2(z-1)}. \tag{4.25}$$

The Lie-point symmetry (4.12) of PIV transforms this to case 2.1.

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