motion of curves

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Motion of curves on two-dimensional surfaces and soliton equations

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Abstract

A connection is established between the soliton equations and curves moving in a three-dimensional space V_3. The signs of the self-interacting terms of the soliton equations are related to the signature of V_3. It is shown that there corresponds a moving curve to each soliton equation. © 1998 Elsevier Science B.V.

Differential geometry and partial differential equations (PDEs) are two different research areas in mathematics. When we study some local properties of surfaces in Euclidean (E_3) or Minkowskian (M_3) 3-spaces we face some known PDEs. For instance the Liouville and sine-Gordon equations describe surfaces of constant Gaussian curvature [1]. Gauss–Codazzi–Mainardi equations describe the surfaces embedded in E_3 or in M_3. These equations are used for the construction of the soliton connection [2–4]. Here differential geometrical tools are utilized to find for example the Bäcklund transformations and prolongation structures [5] of the soliton equations.

During the last two decades another virtue of differential geometry arised in soliton theory. The Serret–Frenet equations for the family of curves (the motion of curves) give certain coupled partial differential equations for the curvature (k) and torsion (τ) scalars of these curves [6–12]. It was shown that some soliton equations like the modified Korteweg–de Vries (mKdV), sine-Gordon and nonlinear Schrödinger (NLS) are among the equations that may arise from the motion of space curves. All these considerations were carried out in Euclidean 3-space E_3. This is why only one version of the nonlinear couplings of the mKdV and NLSEs could be obtained.

In this work we take a 3-space V_3 with signature $1+2\epsilon$, where $\epsilon^2 = 1$. This means that curves in M_3 will also be considered. Self-interacting terms in the evolution equations of the curvature and the torsion of these curves depend upon the signature of the space V_3. The sign difference of the self-interaction terms is due to the signature change of the 3-space. If for instance a curve C is moving in E_3 (or in M_3), focusing (or defocusing) versions of mKdV or NLS equations arise.

The motion of the curve C is described by three functions p, q and w. The function w is determined in terms of the others but the functions p and q are left arbitrary. Each choice of these functions gives a different class

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of curves in V_3. It is in principle possible to convert the differential equations satisfied by the scalars k and τ to any system of two coupled nonlinear PDEs. Here we should remark that not all these equations are integrable. The integrability property of these equations (for each choice of p and q) should be examined. The functions p and q can be suitably chosen to make the evolution equations satisfied by k and τ integrable. So far, for this purpose [6–12] p and q were assumed to be local functions of k and τ. In this way mKdV, NLS, and complex mKdV equations could be obtained.

On the other hand, one may obtain, by a proper choice of p and q (since they are free), all possible integrable equations. This can be done by relaxing the locality assumptions on the functions p and q. The sine-Gordon equation is obtained by assuming that $q = \tau = 0$ and p is a nonlocal function of the curvature k [8,9]. We show that any integrable system of two coupled nonlinear PDEs can be obtained by assuming a nonlocal functional dependence. In this way it is possible to obtain for instance the AKNS [13] hierarchy. Hence, in general there exists a curve C moving in a V_3 corresponding to any integrable nonlinear differential equation (one or two coupled equations).

Some nonlinear partial differential equations, such as the sine-Gordon and the Liouville equations, arise from the surfaces of constant Gaussian curvature. Here we show that such equations and many others may also arise from two-dimensional surfaces with vanishing Gaussian curvature, flat surfaces (see also Ref. [14]).

Let V_3 define a three-dimensional flat space with the line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

where $\mu, \nu = 1, 2, 3$, $x^\mu = (t, y, z)$ and $\eta_{\mu\nu} = \mathrm{diag}(1, \epsilon, \epsilon)$. If $\epsilon = 1$, then $V_3 = E_3$ is a Euclidean 3-space and if $\epsilon = -1$ then $V_3 = M_3$ is a pseudo-Euclidean (Minkowskian) 3-space. Hence, Eq. (1) explicitly takes the form

$$ds^2 = dt^2 + \epsilon dy^2 + \epsilon dz^2. \tag{2}$$

Let S be a surface in V_3 parametrized by $x^\mu(u, v)$, and let C be a curve on S defined by $\alpha : I \to S$ and parametrized by its arc length $s \in I$. An orthonormal frame (t^μ, n^μ, b^μ) at each point of C is defined by (recall that $x^\mu_{,s} = t^\mu$)

$$\eta_{\mu\nu} t^\mu t^\nu = 1, \qquad \eta_{\mu\nu} n^\mu n^\nu = \epsilon, \qquad \eta_{\mu\nu} b^\mu b^\nu = \epsilon, \tag{3}$$

all the other products vanish. The Serret–Frenet equations are $(x^\mu_{,s} = t^\mu)$

$$t^\mu_{,s} = k n^\mu, \tag{4}$$

$$n^\mu_{,s} = -\epsilon k t^\mu - \tau b^\mu, \tag{5}$$

$$b^\mu_{,s} = \tau n^\mu, \tag{6}$$

where k and τ are the curvature and the torsion scalars of the curve C at any point s. The vectors t^μ, n^μ and n^μ are, respectively, the tangent, normal and bi-normal vectors to the curve at any point s [15].

A curve on S is given by $\alpha^\mu(s) = x^\mu(u(s), v(s))$. This curve may be considered as a member of the family of curves $\beta^\mu_\sigma = x^\mu(u(s,\sigma), v(s,\sigma))$ for a fixed value of σ. The change (motion) of the curve with respect to the parameter σ (on S) is given by

$$x^\mu_{,\sigma} = p n^\mu + w t^\mu + q b^\mu, \tag{7}$$

where the p, q and w are functions of s and σ. By using the equation $x^\mu_{,s} = t^\mu$ and (7) we get $w_{,s} = \epsilon k p$ and $t^\mu_{,\sigma}$ (partial derivative of the vector t^μ with respect to σ). Using $t^\mu_{,\sigma}$ obtained this way and the first of the Serret–Frenet equations, Eq. (4), one obtains $k_{,\sigma}$ and $n^\mu_{,\sigma}$. Following a similar approach one finds derivatives of the scalars (k,τ) and vectors (t^μ, n^μ, b^μ). They are given by

$$t^\mu_{,\sigma} = (p_{,s} + kw + \tau q) n^\mu + (q_{,s} - \tau p) b^\mu, \tag{8}$$

$$n^{\mu}_{,\sigma} = -\epsilon(p_{s} + kw + \tau q)t^{\mu} + \frac{1}{k}[(q_{,s} - \tau p)_{,s} - \tau(p_{,s} + kw + \tau q)]b^{\mu}, \tag{9}$$

$$b^{\mu}_{,\sigma} = -\frac{1}{k}[(q_{,s} - \tau p)_{,s} - \tau(p_{,s} + kw + \tau q)]n^{\mu} - \epsilon(q_{,s} - \tau p)t^{\mu}. \tag{10}$$

The compatibility conditions give $w_{,s} = \epsilon kp$ and

$$k_{,\sigma} = (p_{,s} + kw + \tau q)_{,s} + \tau(q_{s} - \tau p), \tag{11}$$

$$\tau_{,\sigma} = -\left[\frac{1}{k}[(q_{,s} - \tau p)_{,s} - \tau(p_{,s} + kw + \tau q)]\right]_{,s} - \epsilon k(q_{,s} - \tau p). \tag{12}$$

These equations may be written in the compact form

$$\begin{pmatrix} k \\ \tau \end{pmatrix}_{,\sigma} = \mathcal{R}\begin{pmatrix} p \\ q \end{pmatrix}, \tag{13}$$

where

$$\mathcal{R} = \begin{pmatrix} D^2 + \epsilon k^2 - \tau^2 + \epsilon k_{,s}D^{-1}k & (D\tau + \tau D) \\ D[(1/k)(D\tau + \tau D) + \epsilon k\tau + \epsilon\tau D^{-1}k] & -D[(1/k)D^2 - \tau^2/k] - \epsilon kD \end{pmatrix}. \tag{14}$$

In the special case $\tau = q = 0$, which means that C is a plane curve, we have

$$k_{,\sigma} = \mathbf{R}p, \tag{15}$$

where $\mathbf{R}$ is the recursion operator of the mKdV equation $k_{,\sigma} = \mathbf{R}k_{,s}$ given by

$$\mathbf{R} = D^2 + \epsilon k^2 + \epsilon k_{,s}D^{-1}k. \tag{16}$$

Here D denotes the total derivative with respect to s and D^{-1} is its inverse. Choosing, for instance, $p = k_{,s}$ reduces Eq. (15) to mKdV. The choices of the geometry $\epsilon = \pm 1$ yield focusing and defocusing versions of the mKdV equations. Choosing $p = \mathbf{R}^n k_{,s}$ with $n = 0, 1, 2, \ldots$ we obtain the infinite integrable hierarchy of the mKdV equations. For other local choices we need to write Eqs. (11) and (12) in a complexified form,

$$\phi_{,\sigma} = (D^2 + i\eta\epsilon\phi D^{-1}\tau\phi^* + |\phi|^2 + \phi_{,s}D^{-1}\phi^*)(p\rho) + (-i\eta D^2 - i\eta\epsilon|\phi|^2 - \epsilon\phi D^{-1}\tau\phi^* + i\eta\epsilon\phi D^{-1}\phi^*_{,s})(q\rho), \tag{17}$$

where $\eta^2 = 1$, $\rho = e^{i\eta(D^{-1}\tau)}$ and $\phi = k\rho$ and ϕ^* is the complex conjugate of ϕ. When $p = 0$ and $q = k$, we have the nonlinear Schrödinger (NLS) equation of both versions ($\epsilon = \pm 1$),

$$i\eta\phi_{,\sigma} = D^2\phi + \frac{\epsilon}{2}|\phi|^2\phi. \tag{18}$$

Another example is obtained by letting $p = k_{,s}$ and $q = -k\tau$. This is the complex mKdV,

$$\phi_{,\sigma} = D^3\phi + \tfrac{3}{2}|\phi|^2\phi_{,s}. \tag{19}$$

In all these cases the function p is chosen as a local function of the k. This means that p is a function of k and its partial derivatives with respect to s and σ to all orders. Other local choices of the function p in terms of k may or may not give integrable nonlinear partial differential equations (equations admitting infinitely many generalised symmetries). For each choice of p one must check whether the resulting equation is integrable [16–18]. The main motivation for choosing integrable equations is their position in mathematics and physics.

It is also possible to choose the function p as a nonlocal function of k. Choosing for instance $p = \mathbf{R}^{-2}k_{,s}$ and letting $k = \theta_{,s}$ we obtain the sine-Gordon equation $\theta_{,s\sigma} = \sin(\theta)$ [9]. Another choice for instance may be $p = \mathbf{R}^{-1}(R_{\text{KdV}})^n k_{,s}$, where $R_{\text{KdV}} = D^2 + 4k + 2k_{,s}D^{-1}$ is the recursion operator of the KdV equation $k_{,\sigma} = k_{,sss} + 6kk_{,s}$. This choice will give the hierarchy of the KdV equation $k_{,\sigma} = (R_{\text{KdV}})^n k_s$, for $n = 0, 1, 2, \ldots$ It is clear from these examples that since p is an arbitrary function, Eq. (15) may be reduced to any nonlinear partial differential equation. One can properly choose p so that all scalar integrable nonlinear PDE can be obtained from Eq. (15).

In the general case, by choosing p and q properly Eq. (14) can be reduced to any system of two coupled nonlinear PDEs. As an example, let

$$\begin{pmatrix} p \\ q \end{pmatrix} = \mathcal{R}^{-1}(R_{\text{AKNS}})^n \begin{pmatrix} k \\ \tau \end{pmatrix}_{,s}, \tag{20}$$

where R_{AKNS} is the recursion operator of the AKNS system of equations given by

$$R_{\text{AKNS}} = \begin{pmatrix} D + 2kD^{-1}\tau & 2kD^{-1}k \\ -2\tau D^{-1}\tau & -D - 2\tau D^{-1}k \end{pmatrix}. \tag{21}$$

Eq. (13) reduces to the AKNS hierarchy for $n = 0, 1, 2, \ldots$ Hence there corresponds a class of moving curves in V_3 to each system of two coupled soliton equations,

$$\begin{pmatrix} k \\ \tau \end{pmatrix}_{,\sigma} = (R_{\text{AKNS}})^n \begin{pmatrix} k \\ \tau \end{pmatrix}_{,s} \tag{22}$$

The derivatives of the vectors in the frame $e_a^\mu = (t^\mu, n^\mu, b^\mu)$ may be written in the more familiar form $\mathrm{d}e_a^\mu = \Omega_a^b e_b^\mu$, where in matrix notation Ω is a matrix-valued 1-form. Here $a, b = 1, 2, 3$ and $\Omega = \Omega_s \mathrm{d}s + \Omega_\sigma \mathrm{d}\sigma$, where

$$\Omega_s = \begin{pmatrix} 0 & k & 0 \\ -\epsilon k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, \qquad \Omega_\sigma = \begin{pmatrix} 0 & w_1 & w_0 \\ -\epsilon w_1 & 0 & w_2 \\ -\epsilon w_0 & -w_2 & 0 \end{pmatrix} \tag{23}$$

with

$$w_0 = q_{,s} - \tau p, \qquad w_1 = p_{,s} + kw + \tau q, \tag{24}$$

$$w_2 = \frac{1}{k}[(q_{,s} - \tau p)_{,s} - \tau(p_{,s} + kw + \tau q)]. \tag{25}$$

The 1-form Ω defines a connection with zero curvature. This is due to the flatness of the space V_3. Vanishing of the curvature of Ω, i.e. $\mathrm{d}\Omega - \Omega\Omega = 0$, is due to the evolution equations given in (13). In order to compare this connection 1-form with the soliton connection 1-form we write it in the more suitable form [2,3]

$$\Omega = \begin{pmatrix} 0 & \pi_0 & \pi_1 \\ -\epsilon\pi_0 & 0 & \pi_2 \\ -\epsilon\pi_1 & -\pi_2 & 0 \end{pmatrix}, \tag{26}$$

where the 1-forms π_0, π_1 and π_2 are given by

$$\pi_0 = k\mathrm{d}s + w_1\mathrm{d}\sigma, \qquad \pi_1 = w_0\mathrm{d}\sigma, \qquad \pi_2 = -\tau\mathrm{d}s + w_2\mathrm{d}\sigma. \tag{27}$$

These 1-forms satisfy (from the zero-curvature condition)

$$d\pi_0 + \pi_1\pi_2 = 0, \qquad d\pi_1 - \pi_0\pi_2 = 0, \qquad d\pi_2 + \epsilon\pi_0\pi_1 = 0. \tag{28}$$

An $SL(2,\mathbb{R})$-valued soliton connection 1-form Γ may be given in terms of the 1-forms π_0, π_1 and π_2,

$$\Gamma = \begin{pmatrix} \theta_0 & \theta_1 \\ \theta_2 & -\theta_0 \end{pmatrix}, \tag{29}$$

where

$$\theta_0 = \alpha\pi_0, \qquad \theta_1 = \alpha_1\left(\pi_1 + \frac{1}{2\alpha}\pi_2\right), \qquad \theta_2 = \alpha_2\left(\pi_1 - \frac{1}{2\alpha}\pi_2\right). \tag{30}$$

Here we have $4\alpha^2 + \epsilon = 0$, $\alpha_1\alpha_2 = \alpha^2$. Let Ψ be a 2×2 matrix-valued (0-form) function of s and τ. Then $d\Psi = \Gamma\Psi$ defines the Lax equation without a spectral parameter. Such a constant may be introduced by performing a gauge transformation $\Gamma' = S\Gamma S^{-1} + dS\,S^{-1}$. Here S is 2×2 matrix-valued function of s, σ and the spectral parameter. In this way we set up a correspondence between a curve C moving in a space V_3 with a soliton connection.

The line element (1) on the surface S, using the parameters (s,σ) of the moving C, reduces to

$$ds^2 = (ds + w\,d\sigma)^2 + \epsilon(p^2 + q^2)d\sigma^2. \tag{31}$$

The Gaussian curvature K of S with the first fundamental form given in (31) is different from zero in general. On the other hand, by the choice $\tau = q = 0$, the line element becomes

$$ds^2 = (ds + w\,dt)^2 + \epsilon p^2 dt^2. \tag{32}$$

The Gaussian curvature K becomes

$$K = \frac{1}{4p}(k_{,\sigma} - \mathbf{R}p) \tag{33}$$

which vanishes by virtue of Eq. (15). Hence all the curves related to Eq. (15) trace flat 2-surfaces. It was usually believed that integrable equations arise from the curved surfaces. For instance the sine-Gordon equation arises from the surface with the line element $ds^2 = \cos^2(\theta)d\sigma^2 + \sin^2(\theta)ds^2$, which describes surfaces of constant negative Gaussian curvature [1]. Here we show that all integrable equations including the sine-Gordon equation may also arise from flat 2-surfaces (for mKdV see Ref. [14]).

In this work we considered the motion of a curve in a 3-space V_3. This condition may be relaxed, but for an arbitrary V_n where $n > 3$ the evolution equations corresponding to the geometrical scalars (k, $\tau,\ldots$) of the curves become quite complicated. It is perhaps more physical and significant to consider the case $n = 4$. This corresponds to classical strings moving in four-dimensional Minkowskian space. Hence it will be quite interesting to see the correspondence between strings and the soliton equations with four dependent variables. Let $x^\mu(s,\sigma)$ denote the strings in M_4. In a similar manner we define curves $x^\mu(s)$ parametrised with arc length s and its variations $x^\mu(s,\sigma)$. We have the orthonormal tetrad $(t^\mu, n^\mu, b_1^\mu, b_2^\mu)$ with

$$\eta_{\mu\nu}t^\mu t^\nu = 1, \qquad \eta_{\mu\nu}n^\mu n^\nu = \epsilon, \tag{34}$$

$$\eta_{\mu\nu}b_1^\mu b_1^\nu = \epsilon, \qquad \eta_{\mu\nu}b_2^\mu b_2^\nu = \epsilon, \tag{35}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,\epsilon,\epsilon,\epsilon)$, and Greek letters run from 1 to 4. Here $\epsilon = -1$ but we keep it to compare the results obtained here with previous sections. The Serret–Frenet equations governing the motion of the tetrad are as follows,

$$t^\mu_{,s} = kn^\mu, \tag{36}$$

$$n^\mu_{,s} = -\epsilon k t^\mu - \tau_1 b_1^\mu - \tau_2 b_2^\mu, \tag{37}$$

$$b^\mu_{1,s} = \tau_2 n^\mu + \tau_3 b_2^\mu, \tag{38}$$

$$b^\mu_{2,s} = \tau_1 n^\mu - \tau_3 b_1^\mu \tag{39}$$

where $k, \tau_1, \tau_2, \tau_3$ are the geometrical scalars describing the curvature and torsions in each 3-space direction, respectively. Here we have $t^\mu = x^\mu_{,s}$. Letting

$$x^\mu_{,\sigma} = p n^\mu + w t^\mu + q_1 b_1^\mu + q_2 b_2^\mu, \tag{40}$$

where p, w, q_1, q_2 are functions of s and σ. Here it is clear that when τ_2, τ_3 and q_2 vanish we get the same equations obtained in the previous sections for three-dimensional spaces. Hence the strings in four dimensions has a very direct correspondence with the integrable evolution equations. This connection and further progress on the motion of curves in a four-dimensional space will be communicated elsewhere.

In this work we established a connection between the curves moving in a 3-space with arbitrary signature (-1 or 3) and soliton equations. We showed that to each soliton (integrable) equation there exists a class of curves moving either in a Euclidean (E_3) or pseudo-Euclidean (M_3) 3-space. The signature of V_3 and the sign of the self-interacting terms in the soliton equations are directly related. We also showed that many integrable nonlinear PDEs may also arise from flat surfaces contrary to the common belief so far [14].

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