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Stable bundles, representation theory and Hermitian operators

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Abstract. We give an interpretation and a solution of the classical problem of the spectrum of the sum of Hermitian matrices in terms of stable bundles on the projective plane.

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In this paper we'll deal with three apparently disjoint problems:

(1) The spectrum of a sum of Hermitian operators,
(2) Components of tensor product of irreducible representations of the group $\mathrm{G}L_n(\mathbb{C})$,
(3) Structure of the moduli space of stable bundles on the projective plane $\mathbb{P}^2$.

We begin with the most familiar subject

1. Hermitian operators

Let $A : E \to E$ be a Hermitian operator in a unitary space E of finite dimension n and let

$$\lambda(A) \;:\; \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$$

be its spectrum. We have the classical

1.1. Problem. *What are the relations between the spectra $\lambda(A)$, $\lambda(B)$ and $\lambda(A+B)$?*

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Here are some of them. First of all we have

(0) *Trace identity*

$$\sum_i \lambda_i(A+B) = \sum_i \lambda_i(A) + \sum_i \lambda_i(B)$$

and a number of classical inequalities [M-O], due to

(1) *Herman Weyl*

$$\lambda_{i+j-1}(A+B) \le \lambda_i(A) + \lambda_j(B), \text{ for } i+j \le n+1,$$
$$\lambda_{i+j-n}(A+B) \ge \lambda_i(A) + \lambda_j(B), \text{ for } i+j \ge n+1,$$

(2) *Ky Fan*

$$\sum_{i\le p} \lambda_i(A+B) \le \sum_{i\le p} \lambda_i(A) + \sum_{i\le p}(B),$$

(3) *Lidskii and Wielandt*

$$\sum_{i\in I} \lambda_i(A+B) \le \sum_{i\in I} \lambda_i(A) + \sum_{i\le p}(B),$$

where I is any subset of $\{1,2,\dots,n\}$ of cardinality p,

and more.

The inequalities (1)–(3) give a complete list of restrictions for $\dim E \le 3$. But in higher dimensions there are a lot of others. It turns out (this is one of the results of this paper) that all of them are of the form

$$\sum_{k\in K} \lambda_k(A+B) \le \sum_{i\in I} \lambda_i(A) + \sum_{j\in J}(B) \tag{IJK}$$

for some triple of subsets $I, J, K \subset \{1,2,\dots,n\}$ of the same cardinality. To describe these triples precisely, let us fix a decomposition $n = p+q$. We have a bijection between subsets $I \subset \{1,2,\dots,n\}$ of cardinality $p = |I|$ and Young diagrams $\sigma = \sigma_I$ in a rectangular box of dimension p (North) by q (East) given as follows.

Let $\Gamma = \Gamma_I$ be a polygonal line with unit edges that runs from the South-West corner of the box to the East-North corner with the i-th edge running to the North for $i \in I$ and to the East otherwise. The line $\Gamma = \Gamma_I$ cuts out from the box a Young diagram $\sigma = \sigma_I \subset p \times q$ situated in its North-West angle. The diagram σ_I in the usual way [G-H] corresponds to a Schubert cycle s_I in a Chow ring of the Grassmannian

$$Gr(p,q) = \{V \subset E \mid \dim V = p, \ \operatorname{codim} V = q\}.$$

1.2. Theorem. *Consider a triple of subsets $I, J, K \subset \{1, 2, \dots, n\}$ such that the Schubert cycle s_K is a component of $s_I \cdot s_J$. Then*

i) *The inequality* (IJK) *holds.*

ii) *In union with the trace identity, this inequalities form a complete and independent set of restrictions on spectra of A, B and $A + B$.*

1.2.1. Remark. The first version of this theorem appears in [Kl6]. See also paper of Helmke and Rosenthal [H-R], which contains a proof of part i) of Theorem 1.2. Some restrictions on the spectrum of the *product* of positive Hermitian matrices was found by Berezin and Gelfand [B-G]. Combining results of this paper and methods of [B-G], one can prove that the inequalities of the theorem, being applied to logarithms of *singular values* $\sigma(A) := \lambda(AA^*), \sigma(B)$ and $\sigma(AB)$, give a complete set of restrictions on the singular spectrum of product AB of nondegenerate complex matrices. There is a similar description of spectra of the product of unitary matrices, providing spectra of multipliers are close to 1. These results need in another technique and will be published separately.

We postpone the proof until section 3 and consider some

1.3. Examples.

(1) Let us take $p = 1$. Then $Gr(p, q) = \mathbb{P}^{n-1}$. In this case the Schubert cycle s_i corresponding to a one element subset $I = \{i\}$ is just H^{i-1}, where H is the class of a hyperplane. So we have the equation

$$s_i \cdot s_j = s_{i+j-1} \text{ for } i + j \leq n + 1,$$

which implies the Weyl inequality $\lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B)$. Taking $q = 1$, we get in a similar way the inequality

$$\lambda_{i+j-n}(A + B) \geq \lambda_i(A) + \lambda_j(B) \text{ for } i + j \geq n + 1,$$

also due to Weyl.

(2) Now let p be arbitrary and

$$I = J = K = \{1, 2, \dots, p\}.$$

Then $\sigma_I = \sigma_J = \sigma_K = \emptyset$ and

$$s_I = s_J = s_K = (\text{the fundamental class of } Gr(p, q)).$$

Hence $s_K = s_I \cdot s_K$, and we get the Ky Fan inequality.

(3) We can extend the previous example by taking $I = \{1, 2, \dots, p\}$ to be the initial interval and $J = K$ to be arbitrary. Then again s_I is the fundamental cycle and therefore

$$s_K = s_I \cdot s_J.$$

This gives us the Lidskii-Wielandt inequality.

For a fixed dimension $n = \dim E$, one can easily find all the components s_K of the product $s_I \cdot s_J$ by making use of the Littlewood-Richardson rule [Jam], [Mac] and then write down the corresponding inequalities (IJK). Most of them seem to be new. Here are some special cases.

1.3.1. Complementary cycles. Let us take a pair of complementary diagrams σ_I and σ_J, so that the central symmetry of the $p \times q$-box maps σ_I onto the complement of σ_J. In this case

$$s_I \cdot s_J = (\text{class of a point}) = s_{\{q+1,q+2,\dots,n\}}.$$

Complementary diagrams σ_I, σ_J correspond to subsets

$$I = \{i_1 < i_2 < \cdots < i_p\}$$
$$J = \{j_1 > j_2 > \cdots > j_p\}$$

such that

$$i_k + j_k = n + 1.$$

Theorem 1.2 gives us in this case the inequality

$$\sum_{k>q} \lambda_k(A+B) \le \sum_{i\in I} \lambda_i(A) + \sum_{i\in I} \lambda_{n+1-i}(B)$$

for any subset $I \subset \{1, 2, \dots, n\}$ of cardinality $p = n - q$.

1.3.2. Pieri's formula. Let a subset $I \subset \{1, 2, \dots, n\}$ be a union of nonadjacent intervals in $\mathbb{Z}$,

$$\begin{aligned} I =& [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_m, b_m] \\ =& I_1 \cup I_2 \cup \ldots \cup I_m \end{aligned}$$

that is, $a_{i+1} - b_i \ge 2$. Let us call a *splitting* of an interval $[a, b]$ the union of two intervals

$$[a, c-1] \cup [c+1, b+1],\ a \le c \le b,$$

(the first interval may be empty). For example, the basic Schubert cycle s_k (=characteristic class of the tautological bundle) corresponds to the following splitting of the initial interval $[1, p]$

$$[1, p] = [1, p-k] \cup [p-k+2, p+1].$$

In this notation Pierie's formula [G-H] may be written as

$$s_I \cdot s_k = \sum_{I'} s_{I'},$$

where summation runs over all sets I' obtained from I by splitting k intervals. Theorem 1.2 deduces from this decomposition the following inequality

$$\sum_{j\in I'}\lambda_j(A+B)\le\sum_{i\in I}\lambda_i(A)+\sum_{\substack{1\le i\le p+1\\ i\ne p-k+1}}\lambda_i(B),$$

valid for any set I' obtained from $I,\ |I|=p$ by splitting k intervals.

2. Representations of the general linear group

Before proving Theorem 1.2 we discuss briefly its connection with another problem. Let us consider an integer spectrum

$$\alpha\ :\ a_1\ge a_2\ge\cdots\ge a_n,\quad a_i\in\mathbb{Z},$$

and associate with it the following dominant weight of general linear group $GL_n(\mathbb{C})$

$$\omega^\alpha:\mathrm{diag}(x_1.x_2,\dots,x_n)\mapsto x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}.$$

The weight ω^α in the usual way corresponds to an irreducible representation $V(\omega^\alpha)$ of the group $GL_n(\mathbb{C})$ with highest weight ω^α. This gives rise to the following

2.1. Problem. *Which irreducible representation $V(\omega^\gamma)$ appears as a component of tensor product $V(\omega^\alpha)\otimes V(\omega^\beta)$?*

This problem has been studied by many authors; see for example [Ela] and references therein. Berenstein and Zelevinsky [B-Z] describe admissible triples $\omega^\alpha,\omega^\beta,\omega^\gamma$ as a projection of integer points in some convex cone in a space of so called Gelfand-Tzetlin patterns.

Our result is that the inequalities (IJK) of Theorem 1.2 answer this question too. More precisely,

2.2. Theorem. *The irreducible representation $V(\omega^{N\gamma})$ is a component of tensor product $V(\omega^{N\alpha})\otimes V(\omega^{N\beta})$ for some positive N if and only if α,β and γ are spectra of Hermitian operators A,B and $C=A+B$.*

2.2.1. Remark. It is not clear whether one can always take N=1. This depends on the very ampleness of some ample sheaves (see section 3 below). Anyway Problem 1.1 is manifestly homogeneous with respect to spectra, while the homogeneity of Problem 2.1 with respect to the weights $\omega^\alpha,\omega^\beta,\omega^\gamma$ is not self evident.

Theorems 1.2 and 2.2 both appear as a byproduct of study stable bundles on the projective plane. We turn to this subject in the next section. Since these theorems are of independent interest, we'll try to keep an exposition self-contained and as elementary as possible.

3. Stable bundles

Let

$$\mathbb{P}^2 = \{(x^\alpha : x^\beta : x^\gamma) | x \in \mathbb{C}\}$$

be homogeneous coordinates on the plane $\mathbb{P}^2$ and let

$$T = \{(t^\alpha : t^\beta : t^\gamma) | t \in \mathbb{C}^*\}$$

be the diagonal torus acting on $\mathbb{P}^2$ by the formula

$$t \cdot x = (t^\alpha x^\alpha : t^\beta x^\beta : t^\gamma x^\gamma).$$

The objects of our interest are T-*equivariant* (or *toric* for short) vector bundles $\mathcal{E}$ over $\mathbb{P}^2$. This means that $\mathcal{E}$ is endowed with an action $T : \mathcal{E}$ that is linear on the fibers and makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{t} & \mathcal{E} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^2 & \xrightarrow{t} & \mathbb{P}^2 \end{array} \qquad t \in T.$$

3.1. Why toric bundles?

The answer to this question is not very essential for the rest of the paper. Nevertheless we'll try to explain here why toric bundles are important (and even crucial) for understanding structure of all bundles on $\mathbb{P}^2$ (and other toric varieties).

First of all we need a notion of stability [Mum 63], [OSS].

3.1.1. Definition. A vector bundle $\mathcal{E}$ (or more generally torsion free sheaf) on $\mathbb{P}^2$ is said to be Mumford-Takemoto *stable* if, for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$, the following inequality holds:

$$\frac{c_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} < \frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}, \tag{3.1}$$

and *semistable* if weak inequalities hold (with sign $\leq$ instead of $<$). Here $c_1(\mathcal{E}) = \deg \det \mathcal{E}$ is the first Chern class of $\mathcal{E}$. We will refer to the right hand side of this inequality as *slope* of $\mathcal{E}$ and denote it by $\mu(\mathcal{E})$.

There are at least two facts which make the notion of stability of great geometric meaning:

(1) *Existence of moduli space* $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ of stable vector bundles $\mathcal{E}$ of given rank $r = \mathrm{rk}(\mathcal{E})$ and Chern classes $c_i = c_i(\mathcal{E})$ [Mar]. This moduli space

has a natural compactification $\overline{\mathcal{M}}$ by (equivalence classes) of semistable sheaves.

(2) *Donaldson theorem* [Don]: A vector bundle $\mathcal{E}$ admits an Einstein-Hermitian metric ($\Leftrightarrow$ metric of scalar Ricci curvature) if and only if it is a direct sum of stable bundles of the same slope.

We will see later that Theorem 1.2 essentially is just a toric variant of the Donaldson theorem.

On the other hand, the structure of the moduli space of vector bundles is of independent interest. Let us discuss it briefly. The action $T : \mathbb{P}^2$ induces a natural action of T on the moduli spaces $\mathcal{M}$ and $\overline{\mathcal{M}}$. Suppose for simplicity that the last space is nonsingular. Then we can apply the results of Bialynicki-Birula [B-B] to reduce the topological and birational structure of $\overline{\mathcal{M}}$ to the corresponding problems for the fixed point set $\overline{\mathcal{M}}^T$.

To be more precise, we are given an action of an algebraic torus T on a smooth projective variety X. Then we have a decomposition of the fixed point set X^T in disjoint union of connected components

$$X^T = \bigsqcup_i Y_i.$$

Fix a sufficiently general one parameter subtorus $\mathbb{C}^* \subset T$ such that

$$X^{\mathbb{C}^*} = X^T,$$

and consider the *Bialynicki-Birula stratification*

$$X^T = \bigsqcup_i X_i$$

given by

$$X_i = \{x \in X \,|\, \lim_{z \to 0} z \cdot x \in Y_i,\ z \in \mathbb{C}^*\}.$$

This stratification is locally closed and T-invariant (but it *depends* on the choice of the one parameter subgroup $\mathbb{C}^* \subset T$). Here are some of its properties.

i) $Y_i = X_i^T$.

ii) The natural projection

$$\begin{aligned} \pi_i : X_i &\to Y_i \\ x &\mapsto \lim_{\substack{z \to 0 \\ z \in \mathbb{C}^*}} z \cdot x \end{aligned}$$

has a structure of affine bundle with fiber $\mathbb{A}^{n_i}$. The rank n_i may be determined as follows. Consider the representation of $\mathbb{C}^* \subset T$ in the normal space N_i to $Y_i \subset X$ in some point $y_i \in Y_i$. Let $e_j \in N_i$ be the eigenbasis of this representation so that

$$z \cdot e_j = z^{a_j} e_j,\ a_j \in \mathbb{Z},\ z \in \mathbb{C}^*,$$

where on the right hand side z^{a_j} is considered as a complex number. Then

$$n_i = \#\{j | a_j > 0\}.$$

iii) The stratification is pure and gives decomposition of the Hodge cohomology groups [Gin]

$$H^{pq}(X) = \bigoplus_i H^{p-n_i, q-n_i}(Y_i).$$

This implies the equality of the Euler characteristics

$$\chi(X) = \chi(X^T).$$

The last equation is valid for noncompact X and *homological* Euler characteristic.

Informally we may summarize these properties as follows.

3.1.2. Localization principle. *Let $T : X$ be a torus action on a smooth projective variety X. Then the fixed point components $Y_i \subset X^T$ in combination with the torus representations in their normal bundles $T : N_i$ form a kind of skeleton of the variety X. These data allow us to restore the topology and birational geometry of X.*

It seems that the first time this approach has been applied to study moduli spaces was by Bertin and Ellengzwajg [B-E] (see also [E-S] and [Kl1–Kl4]).

For moduli of stable bundles on toric varieties, the fixed point scheme $\mathcal{M}^T$ has a natural interpretation

$$\mathcal{M}^T = \begin{pmatrix} \textit{Moduli space of stable toric bundles } \mathcal{E} \textit{ modulo} \\ \textit{twisting } \mathcal{E} \mapsto \mathcal{E} \otimes \chi \textit{ with a character } \chi \textit{ of } T \end{pmatrix}.$$

Thus the localization principle essentially reduces the study of stable bundles and sheaves on $\mathbb{P}^2$ to the corresponding equivariant objects. Most of the known results on structure of toric bundles and sheaves are contained in papers [Kl1–Kl4]. We'll give a brief exposition in the next item.

3.2. Structure of toric bundles

Let $\mathcal{E}$ be a toric vector bundle on $\mathbb{P}^2$. Let us fix a generic point $p_0 \in \mathbb{P}^2$ not in the coordinate lines and denote by

$$E := \mathcal{E}(p_0)$$

the corresponding generic fiber. There is no action of T on E, but nevertheless the equivariant structure of $\mathcal{E}$ gives us some distinguished subspaces in E in the following way.

Let us choose a coordinate line

$$X^\alpha : x^\alpha = 0$$

and a generic point $p_\alpha \in X^\alpha$ in it. Since the T-orbit of p_0 is dense in $\mathbb{P}^2$, we can vary $t \in T$ so that tp_0 tends to p_α. Then for any vector $e \in E = \mathcal{E}(p_0)$, we have $te \in \mathcal{E}(tp_0)$ and can try the limit

$$\lim_{tp_0 \to p_\alpha} (te)$$

which may or not exists. Let us denote by $E^\alpha(0)$ the set of vectors $e \in E$ for which this limit exists:

$$E^\alpha(0) := \{e \in E | \lim_{tp_0 \to p_\alpha} (te) \quad \text{exists}\}.$$

It is easy to see that $E^\alpha(0)$ is a vector subspace in E.

We can slightly extend the previous construction and define, for any $i \in \mathbb{Z}$, the subspace

$$E^\alpha(i) := \{e \in E | \lim_{tp_0 \to p_\alpha} f(tp_0) \cdot (te) \quad \text{exists}\},$$

where $f(p)$ is any rational function on $\mathbb{P}^2$ with pole of order i on the coordinate line $X^\alpha : x^\alpha = 0$. These subspaces form a nonincreasing $\mathbb{Z}$-filtration which we denote by E^α:

$$E^\alpha : \cdots \supset E^\alpha(i-1) \supset E^\alpha(i) \supset E^\alpha(i+1) \supset \cdots .$$

The filtration E^α is exhaustive, i.e.

$$E^\alpha(i) = 0, \text{ for } i \gg 0$$
$$E^\alpha(i) = E, \text{ for } i \ll 0.$$

So we can associate with a toric bundle $\mathcal{E}$ a triple of filtrations $E^\alpha, E^\beta, E^\gamma$ (one for each coordinate line) of the generic fiber $E = \mathcal{E}(p_0)$. Now we have

3.2.1. Theorem [Kl2]**.** *The correspondence*

$$\mathcal{E} \mapsto (E^\alpha, E^\beta, E^\gamma)$$

gives an equivalence of the category of toric vector bundles on $\mathbb{P}^2$ and a category of vector spaces E endowed with a triple of nonincreasing $\mathbb{Z}$-filtrations $E^\alpha, E^\beta, E^\gamma$. □

The theorem tells us that any property or invariant of a vector bundle $\mathcal{E}$ has a counterpart on the level of filtrations. Here we are mainly interested in the property of stability. In terms of filtrations it may be stated as follows.

3.2.2. Theorem. *Let an equivariant vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ correspond by Theorem 3.2.1 to a triple of filtrations $E^\alpha, E^\beta, E^\gamma$. Then $\mathcal{E}$ is semistable if and only if for any nontrivial subspace $0 \neq F \subset E$, the following inequality holds*

$$\frac{1}{\dim F} \sum_{\substack{i\in\mathbb{Z} \\ \sigma=\alpha,\beta,\gamma}} i \dim F^{[\sigma]}(i) \leq \frac{1}{\dim E} \sum_{\substack{i\in\mathbb{Z} \\ \sigma=\alpha,\beta,\gamma}} i \dim E^{[\alpha]}(i) \tag{3.2}$$

where $F^\alpha = F \cap E^\alpha$ is the induced filtration and $E^{[\alpha]}(i) = E^\alpha(i)/E^\alpha(i+1)$ are composition factors of the filtration E^α.

Proof. Let us first of all remark that the sum in the right hand side of the inequality (3.2) is just the first Chern class [Kl2] of the vector bundle $\mathcal{E}$

$$c_1(\mathcal{E}) = \sum_{\substack{i\in\mathbb{Z} \\ \sigma=\alpha,\beta,\gamma}} i \dim E^{[\alpha]}(i).$$

Hence the theorem means that for a toric bundle $\mathcal{E}$ it suffices to check the semistability inequality

$$\frac{c_1(\mathcal{F})}{\operatorname{rk}(\mathcal{F})} \leq \frac{c_1(\mathcal{E})}{\operatorname{rk}(\mathcal{E})}$$

for *toric* subsheavs $\mathcal{F} \subset \mathcal{E}$ only. This in turn follows from existence of so called *Harder-Narasimhan filtration* [H-N]

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}$$

such that

i) The composition factors $\mathcal{E}_{[i]} = \mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable,
ii) Their slopes $\mu(\mathcal{E}) = c_1(\mathcal{E})/\operatorname{rk}(\mathcal{E})$ strictly decrease

$$\mu(\mathcal{E}_{[1]}) > \mu(\mathcal{E}_{[2]}) > \cdots > \mu(\mathcal{E}_{[m]}).$$

The Harder-Narasimhan filtration is unique and it is trivial only for semistable bundles. The uniqueness implies that for a toric bundle $\mathcal{E}$, the filtration should be T-invariant. Hence the destabilizing subsheaf $\mathcal{F} \subset \mathcal{E}$ may be taken as T-stable. □

3.3. Invariant theoretical interpretation

The inequalities (3.2) of Theorem 3.2.2 have an important interpretation in the framework of geometric invariant theory. To explain it let us denote by

$$(\mathcal{F}^\alpha, \mathcal{F}^\beta, \mathcal{F}^\gamma)$$

the triple of flags corresponding to a triple of filtrations

$$(E^\alpha, E^\beta, E^\gamma).$$

To simplify the notations, we will sometimes tacitly suppose the flags to be complete. This does not affect the final results.

The filtration E^α define a function $m_\alpha : \mathbb{Z} \to \mathbb{N}$ given by

$$m_\alpha : i \mapsto \dim E^{[\alpha]}(i).$$

This function has finite support and we will treat its values as *multiplicities*. The sequence

$$\lambda^\alpha \ : \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \tag{3.3}$$

consisting of all points in the support of m_α, each taken according to its multiplicity, said to be the *spectrum* of the filtration E^α. At last we define the dominant weight ω^α of the general linear group $G = GL_n(\mathbb{C})$ by the formula

$$\omega^\alpha : \operatorname{diag}(x_1, x_2, \dots, x_n) \mapsto x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}. \tag{3.4}$$

The weight ω^α in turn defines a line bundle $\mathcal{L}(\omega_\alpha)$ on the variety of complete flags $\mathrm{Flag}_n = G/B$ induced by the character ω^α of the Borel subgroup $B \subset G$. If the flag $\mathcal{F}^\alpha$ is complete, then the bundle $\mathcal{L}(\omega^\alpha)$ is ample. Otherwise it induces an ample bundle on the variety of noncomplete flags of type $\mathcal{F}^\alpha$.

3.3.1. Observation. *A triple of filtrations* $(E^\alpha, E^\beta, E^\gamma)$ *is (semi)stable (i.e. satisfies inequalities* (3.2)*) if and only if the corresponding triple of flags*

$$(\mathcal{F}^\alpha, \mathcal{F}^\beta, \mathcal{F}^\gamma) \in \mathrm{Flag}(E) \times \mathrm{Flag}(E) \times \mathrm{Flag}(E)$$

is stable with respect to the diagonal action of $SL(E)$ *and the polarization*

$$\mathcal{L} = \mathcal{L}(\omega^\alpha) \boxtimes \mathcal{L}(\omega^\beta) \boxtimes \mathcal{L}(\omega^\gamma)$$

in the sense of geometric invariant theory [Mum 65].

Proof. This is a straightforward generalization of a favorite Mumford example of a configuration of subspaces $V^\alpha \subset E$ of *the same* dimension $p = \dim V^\alpha$ with respect to the usual Plücker embedding of the Grassmannian $Gr(p, q)$ [Mum 61], [Mum 63], [Mum 65]. □

Let us recall that the semistability of a point $x \in X$ of a G-variety X with respect to an invertible G-sheaf $\mathcal{L}$ means that there exists a G-invariant section

$$s \in \Gamma(X, \mathcal{L}^{\otimes N})^G$$

which is nonzero at x. For

$$X = \mathrm{Flag}(E) \times \mathrm{Flag}(E) \times \mathrm{Flag}(E)$$

and

$$\mathcal{L} = \mathcal{L}(\omega^\alpha) \boxtimes \mathcal{L}(\omega^\beta) \boxtimes \mathcal{L}(\omega^\gamma),$$

we have by the Borel-Weil theorem [Bot]

$$\Gamma(X, \mathcal{L}) = V(\omega^\alpha) \otimes V(\omega^\beta) \otimes V(\omega^\gamma),$$

where $V(\omega)$ is an irreducible representation of $GL(E)$ with highest weight ω. As result we get

3.3.2. Corollary. *A triple of filtrations $E^\alpha, E^\beta, E^\gamma$ in general position is semistable if and only if, for some $N > 0$, the tensor product of irreducible representations*

$$V(N\omega^\alpha) \otimes V(N\omega^\beta) \otimes V(N\omega^\gamma)$$

contains a nonzero $SL(E)$ invariant.

Proof. As it was explained above, the stability of general triple of flags $(\mathcal{F}^\alpha, \mathcal{F}^\beta, \mathcal{F}^\gamma)$ is equivalent to the existence of a nonzero G-invariant section $s \in \Gamma(X, \mathcal{L}^{\otimes N}) = V(N\omega^\alpha) \otimes V(N\omega^\beta) \otimes V(N\omega^\gamma)$. □

Moving a term from left to right, one can readily restate the corollary as an assertion on irreducible components

$$V(N\omega^\gamma) \subset V(N\omega^\alpha) \otimes V(N\omega^\beta)$$

of the tensor product. Hence the problem of describing these components is equivalent to the problem of stability for a triple of filtrations in a general position. In the next theorem we give a criterion for this by applying the Schubert calculus to the inequalities (3.2).

3.3.3. Theorem. *Let $E^\alpha, E^\beta, E^\gamma$ be a triple of filtrations in general position with spectra* (3.3) *$\lambda^\alpha, \lambda^\beta, \lambda^\gamma$. Then the following conditions are equivalent:*

i) *The triple $(E^\alpha, E^\beta, E^\gamma)$ is semistable;*
ii) *For any triple of Schubert cycles s_I, s_J, s_K (see section 1) with nonzero product in the Chow ring of the Grassmannian, the following inequality holds:*

$$\frac{1}{p}\left(\sum_{i\in I}\lambda_i^\alpha + \sum_{j\in J}\lambda_j^\beta + \sum_{k\in K}\lambda_k^\gamma\right) \le \frac{1}{n}\left(\sum_{i=1}^n\lambda_i^\alpha + \sum_{j=1}^n\lambda_j^\beta + \sum_{k=1}^n\lambda_k^\gamma\right) \tag{3.5}$$

where $p = \#I = \#J = \#K$.

Proof. The inequality of (semi)stability (3.2) depends only on dimensions of the intersections of a test subspace $F \subset E$ with the filtrations $E^\sigma, E^\beta, E^\gamma$. These dimensions determine relative positions of F with respect to these filtrations, i.e. three Young diagrams $\sigma_I, \sigma_J, \sigma_K$ (in the notation of section 1). The inequality (3.5) is just the stability inequality (3.2) for a subspace F in positions $\sigma_I, \sigma_J, \sigma_K$ with respect to $E^\sigma, E^\beta, E^\gamma$.

So to write down all stability inequalities (3.2), we have to decide which relative positions $\sigma_I, \sigma_J, \sigma_K$ are possible, i.e. which Schubert cells c_I, c_J, c_K in general position have a nonempty intersection. Because of the generality position, the question reduces to nontriviality of the intersection of the corresponding Schubert cycles s_I, s_J, s_K, which proves the theorem. □

Remark. If the product $s_I \cap s_J \cap s_K$ has positive dimension, then it should intersect the boundary component $s_I \backslash c_I$, since the Schubert cell c_I is affine. This allows us to change the cycle s_I to one of its degenerations $s_{I'} \subset s_I$ which *improves* the inequality (3.5). As a result, the inequality (3.5) suffices to check for Schubert cycles with zero-dimensional intersection.

Combining this remark with Corollary 3.3.2 and moving some terms from left to right, we get the following result.

3.3.4. Corollary. *Let as before $\lambda^\alpha, \lambda^\beta, \lambda^\gamma$ be the spectra* (3.3) *and let $\omega^\alpha, \omega^\beta, \omega^\gamma$ be the corresponding dominant weights of the group $GL(E)$. Then the following conditions are equivalent:*

i) *For some positive N the irreducible representation $V(N\omega^\gamma)$ is a component of $V(N\omega^\alpha) \otimes V(N\omega^\beta)$;*

ii) *The spectra $\lambda^\alpha, \lambda^\beta, \lambda^\gamma$ satisfy the trace identity*

$$\sum_{k=1}^{n} \lambda_k^\gamma = \sum_{i=1}^{n} \lambda_i^\alpha + \sum_{j=1}^{n} \lambda_j^\beta,$$

and for any triple of subsets I, J, K such that the Schubert cycle s_K is a component of $s_I \cdot s_J$, the following inequality holds:

$$\sum_{k \in K} \lambda_k^\gamma \leq \sum_{i \in I} \lambda_i^\alpha + \sum_{j \in J} \lambda_j^\beta. \tag{IJK}$$

Proof. The trace identity appears because we deal with the general linear group $GL(E)$ rather than $SL(E)$, as in the Corollary 3.3.2. The inequality (IJK) follows from this trace identity and the corresponding inequality of the theorem. □

3.3.5. Corollary. *Theorem* 2.2 *follows from Theorem* 1.2. □

3.4. Back to Hermitian operators

Now let E be a unitary space and $H : E \to E$ a Hermitian operator. Consider *the spectral filtration* of H given by

$$E^H(x) = \begin{pmatrix} \textit{sum of eigenspaces of } H \\ \textit{with eigenvalues} \geq x \end{pmatrix}.$$

It is well known that the Hermitian operator may be reconstructed from its spectral filtration by means of the *spectral decomposition*

$$H = \int_{-\infty}^{\infty} x dP_H(x),$$

where $P_H(x)$ is the orthogonal projector with kernel $E^H(x)$. So in a unitary space E we have equivalence

$$(\mathbb{R}\textit{-filtrations}) \Leftrightarrow (\textit{Hermitian operators}).$$

Let us define the *multiplicity* of an $\mathbb{R}$-filtration E^α in a point $x \in \mathbb{R}$ by the formula

$$m_E^\alpha(x) := \dim E^\alpha(x) - \dim E^\alpha(x + 0).$$

This is an integer valued function with finite support, which is said to be the *spectrum* of the filtration E^α. We use it to extend the notion of stability from $\mathbb{Z}$- to $\mathbb{R}$-filtrations.

3.4.1. Definition. A family of $\mathbb{R}$-filtrations $E^\alpha, \alpha \in A$, is said to be *semistable* if for any proper subspace $F \subset E$ with induced filtrations $F^\alpha = F \cap E^\alpha$, the following inequality holds:

$$\frac{1}{\dim F} \sum_{\substack{x \in \mathbb{R} \\ \alpha \in A}} m_F^\alpha(x) \leq \frac{1}{\dim E} \sum_{\substack{x \in \mathbb{R} \\ \alpha \in A}} m_E^\alpha(x).$$

If the strict inequalities are valid, the family $E^\alpha, \alpha \in A$, is said to be *stable*.

On the right hand side, we readily recognize the trace of Hermitian operator H^α with spectral filtration E^α. So the stability inequality may be rewritten as

$$\frac{1}{\dim E} \sum_{\alpha \in A} \mathrm{Tr}(H^\alpha) \geq \frac{1}{\dim F} \sum_{\substack{x \in \mathbb{R} \\ \alpha \in A}} m_F^\alpha(x). \tag{3.6}$$

Let us define the restriction $H|F$ of a Hermitian operator H onto a subspace $F \subset E$ as the the composition

$$H|F : \; F \hookrightarrow E \xrightarrow{H} E \longrightarrow F$$

of the inclusion $F \hookrightarrow E$, the operator $H : E \to E$ and orthogonal projection $E \to F$. Then we have

3.4.2. Proposition. *In the previous notation the following inequality holds:*

$$\operatorname{Tr}(H^\alpha|F) \geq \sum_{x\in\mathbb{R}} x \cdot m_F^\alpha(x). \tag{3.7}$$

The equality take place for H-stable subspace $F \subset E$ only.

Proof. Let us consider the spectrum of H^α

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

and the corresponding orthonormal eigenbasis

$$e_1, e_2, \ldots, e_n.$$

Denote by $E_i = \langle e_1, e_2, \ldots, e_i \rangle$ the subspace spanned by the first i eigenvalues and put $F_i = F \cap E_i$. In this notation the inequality (3.7) may be rewritten as

$$\operatorname{Tr}(H^\alpha|F) \geq \sum_{i=1}^{n} \lambda_i \dim F_i/F_{i-1} = \sum_{j\in J} \lambda_j,$$

where J is the set of the indices for which $F_j \neq F_{j-1}$. Let F_j, $j \in J$ be an orthonormal basis of the space F consistent with the filtration F_i, i.e. $f_j \in F_j \cap F_{j'}^\perp$ where $j' \in J$ is the immediate predecessor of j. From the extremal property of eigenvalues,

$$\lambda_j = \min_{x\in E_j} \frac{(Hx, x)}{(x, x)}$$

and inclusions $f_j \in F_j \subset E_j$ it follows that $(Hf_j, f_j) \geq \lambda_j$. Hence

$$\operatorname{Tr}(H|F) = \sum_{j\in J} (Hf_j, f_j) \geq \sum_{j\in J} \lambda_j = \sum_{i=1}^{n} \lambda_i \dim F_i/F_{i-1}.$$

This is equivalent to (3.7).

Looking at the last equation, we see that equality in (3.7) may occur only if $(Hf_j, f_j) = \lambda_j$ for all $j \in J$. In this case $Hf_j = \lambda_j f_j$ and subspace $F \subset E$ is invariant under the operator H. □

Now we are in position to prove the first result on stable families of filtrations.

3.4.3. Theorem. *Let H^α be a family of Hermitian operators with scalar sum*

$$\sum_\alpha H^\alpha = \lambda \cdot \mathbb{I}. \tag{3.8}$$

Then the corresponding family of spectral filtrations E^α is semistable.

Proof. Let us apply the orthogonal projector $\pi : E \to F$ to both sides of (3.8) and take the trace. Then making use of (3.7) we get

$$\lambda = \frac{1}{\dim F}\sum_\alpha \operatorname{Tr}(H^\alpha|F) \geq \frac{1}{\dim F}\sum_\alpha \sum_{x\in\mathbb{R}} x\cdot m_F^\alpha(x).$$

On the other hand, taking the trace of (3.8), we get

$$\lambda = \frac{1}{\dim E}\sum_\alpha \operatorname{Tr} H^\alpha.$$

Combining these two relations, we get the semistability inequality in the form (3.6). □

3.4.4. Remark. Suppose that under conditions of the theorem in the semistability inequalities (3.6), we have an equality for a subspace $F \subset E$. Then by Proposition 3.4.2, the subspace F and its orthogonal complement $F^\perp$ should be invariant with respect to all operators H^α. This means that the configuration of spectral filtrations E^α splits into direct sum $F \oplus F^\perp$. Continuing in this way, we can decompose the configuration of spectral filtrations in a direct sum of *stable* configurations of the same *slope* (= left hand side of (3.6)).

3.4.5. Remark. The simplest example of the operators with scalar sum are orthogonal projectors $P^\alpha : V \to V_\alpha$ of some orthogonal decomposition

$$V = \bigoplus_\alpha V_\alpha.$$

A less trivial example give the restrictions $H^\alpha = P^\alpha|E$ of these projectors on a subspace $E \subset V$. The theorem of Naimark [Nai] asserts that up to an affine transformations $H^\alpha \mapsto aH^\alpha + b\mathbb{I}$, the last example is universal. More precisely, for a family of *positive* operators $H^\alpha : E \to E$ such that

$$\sum_{\alpha\in I} H^\alpha = \mathbb{I},$$

there exists a unitary extension

$$E \subset \bigoplus_\alpha V_\alpha$$

(orthogonal sum) such that $H^\alpha = P^\alpha|E$. The extension may be taken *subdirect product*, i.e. with surjective projections $E \to V_\alpha$, and in this case it is unique.

3.5. Einstein-Hermitian metric

In this section we will deal with the following question which arises from Theorem 3.4.3.

3.5.1. Problem. *Let E^α be a stable family of filtrations. Does there exist a Hermitian metric on E such that sum of Hermitian operators H^α with spectral filtrations E^α is a scalar?*

It turns out that the answer is positive. This metric should be considered as a toric analog of the Einstein-Hermitian metric of scalar Ricci curvature on a stable vector bundle $\mathcal{E}$ [Don]. This motivates the following definition.

3.5.2. Definition. Let $E^\alpha, \alpha \in A$ be a family of filtrations of a complex vector space E. A Hermitian metric on E is said to be an *Einstein-Hermitian metric* for this family if the sum of Hermitian operators H^α with the spectral filtrations E^α is a scalar operator, i.e.

$$\sum_\alpha H^\alpha = \lambda \cdot \mathbb{I}.$$

A stable configuration of filtrations E^α has only scalar endomorphisms. On the other hand, a unitary isomorphism between two Einstein-Hermitian metrics gives such an endomorphism. Hence the Einstein-Hermitian metric, if it exists, is unique up to proportionality.

We deduce the existence of this metric from the unitary trick of Kemf and Ness [K-N].

3.5.3. Theorem. *For any stable system of filtrations $E^\alpha, \alpha \in A$ in a complex vector space E, there exists an Einstein-Hermitian metric on E.*

Proof. Since the stability condition is open, we may restrict ourselves to the filtrations E^α with *rational* spectra (= spectra of the corresponding Hermitian operators H^α for *any* metric on E). An appropriate homothety $E^\alpha(x) \mapsto E^\alpha(ax)$ further reduces the problem to $\mathbb{Z}$-filtrations E^α. This allows us to give a purely algebraic proof (for irrational spectra one have to use Kählerian methods).

Next, the stability condition is invariant under affine transformations

$$E^\alpha(x) \mapsto E^\alpha(ax + b_\alpha).$$

Hence we may suppose the spectrum (3.3) of each filtration E^α to be traceless

$$\sum_i \lambda_i^\alpha = 0.$$

Then the corresponding dominant weight (3.4)

$$\omega^\alpha : \mathrm{diag}(x_1, x_2, \dots, x_n) \mapsto x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

comes from the group PGL_n. It is convenient for us to identify a complete flag $\mathcal{F}$ in E with the Borel subgroup

$$B = \mathrm{Aut}(\mathcal{F}) \subset GL(E).$$

Then by the observation 3.3.1, a stable configuration of filtrations E^α of weights ω^α may be identified with a stable configuration of Borel subgroups

$$\prod_\alpha B^\alpha \in \prod_\alpha \mathcal{B}^\alpha$$

with respect to the polarization

$$\mathcal{L} = \bigotimes_\alpha \mathcal{L}(\omega^\alpha).$$

Here $\mathcal{B}^\alpha$ is just an exemplar of variety of Borel[1] subgroups in $GL(E)$.

Now by the theorem of Borel and Weil [Bot], the global sections of the sheaf $\mathcal{L}(\omega^\alpha)$

$$\Gamma(\mathcal{B}^\alpha, \mathcal{L}(\omega^\alpha)) = V(\omega^\alpha)$$

form a space of an irreducible representation $V(\omega^\alpha)$ of the group $SL(E)$ with the highest weight ω^α. So we have a map

$$\varphi : \mathcal{B}^\alpha \to \mathbb{P}(V(\omega^\alpha))$$

which assigns to a Borel subgroup $B^\alpha \in \mathcal{B}^\alpha$ the highest vector

$$v(B^\alpha) \in V(\omega^\alpha)$$

with respect to B^α. In a similar way the sheaf $\mathcal{L} = \bigotimes_\alpha \mathcal{L}(\omega^\alpha)$ defines a morphism

$$\varphi : \prod_\alpha \mathcal{B}^\alpha \to \mathbb{P}\big(\bigotimes_\alpha V(\omega^\alpha)\big)$$
$$\prod_\alpha B^\alpha \mapsto \bigotimes_\alpha v(B^\alpha).$$

Let us now choose an arbitrary Hermitian metric on E and a unitary invariant metric on each irreducible representation $V(\omega^\alpha)$. Then we may apply the theorem of Kempf and Ness [K-N] which in our settings means that, for a $\mathcal{L}$-stable configuration of Borel subgroups

$$\prod_\alpha B^\alpha \in \prod_\alpha \mathcal{B}^\alpha,$$

[1] or rather parabolic conjugate to $\mathrm{Aut}(E^\alpha)$.

the $SL(E)$ orbit of the vector

$$\bigotimes_\alpha v(B^\alpha) \in \bigotimes_\alpha V(\omega^\alpha)$$

contains a vector of minimal length and it is unique up to the action of the unitary group $U(E)$.

Let us suppose that $\bigotimes_\alpha v(B^\alpha)$ is just this vector of minimal length

$$\left|\bigotimes_\alpha v(B^\alpha)\right|^2 = \prod_\alpha (v(B^\alpha), v(B^\alpha)) = \min .$$

Then the extremum condition gives the equation

$$\sum_\alpha \frac{(g\cdot v(B^\alpha), v(B^\alpha)) + (v(B^\alpha), g\cdot v(B^\alpha))}{(v(B^\alpha), v(B^\alpha))} = 0, \tag{3.8}$$

where $g \in \mathfrak{sl}(E)$ is any traceless transformation. Let us denote the summands of this formula by

$$\ell^\alpha(g) := \frac{(g\cdot v(B^\alpha), v(B^\alpha)) + (v(B^\alpha), g\cdot v(B^\alpha))}{(v(B^\alpha), v(B^\alpha))}.$$

These are real linear forms on $\mathfrak{sl}(E)$ with the following properties:

i) $\ell^\alpha(g) = \ell^\alpha(g^*)$, *i.e.* ℓ^α *depends only on the Hermitian part of* g. □

Moreover

ii) $\ell^\alpha(g)$ *depends only on the diagonal part of* g *with respect to the Borel subgroup* B^α.

Let $U^\alpha \subset B^\alpha$ be the unipotent radical and $\mathcal{U}^\alpha \subset \mathfrak{sl}(E)$ the corresponding nilpotent subalgebra. Since $\mathcal{U}^\alpha$ annihilates the highest vector $v(B^\alpha)$, then

$$\ell^\alpha(\mathcal{U}^\alpha) = 0.$$

By i) the same is true for the opposite unipotent subalgebra $\mathcal{U}^\alpha_- = (\mathcal{U}^\alpha)^*$. □

iii) $\ell^\alpha(g) = \operatorname{Tr}((g + g^*)\cdot H^\alpha)$ *where* H^α *is the Hermitian operator with the spectral filtration* E^α.

Treating $\mathcal{U}^\alpha$ and $\mathcal{U}^\alpha_-$ as upper and lower triangular matrices, we find that H^α is a diagonal matrix. Hence by ii) both sides of iii) depend only on the diagonal part of g. Since $v(B^\alpha)$ is an eigenvector of weight ω^α for the Cartan (=diagonal) subalgebra, then for diagonal element g the equation iii) is a tautology. □

The proof of the theorem is now straightforward. Making use of iii) we can rewrite the extremum equation (3.8) in the form

$$\sum_\alpha \operatorname{Tr}((g+g^*)\cdot H^\alpha)=0,\ \forall g\in\mathfrak{sl}(E).$$

Since the trace form is nondegenerate, this implies

$$\sum_\alpha H^\alpha = 0.$$

□

This theorem in combination with Theorem 3.4.3 and Remark 3.4.4 implies

3.5.5. Corollary. *A system of filtrations $E^\alpha, \alpha\in A$ admits an Einstein-Hermitian metric if and only if it is a direct sum of stable systems with the same slope.* □

For a system of filtrations in general position, there are the alternative descriptions of stability in Theorem 3.3.3 and Corollaries 3.3.2, 3.3.4. Combining them with the previous theorem, we may characterize spectra of Hermitian matrices A, B and $A+B$ by the inequalities (IJK) of Corollary 3.3.4. As result we find that

3.5.6. Corollary. *Theorem* 1.2 *is valid.* □

4. Complex and Kähler structures

Leaving aside a rather popular presentation of Theorems 1.2 and 2.2, one can see that the genuine problem is in understanding the structure of the moduli space

$$\mathcal{M}=\mathcal{M}(\lambda^\alpha \mid \alpha\in A)$$

of semistable families of filtrations $E^\alpha, \alpha\in A$ with given spectra λ^α. This problem naturally arises from the study of the moduli space of vector bundles on $\mathbb{P}^2$ (see section 3.1). Theorems 1.2 and 2.2 are just criteria for this moduli space to be nonempty. In [Kl7] we have found its Betti numbers. Here we are interested mainly in description of its complex and Kähler strucures in terms of Hermitian operators.

There are two approachs to the moduli space $\mathcal{M}$.

1) It may be defined *algebraically* as an invariant-theoretical factor

$$\mathcal{M}(\lambda^\alpha \mid \alpha\in A)=\Big(\prod_\alpha \mathcal{B}^\alpha\Big)/\!\!/GL(E) \tag{4.1}$$

with respect to the polarization $\mathcal{L} = \bigotimes_\alpha \mathcal{L}(\omega^\alpha)$ from the proof of Theorem 3.5.3, where we borrow the notations.

2) The other *transcendent* approach based on the existence and uniqueness of the Einstein-Hermitian metric and leads to an identification (cf. [Nes])

$$\mathcal{M}(\lambda^\alpha \mid \alpha \in A) = \begin{pmatrix} \text{solutions of the equation } \sum_\alpha H^\alpha = 0 \\ \text{up to a unitary transformation} \end{pmatrix} \tag{4.2}$$

Here and further H^α is a Hermitian operator with spectrum λ^α.

The first definition makes sense only for *integer* spectra λ^α. In this case $\mathcal{M}$ is a manifestly projective algebraic variety, and hence carries a canonical complex structure and Kähler metric.

The second approach works for irrational spectra as well, but apparently it defines $\mathcal{M}$ only as a smooth variety (may be with singularities).

This section appears from an attempt to recover the complex and Kähler structure in the framework of the transcendent approach. It turns out to be possible and the construction looks very natural and elegant. Let me explain it first in the simplest example.

4.1. Spatial polygons

Let $\dim E = 2$ and H^α be a traceless Hermitian operator with fixed spectrum

$$\operatorname{spec}(H^\alpha) = (m_\alpha, -m_\alpha), \quad m_\alpha > 0.$$

It is a good idea to represent a Hermitian 2×2-matrix by a vector in Euclidean 3-space $\mathbb{E}^3$

$$H^\alpha = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \mapsto a_\alpha = ai + bj + ck \in \mathbb{E}^3.$$

Under this identification unitary conjugation $H^\alpha \mapsto UH^\alpha U^{-1}$ looks as a rotation in $\mathbb{E}^3$ and the positive eigenvalue m_α of H^α is just the length of the corresponding vector a_α. So the equation in the right hand side of (4.2) takes the form

$$a_1 + a_2 + \ldots + a_n = 0, \quad |a_i| = m_i \qquad a_i \in \mathbb{E}^3. \tag{4.3}$$

The moduli space (4.2) in this case has a transparent interpretation

$$\mathcal{M}(m_i \mid i = 1, \ldots, n) = \begin{pmatrix} \text{spatial } n\text{-gons with sides of given} \\ \text{length } m_i \text{ up to a motion in } \mathbb{E}^3 \end{pmatrix}.$$

To define a complex structure on $\mathcal{M}$, let us look at its tangent space. Let $a_i = a_i(t)$ be a one parameter deformation of the polygon. A tangent vector to $\mathcal{M}$ may be represented by the derivatives $v_i = \dot{a}_i$ subject to the following conditions:

i) $(v_i, a_i) = 0, \quad \forall i$;
ii) $v_1 + v_2 + \ldots + v_n = 0$;
iii) *vectors* v_i *and*

$$u_i = v_i + [\omega, a_i], \quad \omega \in \mathbb{E}^3 \tag{4.4}$$

represent the same tangent vector to $\mathcal{M}$.

The first two conditions follow from equations (4.3) and transformation (4.4) corresponds to an infinitesimal rotation of the polygon as a whole. To exclude the ambiguity (4.4), one may impose on v_i an additional calibration equation. We will use the following one

$$\frac{v_1^2}{m_1} + \frac{v_2^2}{m_2} + \cdots \frac{v_n^2}{m_n} = \min,$$

which implies

$$\frac{[a_1, v_1]}{m_1} + \frac{[a_2, v_2]}{m_2} + \cdots \frac{[a_n, v_n]}{m_n} = 0, \tag{4.5}$$

where the brackets mean the vector product in $\mathbb{E}^3$. It turns out that if not all the vectors a_i are collinear, then among systems $v_i \in \mathbb{E}^3$ representing the same tangent vector to $\mathcal{M}$, there is exactly one satisfying (4.5) [Kl5]. Such a system v_i is said to be *calibrated*.

Now we find out that the tangent space to $\mathcal{M}$ has a natural complex structure J given by the rotation of each component v_i of the *calibrated* tangent vector on angle $\frac{\pi}{2}$ over the side a_i

$$J : v_i \mapsto u_i = \frac{[a_i, v_i]}{m_i}.$$

The calibration condition (4.5) implies that u_i is again a tangent vector. The almost complex structure on $\mathcal{M}$ defined by the operator J turns out to be integrable.

Thus if the lengths m_i do not allow the polygon to degenerate into a line segment, then $\mathcal{M}$ is a compact smooth complex manifold. It carries a natural simplectic form

$$\omega(u, v) = \sum_i \frac{(u_i, v_i, a_i)}{(a_i, a_i)}$$

invariant under the gauge transformations (4.4) and Kähler metric

$$g(u, v) = \omega(Ju, v).$$

The moduli space of polygons with sides of integer lengths m_i has also an algebraic interpretation (4.1). It turns out to be the moduli space of semistable configurations of n points in the line $p_i \in \mathbb{P}^1$ with multiplicities m_i. Such a configuration is said to be semistable if no more than a half of the points (counted according to the multiplicities) coincide.

4.2. General case

In this section we extend the previous constructions to the moduli (4.2) of Hermitian operators H^α in a space of arbitrary dimension. We'll suppose all the operators to be traceless

$$\mathrm{Tr}(H^\alpha) = \sum_i \lambda_i^\alpha = 0.$$

Besides the spectra λ^α are supposed to be *generic*, so that semistability implies stability. This is the case for spectra with no linear rational relations between eigenvalues λ_i^α independent of the trace identities $\sum_i \lambda_i^\alpha = 0$. For generic spectra, the moduli space (4.2) is smooth.

Let $H(t)$ be a one parameter family of Hermitian matrices. It is well known from the perturbation theory of Hermitian operators that the derivative $\frac{d\lambda}{dt}$ of an eigenvalue λ of H is equal to an eigenvalue of the restriction of the operator $A = \frac{dH}{dt}$ to the eigenspace E_λ (the restriction $A|E_\lambda$ has been defined before the proposition (3.4.2)). Thus we get the first representation of the tangent space to the $\mathcal{M}(\lambda^\alpha \mid \alpha \in I)$.

4.2.1. Proposition. *A tangent vector to the variety $\mathcal{M}(\lambda^\alpha \mid \alpha \in I)$ at a point H^α, $\alpha \in I$ may be represented by a family of Hermitian matrices A_α satisfying the following conditions*

(1) *The restriction $A_\alpha|E_\lambda$ of A_α on any eigenspace E_λ of H^α is zero;*
(2) $\sum_{\alpha\in I} A_\alpha = 0$*;*
(3) *Two such systems represent the same tangent vector iff they are connected by a gauge transformation of the form*

$$A_\alpha \mapsto A_\alpha + [H^\alpha, X] \tag{4.6}$$

for some skew-Hermitian matrix X.

Proof. The first condition, as explained above, means that the deformation $H^\alpha(t)$ is isospectral; the second one follows from the equation $\sum_{\alpha\in I} H^\alpha = 0$. The transformation (4.6) is an infinitesimal form of conjugation by a unitary matrix $\exp tX$. □

The first condition is easier to understand in an orthonormal eigenbasis of the operator H^α. Then

$$H^\alpha = \mathrm{diag}(\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_n^\alpha);\ \lambda_1^\alpha \geq \lambda_2^\alpha \geq \dots \geq \lambda_n^\alpha.$$

Now (1) just means that the the matrix A_α has zero diagonal blocks (which correspond to scalar blocks of the matrix H^α). Let us decompose it in upper and lower triangular parts

$$A_\alpha = A_\alpha^+ + A_\alpha^-.$$

In other words, the operator A_α^+ respects the spectral filtration E^α of H_α and induces zero transformation in its composition factors. The operators A_α^- is conjugate to A_α^+.

4.2.2. Proposition. *There exists a unique representative* $\{A_\alpha,\ \alpha \in I\}$ *of a tangent vector to the variety* $\mathcal{M}(\lambda^\alpha \mid \alpha \in I)$ *satisfying the calibration equations*

$$\sum_{\alpha\in I} A_\alpha^+ = \sum_{\alpha\in I} A_\alpha^- = 0.$$

Proof. Let us consider the linear operator L in the space of traceless skew-Hermitian matrices

$$L : X \mapsto \sum_{\alpha\in I} [H^\alpha, X]^+. \tag{4.7}$$

The right hand side of (4.7) is skew-Hermitian because operators $[H^\alpha, X]^+$ and $[H^\alpha, X]^-$ are conjugate and

$$\sum_{\alpha\in I} [H^\alpha, X]^+ + \sum_{\alpha\in I} [H^\alpha, X]^- = \sum_{\alpha\in I} [H^\alpha, X] = 0.$$

We assert that L is invertible. Actually, if $L(X) = 0$, then

$$\mathrm{Tr}(X \cdot L(X)) = \sum_{\alpha\in I} \mathrm{Tr}(X \cdot [H_\alpha, X]^+) = 0. \tag{4.8}$$

Let us evaluate the summand $\mathrm{Tr}(X \cdot [H^\alpha, X]^+)$ in the eigenbasis of H^α

$$H^\alpha = \mathrm{diag}(\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_n^\alpha),$$
$$\lambda_1^\alpha \geq \lambda_2^\alpha \geq \dots \geq \lambda_n^\alpha.$$

Decompose the skew-Hermitian matrix $X = |x_{ij}|$ as

$$X = X_\alpha^+ + X_\alpha^0 + X_\alpha^-,$$

where X_α^0 consists of diagonal blocks of X corresponding to scalar components of H^α and $X_\alpha^\pm$ are the remaining upper and lower triangular parts of X. In this notation $[H^\alpha, X]^+$ is an upper triangular matrix with elements $x_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)$. Hence

$$\mathrm{Tr}(X \cdot [H^\alpha, X]^+) = \sum_{i<j} x_{ij}(\lambda_i^\alpha - \lambda_j^\alpha) x_{ji} = -\sum_{i<j} |x_{ij}|^2 (\lambda_i^\alpha - \lambda_j^\alpha) \leq 0.$$

Thus the summands in (4.8) are nonpositive and hence they are all zero. It is possible only if both nondiagonal parts $X_\alpha^\pm$ are zero for all α. It follows that X commutes with all H^α. Since stable systems of filtrations have no nonscalar endomorphisms, $X = 0$. Thus $\ker L = 0$ and L is invertible.

To finish the proof, let us consider a tangent vector to the moduli represented by matrices A_α, $\alpha \in I$ from Proposition 4.2.1. Since $\sum_\alpha A_\alpha = 0$, the sum $\sum_{\alpha\in I} A_\alpha^+$ is skew-Hermitian. Writing this sum as $L^{-1}(X)$, we get the equation

$$\sum_{\alpha\in I} A_\alpha^+ = \sum_{\alpha\in I}[H^\alpha, X]^+$$

for some skew-Hermitian matrix X. Hence we can satisfy the calibration equation

$$\sum_{\alpha\in I} A_\alpha^+ = 0$$

by the gauge transformation $A_\alpha \mapsto A_\alpha - [H^\alpha, X]$. □

4.2.3. Corollary. *The tangent space to the variety $\mathcal{M}(\lambda^\alpha \mid \alpha \in I)$ at a point H^α, $\alpha \in I$ is naturally equivalent to the complex space of nilpotent operators U^α, $\alpha \in I$ satisfying the following conditions:*

(1) *U^α respects the spectral filtration of H^α and induces a zero operator on its composition factors;*
(2) *$\sum_{\alpha\in I} U^\alpha = 0$.*

Proof. The operators U^α are the same as A_α^+. □

Remark. Corollary 4.2.3 identifies the tangent space to the moduli $\mathcal{M}(\lambda^\alpha \mid \alpha \in I)$ with *a complex* space and hence defines on $\mathcal{M}$ an almost complex structure J. This structure is in fact integrable. Probably it is not easy to check Newlander-Nierenberg integrability conditions [N-N] directly. Indirect arguments use identification of J for *rational* spectra λ^α with the complex structure coming from algebraic construction (4.1). Then J should be integrable for irrational spectra as well.

Thus $\mathcal{M}(\lambda^\alpha \mid \alpha \in I)$ is a compact complex variety. It carries a natural symplectic form

$$\omega(A, B) = \frac{1}{i}\sum_{\alpha\in I}\frac{([A_\alpha, B_\alpha], H^\alpha)}{(H^\alpha, H^\alpha)} = \frac{1}{i}\sum_{\alpha\in I}\frac{([A_\alpha^+, B_\alpha^-], H^\alpha) + ([A_\alpha^-, B_\alpha^+], H^\alpha)}{(H^\alpha, H^\alpha)}$$

where parentheses denote the trace form $(X, Y) = \mathrm{Tr}(XY)$. Here $A = \{A_\alpha \mid \alpha \in I\}$ and $B = \{B_\alpha \mid \alpha \in I\}$ are *calibrated* tangent vectors, i.e. satisfying the conditions of Proposition 4.2.2.

4.2.4. Proposition. *The form $g(A, B) = \omega(JA, B)$ is a Kähler metric on $\mathcal{M}(\lambda^\alpha \mid \alpha \in I)$.*

Proof. Let us recall that the complex structure J on tangent space acts on the components $A_\alpha^\pm$ as multiplication by $\pm i$. Hence

$$g(A, B) = \omega(JA, B) = \sum_{\alpha\in I}\frac{([A_\alpha^+, B_\alpha^-], H^\alpha) + ([B_\alpha^+, A_\alpha^-], H^\alpha)}{(H^\alpha, H^\alpha)}.$$

Let us check the positivity of the summand in the eigenbasis of the operator H^α

$$H^\alpha = \operatorname{diag}(\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_n^\alpha),$$
$$\lambda_1^\alpha \geq \lambda_2^\alpha \geq \dots \geq \lambda_n^\alpha.$$

Then $A_\alpha^\pm$ is an upper (lower) triangular matrices with elements a_{ij} and $a_{ji} = \bar{a}_{ij}$ respectively. In this notation

$$([A_\alpha^+, A_\alpha^-], H_\alpha) = (A_\alpha^+, [A_\alpha^-, H_\alpha]) = \sum_{i<j} (\lambda_i^\alpha - \lambda_j^\alpha)|a_{ij}|^2 \geq 0.$$

Hence $g(A, A) \geq 0$ and the equality is valid only for $A = 0$. □

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