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‘Density’ Gibbs states and uniqueness conditions in one-dimensional models

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Abstract. Weconstructaone-dimensionalmodelwithtwospinsandauniquegroundstatehaving infinitely many extreme limit Gibbs states. This model is closely related to uniqueness conditions in one-dimensional models.

1. Introduction

The problem of phase transitions in one-dimensional models is an object of constant interest during the last few decades [1–13]. It is well known that if the pair potential U(x) of the model satisfies the condition $\sum_{x\in\mathbb{Z}^1;x>0} xU(x) < \infty$ then the model does not exhibit phase transition [1–3]. In [4] the absence of phase transitions is proved for the antiferromagnetic model with the pair potential $U(x) = \text{constant} \times x^{-1-\alpha}$, where $0 < \alpha < 1$. Based on the methods of [4] in [14] the following conjecture was formulated: any one-dimensional model with discrete (at most countable) spin space and with a unique ground state has a unique limit Gibbs state if the spin space of this model is finite or the potential of this model is translationally invariant.

In this paper we construct a model (1) which disproves this conjecture. We prove that in spite of the fact that the model (1) has a finite spin space and a unique ground state, it has infinitely many extreme limit Gibbs states.

2. The model

Consider a model of the classical statistical mechanics on the one-dimensional integer lattice $\mathbb{Z}^1$ with the Hamiltonian

$$H(\varphi(x)) = \sum_{x\in\mathbb{Z}^1;x<0} U(\varphi(x), \varphi(B_{-n(x)-1})) - \sum_{x\in\mathbb{Z}^1;x\geqslant 0} \varphi(x) \tag{1}$$

where the spin variable $\phi(x)$ takes two values 0 and 1, and $\phi(B_{-n(x)})$ is the restriction of the

configuration= $\phi(x=)$ to the set$P^n B^i{}_{-n(x)}n$ >, B_{-n}, n = 1,2,... is a half-open interval $[-c_n,-c_{n-1})$,

where $_1$ $_{i=1}$ at
$x \in B_{-n(x)}$.
c 0,c_n 10$_{3}$ $_{+1}$ 1, the value of n(x) in (1) is defined by the

condition

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In order to define the potential U of the model, first of all we set two sequences: $a_k = \frac{2}{3} + \sum_{i=1}^{k-1}\left(\frac{1}{4}\right)^i$ and $b_k = \frac{2}{3} + \sum_{i=1}^{k}\left(\frac{1}{4}\right)^i$, and after that we define the sequence of half-open intervals I_k, k = 1,2,...,$I_k = [a_k,b_k)$ and the sequence of positive numbers $P_k = (a_k + b_k)/2$.

The interaction in the model (1) takes place between points x and the left neighbour intervals $B_{n(x)-1}$. The potential $U(\phi(x),\phi(B_{n(x)-1}))$, which specifies the interaction between the spin variable $\phi(x)$ at the point x and the restriction of the configuration $\phi(x)$ to the interval $B_{n(x)-1}$ is defined by the relations:

$$U(\phi(x) = 1,\phi(B_{-n(x)-1})) = 0$$

if

$$_1 X _{n(x)} \quad \phi(x)/(c_n - c_{n-1}) = 1$$

$$x\in Z;x\in B_{--1}$$

$$U(\phi(x) = 0,\phi(B_{-n(x)-1})) = \infty$$

if

$$_1 X _{n(x)} \quad \phi(x)/(c_n - c_{n-1}) = 1$$

$$x\in Z;x\in B_{--1}$$

$$U(\phi(x) = 1,\phi(B_{-n(x)-1})) = -\ln P_k$$

if

$$_1 X _{n(x)} \quad \phi(x)/(c_n - c_{n-1}) \in I_k$$

$x \in Z; x \in B_{--1}$

$U(\phi(x) = 0, \phi(B_{-n(x)-1})) = -\ln(1 - P_k)$

if

$$\sum_{x\in\mathbb{Z}^1; x\in B_{-n(x)-1}} \varphi(x)/(c_n - c_{n-1}) \in I_k$$

$$U(\varphi(x) = 1, \varphi(B_{-n(x)-1})) = -\ln \tfrac{2}{3}$$

if for any k if

$$\sum_{x\in\mathbb{Z}^1; x\in B_{-n(x)-1}} \varphi(x)/(c_n - c_{n-1}) \notin I_k$$

$$U(\varphi(x) = 0, \varphi(B_{-n(x)-1})) = -\ln \tfrac{1}{3}$$

for any k

$$\sum_{x\in\mathbb{Z}^1; x\in B_{-n(x)-1}} \varphi(x)/(c_n - c_{n-1}) \notin I_k .$$

Let $I_V = [-V, V]$ and $[-V, -1] = \bigcup_{i=1}^{r} B_{-i}$. Suppose that the boundary conditions $\phi^k(x)$, $x \in Z^1 - I_V$ are fixed.

The Hamiltonian in the subset I_V is given by

$$H_V(\varphi(x)|\varphi^k(x)) = \sum_{x=-V}^{-1} U(\varphi(x), \varphi(B_{-n(x)-1})) - \sum_{x=0}^{V} \varphi(x).$$

The restriction of the configuration $\phi(x)$ to the interval I_V will be denoted by $\phi_V(x)$ and the set of all configurations $\phi_V(x)$ will be denoted by 8(V).

The finite-volume Gibbs state in 8(V) at inverse temperature $\beta = T^{-1}$ and boundary conditions $\phi^k(x)$ are defined by

$$P_V^k(\varphi_V(x)|\varphi^k(x)) = \Xi_V^{-1} \exp(-\beta H_V(\varphi_V(x)|\varphi^k(x)))$$

where $4_V = P_{\phi_V(x)\in 8(V)} \exp(-\beta H_V(\phi_V(x)|\phi^k(x)))$ is the partition function.

An extreme limit Gibbs state is the weak limit of finite-volume Gibbs states. It is well known that the set of all limit Gibbs states coincides with the closed convex hull of the set of weak limits of finite-volume Gibbs states [16].

A configuration $\phi^{gr}(x)$ is said to be a ground state, if for any finite perturbation $\phi^0(x)$ of the configuration $\phi^{gr}(x)$ the expression $H(\phi^0(x)) - H(\phi^{gr}(x))$ is non-negative.

It follows from the construction of the Hamiltonian that the model (1) can be interpreted as an inhomogeneous Markov chain with two states [16,17] starting at minus infinity, whose transition probabilities are defined by the following rule:

If the point x belongs to the block $B_{-n(x)}$, then the probabilities for the variable $\phi(x)$ depend on the spin variables $\phi(x)$ belonging to the previous block $B_{-n(x)-1}$, namely if the density of particles in $B_{-n(x)-1}$ is 1, then the probability that $\phi(x) = 1$ is 1, if the density of particles in $B_{-n(x)-1}$ belongs to the interval I_k, then the probability that $\phi(x) = 1$ is P_k and if the density of particles in $B_{-n(x)-1}$ does not belong to any interval I_k, then the probability that $= 1$ is $\frac{2}{3}$

$e/(e+1)$. $\phi(x)$. If the point belongs to the interval $[0,\infty)$ then the probability that $\phi(x) = 1$ is

In the next section we prove the following lemma:

Lemma 1. *The model (1) has a unique ground state.*

Obviously, for each k, there exists a configuration $\phi^k(x)$, such that the value of the density of the particles in each block B_n for all sufficiently large $n = n(k)$ belongs to the interval I_k:

$$\sum_{x\in\mathbb{Z}^1;x\in B_{-n}} \varphi(x)/(c_n - c_{n-1}) \in I_k .$$

Let the value of the β be 1. A limit Gibbs state corresponding to the boundary conditions $\phi^k(x)$ will be denoted by P^k.

In spite of the fact that the model (1) has a unique ground state, the set of limit Gibbs states of the model (1) is very rich.

Theorem 1. *At* β *=1 the model (1) has countable number of extreme limit Gibbs states* P^k.

Theorem1showstheexistenceof‘density’limitGibbsstatescharacterizedbythedensities of particles in typical configurations.

3. Proofs

We prove lemma 1 by showing that the only ground state of the model (1) is the configuration $\phi^{gr}(x) = 1$ for all $x \in Z^1$.

Proof of lemma 1. First of all, let us show that the configuration $\phi^{gr}(x)$ is a ground state of model (1). Let a configuration $\phi^0(x)$ be a finite perturbation of the configuration ϕ^{gr}. Then the expression $H(\phi^0(x)) - H(\phi^{gr}(x))$ is non-negative. Indeed,

$$H(\phi^0(x)) - H(\phi^{gr}(x)) = \sum_{x\in Z;x<0} (U(\phi^0(x),\phi^0(B_{-n(x)-1})) - U(\phi^{gr}(x),\phi^{gr}(B_{-n(x)-1})))$$

$$+ \sum_{x\in Z;x>0} (\phi_{gr}(x) - \phi^0(x)) = X^0 + X^{00} .$$

Let$=(U(\phi^0(x),\phi^0(B_{-n(x)-1})) - U(\phi^{gr}(x),\phi^{gr}(B_{-n(x)-1}-)))$ be a non-zero term of$- P^0$. If $\phi^0(x)$

1, then due to the definitions this term is equal to $\ln P_k\, 0$ for some k and hence is positive. If

$\phi^0(x) = 0$, then due to the definitions this term is equal to $\infty - \ln(1 - P_k)$ for some k and again is positive. On the other hand, all non-zero terms of P^{00} are 1. Thus, the configuration $\phi^{gr}(x)$ is a ground state of the model (1).

Let the configuration $\phi^1(x)$ be a ground state of the model (1) and the set $Z(\phi)$ of all points $x^0 \in Z^1$, such that $\phi^1(x^0) = 0$ is not empty.

If $Z(\phi) \cap [0,\infty)$ is not empty and contains a point x^0, we define a configuration $\phi^{1,1}(x)$ by the following rule: $\phi^{1,1}(x^0) = 1$ and $\phi^{1,1}(x) = \phi^1(x)$ for all $x \neq x^0$. Now $\phi^{1,1}(x) - \phi^1(x) = -1$ and we have a contradiction with the fact that $\phi^1(x)$ is a ground state.

define a configurationIf $Z(\phi) \cap (-\infty,-\phi 1]^{1,1}$is not empty, we consider the point(x) by the following rule: $\phi^{1,1}(x^0) = x^0 1 =$ and $\max \phi^1 x'^1 \in (Z(\phi) x) \cap = (-\phi_\infty 1, (-x 1]) x$ for all, and

$x \neq x^0$. Now $H(\phi^{1,1}(x)) - H(\phi^1(x))$ is either $-\ln P_k + \ln(1 - P_k)$ for some k or $-\infty$ and since $P_k > \frac{1}{2}$, the expression $-\ln P_k + \ln(1-P_k) < 0$ and again we have a contradiction with the fact that $\phi^1(x)$ is a ground state. The proof of lemma 1 is completed.

Proof of theorem 1. Let P^k be a limit Gibbs state corresponding to the boundary conditions $\phi^k(x)$. In order to prove the theorem, we show that P^l cannot be represented as a finite linear combination of limit Gibbs states P^{l_i}: for any collections $l_1,...,l_s$ and $\mu_1,...,\mu_s$, where $l_i \neq l$ and $0 < \mu_i \leq 1$,

$$P^l \neq \sum_{i=1}^{s} \mu_i P^{l_i}.$$

For this reason we show that there exists an interval B_{-n}, such that the restriction of the measures P^l and $\sum_{i=1}^{s} \mu_i \boldsymbol{P}^{l_i}$ on B_{-n} are different:

$$P^l[B_{-n}] \neq \sum_{i=1}^{s} \mu_i \boldsymbol{P}^{l_i}[B_{-n}]. \qquad (2)$$

particles in the restrictions of the configurationsWe define B_{-n} as an interval satisfying the conditions$\phi^{l_i}(x)$ and $\phi n > l^l(x)$ to$i,n > l B_{-n}$ belong to the intervalsand the densities of

I_{l_i} and I_l, respectively; that is

$$\sum_{x \in Z^1; x \in B_{-n}} \phi^l(x)/(c_n - c_{n-1}) \in I_l$$

$$\sum_1 \phi^{l_i}(x)/(c_n - c_{n-1}) \in I_{l_i}.$$

$x \in Z; x \in B_{-n}$

Let us define a random variable $\chi_{-n} = \sum_{x \in Z^1; x \in B_{-n}} \phi(x)/(c_n - c_{n-1})$.

We prove relation (2) by showing that for any k and n, n > k and at sufficiently large V ,

$$P_V^k(\chi_{-n} \in I_k) > \frac{3}{4} \tag{3}$$

where $\boldsymbol{P}_V^k$ is the Gibbs distribution corresponding to the boundary conditions $\phi^k(x)$, $x \in Z^1 - [-V,V]$.

Indeed, equation (3) implies (2), since from (3) it follows that if n > l, and n > $\max_i(l_i)$ then $P_V^l(\chi_{-n} \in I_l) > \frac{3}{4}$ and $\sum_{i=1}^s \mu_i \boldsymbol{P}^{l_i}(\chi_{-n} \in I_l) < 1 - \sum_{i=1}^s \mu_i \boldsymbol{P}^{l_i}(\chi_{-n} \in I_{l_i}) < \frac{1}{4}$.

Suppose that $[-V, -1] = \cup_{i=1}^r B_{-i}$.

Itreadilyfollowsfromthedefinitionofthepotentialthatallspinvariables$\phi(x)$,x ∈ [0,∞) are independent (they take 1 and 0 with respective probabilities e/(e+1) and 1/(e+1)). Hence the restriction of the Gibbs distribution P_V^k to the set $\phi(x)$,x ∈ [−V,−1] can be treated as a one-sided inhomogeneous Markov chain with two states starting at minus infinity [16,17].

Thus,

$$P_V^k(\chi_{-n} \in I_k) \geqslant \boldsymbol{P}_V^k(\cap_{i=n}^r \chi_{-i} \in I_k) = \boldsymbol{P}_V^k(\chi_{-r} \in I_k) \prod_{i=r-1}^{n} \boldsymbol{P}_V^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k).$$

Nowweestimatetheexpression$P_V^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k)$. Let $x \in B_{-i}$. Bythedefinition of the potential $P_V^k(\phi(x) = 1 | \chi_{-i-1} \in I_k) = P_k$.

Let us define the sequence of positive numbers $\epsilon_k = 1/2\left(\frac{1}{4}\right)^k$. By the law of large numbers,

$$P_V^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k) \geqslant \boldsymbol{P}_V^k(|\chi_{-i} - P_k| < \epsilon_k | \chi_{-i-1} \in I_k) \geqslant 1 - \frac{1}{|B_{-i}|\epsilon_k{}^2} = 1 - \frac{4^{2k+1}}{10^{3n+1}} > 1 - \frac{1}{10^{3n-2k}}$$

and since n > k

$$P_V^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k) > 1 - 10^{-n}.$$

Finally,

n n

$$P_V^k(\chi_{-r} \in I_k) \prod_{i=-r}^{-1} P_V^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k) > \prod_{i=-r}^{-1} (1-10^{-i}) > \prod_{i=1}^{\infty}(1-10^{-i}) > \frac{3}{4}.$$

Relation (3) and hence relation (2) is proved. Thus, model (1) has at least a countable number of limit Gibbs states corresponding to the boundary conditions $\phi^k(x)$. Since the Gibbs measure P_V^k corresponding to the volume V and the boundary conditions $\phi^k(x)$ by the definition of the potential depends just on the density of particles outside $[-V,V]$ and in the definition of the potential the set of all possible densities is partitioned into the countable number of classes, one can conclude that the set of all extreme limit Gibbs states is countable. The proof of the theorem 1 is completed.

4. Uniqueness conditions in one dimension

Under some natural conditions the conjecture formulated in [14] is correct [5]. Suppose that the model has a unique ground state $\phi^{gr}(x)$ satisfying the following stability condition: for any finite set $A \subset Z^1$ with length $|A|$

$$H(\phi^0(x)) - H(\phi^{gr}(x)) > t|A| \tag{4}$$

where $t > 0$, $|A|$ is the number of sites of A and $\phi^0(x)$ is a perturbation of the ground state ϕ^{gr} on the finite set A, and the potential U(B) satisfies some natural decreasing conditions. Then the model has a unique limit Gibbs state at low temperatures [5].

By a natural decreasing potential we mean the following: for any fixed interval I with the length n, the expression $\sum_{B\subset Z^1; B\cap I\neq\emptyset, B\cap(Z^1-I)\neq\emptyset} U(B)$, grows not faster then n^{α}, $0 < \alpha < 1$.

It can be easily shown that in model (1) this decreasing condition is not satisfied: the order of the influence of the block B_{-n-1} on the block B_{-n} is equal to the length of B_{-n}!

5. Final remarks

In [15] a one-dimensional model having a unique ground state and a countable number of extreme limit Gibbs states was constructed. Since the model in [15] has a countable number of spin variables, theorem 1 can be considered as an improvement of the results of [15]. The result of [5] is extendible to all values of the temperature.

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