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J-APPROXIMATION OF COMPLEX PROJECTIVE SPACES BY LENS SPACES

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In this paper we study the group $J(L^k(n))$ of stable fibre homotopy classes of vector bundles over the lens space, $L^k(n) = S^{2k+1}/\mathbb{Z}_n$ where $\mathbb{Z}_n$ is the cyclic group of order n. We establish the fundamental exact sequences and hence find the order of $J(L^k(n))$. We define a number N_k and prove that the inclusion-map $i : L^k(n) \to P_k(\mathbb{C})$ induces an isomorphism of $J(P_k(\mathbb{C}))$ with the subgroup of $J(L^k(n))$ generated by the powers of the realification of the Hopf-bundle iff n is divisible by N_k. This provides the discrete approximation to the continuous case.

0. Introduction.

Let p be a prime; k, $n \in \mathbb{Z}^+$ and $L^k(p^n) = S^{2k+1}/\mathbb{Z}_{p^n}$ be the lens space where $\mathbb{Z}_{p^n}$ is the cyclic group of order p^n. $L^k(p^n)$ has the structure of a CW-complex $L^k(p^n) = \cup_{j=0}^{2k+1} e^j$ and its $2k$-th skeleton,

$$L_0^k(p^n) = \{[z_0, \dots, z_k] \in L^k(p^n) : \ z_k \text{ is real} \geq 0\}.$$

In this paper we study the group $J(L^k(p^n))$, making use of the already established results in [10] and [12] on $\tilde{K}_{\mathbb{R}}(L^k(p^n))$. We first establish the exact sequences analogous to the ones proved in [4] for $J(P_k(\mathbb{C}))$. Define $\overline{L}^k(p^n) = \begin{cases} L_0^k(p^n) & \text{if } p \text{ is odd} \\ L^k(p^n) & \text{if } p = 2 \end{cases}$. The main difficulty is to prove the injectivity of the map $c^! : J(\overline{L}^{2k}(p^n)/\overline{L}^{2k-2}(p^n)) \to J(\overline{L}^{2k}(p^n))$, whereas the corresponding result, e.g. [4, Lemma 4.9], is trivial for complex projective spaces. We resolve this difficulty by using the transfer map $\tau : \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^n)) \to \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^{n+1}))$ and to make the transfer map suitable for application, we prove a number of preliminary results in Section 2.2 concerning binomial expansions. This leads to Proposition 2.3.3 about the kernel of $(\psi_{\mathbb{R}}^t - 1)$ where t is an integer not divisible by p and $\psi_{\mathbb{R}}^t$ is the Adams operation and which plays a fundamental role in the proofs of Lemma 3.2.1 and Proposition 3.2.2 for the injectivity of $c^!$. Using the exact sequences we establish, we find in Proposition 3.3.4, the order of $J(\overline{L}^{2v}(p^n))$. Let $G(p, k, n)$ be the subgroup of $J(L^k(p^n))$ generated by the powers of the realification of the Hopf-bundle over $L^k(p^n)$

which coincides with $J(L^k(p^n))$, except for p odd and $k \equiv 0 (\mathrm{mod}\, 4)$ and for $p = 2$ and $n \geq 2$ in which case it is a subgroup of $J(L^k(p^n))$ of index 2. Let $i : L^k(p^n) \to P_k(\mathbb{C})$ be the inclusion-map. We define a number N_k in 2.2.9. The main result of the paper; e.g., Theorem 3.4.2 states that $i^!$ maps the p-summand, $J_p(P_k(\mathbb{C}))$ of $J(P_k(\mathbb{C}))$ isomorphically onto $G(p, k, n)$ iff n is greater or equal to the p-exponent of N_k. This provides the discrete approximation to the continuous case. We then conjecture in 3.4.4 a stronger version of this which involves the degree function on the J-groups.

Finally, we observe that the transfer map passes to the quotient and defines a map on the J-groups of the respective lens spaces. We prove in Proposition 3.5.3 that $\tau \circ i^!(x) = px$ for $\forall x \in G(p, k, n + 1)$.

The paper is self-contained as a whole. Only very elementary facts about the $\tilde{K}_{\mathbb{R}}$-groups of lens spaces are used and everything concerning J-groups of lens spaces is developed from scratch.

1. $\tilde{K}_{\mathbb{R}}$-groups of lens spaces.

1.1. Survey of results. Let p be a prime and k, $n \in \mathbb{Z}^+$. Let η be the complex Hopf-bundle over $L^k(p^n)$, $\mu = \eta - 1 \in \tilde{K}_{\mathbb{C}}(L^k(p^n))$ be its reduction and $w = r(\mu) \in \tilde{K}_{\mathbb{R}}(L^k(p^n))$ be the realification of μ. It is (essentially) shown in [**10**] and [**12**] that $\tilde{K}_{\mathbb{C}}(L^k(p^n))$ is generated multiplicatively by μ subject to the relations :
I. $\mu^{k+1} = 0$, II. $\psi_{\mathbb{C}}^{p^n}(\mu) = \mu\psi_{\mathbb{C}}^{p^n}(\mu) = \cdots = \mu^{k-1}\psi_{\mathbb{C}}^{p^n}(\mu) = 0$.

For p odd, $\tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ is generated multiplicatively by w subject to the relations :
I′. $w^{[k/2]+1} = 0$ and II′. The realification of the relations II above.

For $p = 2$ and $n \geq 2$, let ξ be the real line-bundle over $L^k(2^n)$ such that $c(\xi) = \eta^{2^{n-1}}$ where c is the complexification-map. Let $\lambda = \xi - 1 \in \tilde{K}_{\mathbb{R}}(L^k(2^n))$. Then $\tilde{K}_{\mathbb{R}}(L^k(2^n))$ is generated multiplicatively by w and λ subject to:
I′. $w^{[k/2]+1} = 0$ if $k \not\equiv 1$ (mod 4) and $2w^{[k/2]+1} = w^{[k/2]+2} = 0$ if $k \equiv 1$ (mod 4)
II′. The realification of relations II above. Relations II′ in $\tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ for p odd and in $\tilde{K}_{\mathbb{R}}(L^k(2^n))$ for $p = 2$ are equivalent to the periodicity-relations: $\psi_{\mathbb{R}}^{s+p^n}(w) = \psi^s(w), \forall s \in \mathbb{Z}$ or to the single relation obtained by taking $s = -1$; i.e., $\psi_{\mathbb{R}}^{p^n-1}(w) - w = 0$ which by Proposition 1.1.6 is of the form: $p^n(p^n - 2)w + \sum_{j\geq 2} \alpha_j w^j = 0$ or upon multiplication by w^{i-1} : $(i \geq 1), p^n(p^n - 2)w^i + \sum_{j\geq 2} \alpha_j w^{i+j} = 0 (i \geq 1)$ which are equivalent to:

$p^n w^i + \sum_{j\geq 2}\beta_j w^{i+j} = 0$ for p odd and $2^{n+1}w^i + \sum_{j\geq 2}\beta_j w^{i+j} = 0$ for $p = 2$.
III$'$. $2\lambda = \psi_{\mathbb{R}}^{2^{n-1}}(w)$ and IV$'$. $\lambda w = (\psi_{\mathbb{R}}^{2^{n-1}+1} - \psi_{\mathbb{R}}^{2^{n-1}} - 1)(w)$.

For $k \equiv 0 \,(\mathrm{mod}\, 4)$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \mathbb{Z}_2 \oplus \tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ if p is odd and $\tilde{K}_{\mathbb{R}}(L_0^k(2^n)) = \tilde{K}_{\mathbb{R}}(L^k(2^n))/\mathbb{Z}_2\langle 2^{n+k-2}w\rangle$ if $p = 2$ and for $k \not\equiv 0 \,(\mathrm{mod}\, 4)$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \tilde{K}_{\mathbb{R}}(L_0^k(p^n))$.

Lemma 1.1.1. *Let p be a prime; $v, n \in \mathbb{Z}^+$, $n \geq 2$ if $p = 2$. Then*
$$\tilde{K}_{\mathbb{R}}(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)) = \begin{cases} \mathbb{Z}_{p^n} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{2^{n+1}} & \text{if } p = 2 \end{cases}.$$

Lemma 1.1.2. *Let $v, n \in \mathbb{Z}^+$. Then*
$$\tilde{K}_{\mathbb{R}}(L^{4v+1}(p^n)/L^{4v}(p^n)) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \mathbb{Z}_2 & \text{if } p = 2 \end{cases}.$$

Lemma 1.1.3. *Let p be a prime, $v \in \mathbb{Z}^+$. Then*
$$\tilde{K}_{\mathbb{R}}(L^{4v+3}(p^n)/L^{4v+2}(p^n)) = 0.$$

Lemma 1.1.4. *Let p be an odd prime; $k, t \in \mathbb{Z}^+$ such that $(p, t) = 1$. Then $((\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1) = 0$ on $\tilde{K}_{\mathbb{R}}(L_0^k(p))$.*

Proof. By Fermat's Theorem, $t^{p-1} \equiv 1 \,(\mathrm{mod}\, p)$ and thus $t^{\frac{p-1}{2}} \equiv \pm 1 \,(\mathrm{mod}\, p)$. Let η, μ, w be defined as in 1.1. Then $((\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1)\mu = \eta^{t^{\frac{p-1}{2}}} - \eta = \eta^{\pm 1} - \eta$. If we take realification of both sides and note that $r(\eta^{-1}) = r(\eta)$, we obtain $((\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1)w = 0$. $\square$

Definition 1.1.5. For $m, k \in \mathbb{Z}^+$, define the even binomial coefficient $u_m(k) = \frac{k^2(k^2-1)\ldots(k^2-(m-1)^2)}{\frac{1}{2}(2m)!}$. Note that $u_k(k) = 1$ and $u_m(k) = 0$ for $m > k$.

Proposition 1.1.6. *In $\tilde{K}_{\mathbb{R}}(P_\infty(\mathbb{C}))$, $\psi_{\mathbb{R}}^k(w) = \sum_{m=1}^k u_m(k)w^m$.*

Proof. This is [7, Theorem 5.2.4]. $\square$

2. The transfer-map.

2.1. Properties of the transfer-map. Let H be a subgroup of the compact Lie group G of finite index. Then there exists an induction-homomorphism $i_! : R_F(H) \to R_F(G)$ $(F = \mathbb{R}, \mathbb{C})$ on the representation rings as defined in [6, Section 7]. Let P be the top space of a principal G-bundle. Using the induction-homomorphism, one defines a transfer-map, $\tau_F : K_F(P/H) \to K_F(P/G)$ for the fibration $f : P/H \overset{G/H}{\to} P/G$ which in turn induces $\tau_F : \tilde{K}_F(P/H) \to \tilde{K}_F(P/G)$. In the special case, $H = \mathbb{Z}_{p^n}$, $G = \mathbb{Z}_{p^{n+1}}$ where p

is a prime and $P = S^{2k+1}$, we obtain a transfer-map, $\tau_F : \tilde{K}_F(L^k(p^n)) \to \tilde{K}_F(L^k(p^{n+1}))$ and its restriction, $\tau_F : \tilde{K}_F(L_0^k(p^n)) \to \tilde{K}_F(L_0^k(p^{n+1}))$. We now list some fundamental properties of the transfer.

Proposition 2.1.1.

(i) *The transfer-map commutes with the complexification and realification maps, i.e., the following diagrams commute:*

$$\begin{array}{ccc} \tilde{K}_{\mathbb{R}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{R}}} & \tilde{K}_{\mathbb{R}}(L^k(p^{n+1})) \\ \downarrow c & & \downarrow c \\ \tilde{K}_{\mathbb{C}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{C}}} & \tilde{K}_{\mathbb{C}}(L^k(p^{n+1})) \end{array} \qquad \begin{array}{ccc} \tilde{K}_{\mathbb{C}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{C}}} & \tilde{K}_{\mathbb{C}}(L^k(p^{n+1})) \\ \downarrow r & & \downarrow r \\ \tilde{K}_{\mathbb{R}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{R}}} & \tilde{K}_{\mathbb{R}}(L^k(p^{n+1})) \end{array}$$

(ii) *If $t \in \mathbb{Z}^+$ and $(p,t) = 1$ then $\psi_F^t \circ \tau_F = \tau_F \circ \psi_F^t$.*

(iii) $\tau_F \circ f^!(x) = \tau_F(1)x$, $\forall x \in \tilde{K}_F(L^k(p^{n+1}))$.

(iv) $f^! \circ \tau_F(x) = px$, $\forall x \in \tilde{K}_F(L^k(p^n))$.

(v) *Let $F = \mathbb{C}$ and η_n and η_{n+1} be the Hopf-bundles over $L^k(p^n)$ and $L^k(p^{n+1})$ respectively. Then $\tau_{\mathbb{C}}(\eta_n^i) = \sum_{j \equiv i (\mathrm{mod}\ p^n)} \eta_{n+1}^j$.*

Proof. (i) and (iv) follow immediately from the definition of the transfer-map as in [**6**, Section 7]. For (ii) and (iii) we refer the reader to [**14**, Lemma 2.2]. (v) is [**3**, Lemma 6.5.8]. □

Lemma 2.1.2. *Let $\mu \in \tilde{K}_{\mathbb{C}}(P_\infty(\mathbb{C}))$ and $w \in \tilde{K}_{\mathbb{R}}(P_\infty(\mathbb{C}))$ be the multiplicative generators and $r : \tilde{K}_{\mathbb{C}}(P_\infty(\mathbb{C})) \to \tilde{K}_{\mathbb{R}}(P_\infty(\mathbb{C}))$ be the realification-map. Then $r(\mu^k) = \sum_{i=[\frac{k+1}{2}]}^{k} a_i w^i$ $(a_i \in \mathbb{Z})$.*

Proof. This can be proved by induction on k using the relation $r(\psi_{\mathbb{C}}^k(\mu)) = \psi_{\mathbb{R}}^k(w)$. □

We shall now drop the subscript and write down τ for $\tau_{\mathbb{R}}$.

Proposition 2.1.3. *Let $k \in \mathbb{Z}^+$ and assume that $n \geq 2$ if $p = 2$. Then $\tau(w^k) = \sum_{i \geq 1} a_i w^{k+i-1}$ in $\tilde{K}_{\mathbb{R}}(L_0^k(p^{n+1}))$ where $a_1 = p$ and p/a_i for $2 \leq i \leq p$.*

Proof. It suffices to prove it for $k = 1$ since by (iii) of Proposition 2.1.1 $\tau(w^k) = \tau(w)w^{k-1}$.

That $a_1 = p$ follows from (iv) of Proposition 2.1.1.

For $p = 2$, $\tau_{\mathbb{C}}(1) = 1 + (1+\mu)^{2^n} = 2 + \sum_{i=1}^{2^n-1} \binom{2^n}{i} \mu^i + \mu^{2^n}$, by (v) of Proposition 2.1.1, where $2/\binom{2^n}{i}$ for $1 \leq i \leq 2^n - 1$. $\tau_{\mathbb{C}}(\mu) = (\tau_{\mathbb{C}}(1))\mu = 2\mu + \sum_{i=1}^{2^n-1} \binom{2^n}{i} \mu^{i+1} + \mu^{2^n+1}$ by (iii) of Proposition 2.1.1.

We take realification of both sides and using Lemma 2.1.2 and commutativity of the second diagram in (i) of Propositon 2.1.1, we obtain

$\tau(w) = \sum_{i\geq 1} a_i w^i$ where $2/i$ for $2 \leq i \leq 2^{n-1}$ and since $n \geq 2$ this yields the result for $p = 2$.

For p odd, $\tau_{\mathbb{C}}(1) = 1 + (1+\mu)^{p^n} + (1+\mu)^{2p^n} + \cdots + (1+\mu)^{(p-1)p^n} = p + \sum_{i\geq 1} b_i \mu^i$.

For $1 \leq i \leq p^n - 1$, $b_i = \binom{p^n}{i} + \binom{2p^n}{i} + \cdots + \binom{(p-1)p^n}{i}$ and hence p/b_i.

For $i = p^n$, $b_{p^n} = 1 + \binom{2p^n}{p^n} + \cdots + \binom{(p-1)p^n}{p^n}$.

For $1 \leq s \leq p-1$, $\binom{sp^n}{p^n} = s \prod_{m=1}^{p^n-1} \frac{sp^n - p^n + m}{m} \equiv s \pmod p$. Thus $b_{p^n} \equiv (1 + 2 + \cdots + (p-1)) (\mathrm{mod}\ p) \equiv \frac{p(p-1)}{2} \equiv 0 (\mathrm{mod}\ p)$, i.e., $p \,/\, b_{p^n}$.

For $p^n + 1 \leq i \leq 2p^n - 1$, $a_i = \binom{2p^n}{i} + \cdots + \binom{(p-1)p^n}{i}$ and hence p/a_i. Thus p/b_i for $0 \leq i \leq 2p^n - 1$. $\tau_{\mathbb{C}}(\mu) = (\tau_{\mathbb{C}}(1))\mu = p\mu + \sum_{j\geq 1} b_j \mu^{j+1} = p\mu + \sum_{j\geq 2} b_{j-1}\mu^j$ $\tau(w) = \tau(r(\mu)) = r(\tau_{\mathbb{C}}(\mu)) = pw + \sum_{j\geq 2} b_{j-1} r(\mu^j)$ by the commutativity of the 2^{nd}-diagram in (i) of Proposition 2.1.1 $r(\mu^j) = \sum_{i=[\frac{j+1}{2}]}^{j} c_i^j w^i$ ($c_i^j \in \mathbb{Z}$) by Lemma 2.1.2.

Thus $\tau(w) = pw + \sum_{j\geq 2} \sum_{i=[\frac{j+1}{2}]}^{j} b_{j-1} c_i^j w^i = pw + \sum_{i\geq 1} a_i w^i$ where $a_i = \sum_{[\frac{j+1}{2}]\leq i\leq j} b_{j-1} c_i^j = \sum_{j=i}^{2i} b_{j-1} c_i^j$. Let $i \leq p^n$. Then in the second sum above, $j \leq 2i \leq 2p^n$ and $p|b_{j-1}$ by the first part of the proof. Hence $p|a_i$ for $2 \leq i \leq p^n$ and hence for $2 \leq i \leq p$. □

2.2. Preliminaries on binomial expansions. Section 2.2 is a technical section aimed at proving Proposition 2.2.2.

If p is a prime and $n \in \mathbb{Z}^+$, $v_p(n)$ will denote the exponent of p in the prime factorization of n.

Definition 2.2.1. For $p^{n-1} \leq k \leq p^n - 1$, define $\Phi(k) = n + [\frac{p^n - k - 1}{p}]$.

If we arrange the integers in decreasing fashion from $k = p^n - 1$ to $k = p^{n-1}$ in blocks B_j of p consecutive integers then Φ is the step function which is constant on each block, increases by 1 with each increasing block and takes the value n on B_1.

Proposition 2.2.2. *Let $S_{n,p} = \sum_{j\geq p^{n-1}} c_j [\psi_{\mathbb{R}}^p(w)]^j$ in $\tilde{K}_{\mathbb{R}}(P_\infty(\mathbb{C}))$. If we expand $S_{n,p} = \sum_{k\geq p^{n-1}} a_k w^k$ then $v_p(a_k) \geq \Phi(k)$ $(p^{n-1} \leq k \leq p^n - 1)$.*

Proposition 2.2.2 is essential for the inductive proof of Proposition 2.3.1 which in turn is essential for the proof of Lemma 3.2.1 for the injectivity of the homomorphism $c^!$. Proposition 2.3.1 asserts for p odd and t prime to p, the existence of a series in w^k that starts at w^j (for any j) and which belongs to Ker $[(\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$, the exponents of whose coefficients have a lower-bound given by a certain function $\psi(j,k)$ which is attained for $k = j$. For $v_p(j) \leq n-2$, the result follows by applying the transfer-map to the series we have in Ker $[(\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^{n-1}))$ by the induction-hypothesis and by applying Proposition 2.1.3 to the coefficients. The difficult case is the one for $j = p^{n-1}$ (the more general case, $v_p(j) \geq n-1$ easily follows from this) and this is where Proposition 2.2.2 comes into play. For $j = p^{n-2}$, we have two series to compare; one that we have by the case $v_p(j) \leq n-2$ and another one that we obtain by applying the homomorphism $f!$ induced by the p-th power map, $f : L_0^{2v}(p^n) \to L_0^{2v}(p^{n-1})$ to the series that we have by the induction-hypothesis for $j = p^{n-2}$. By noting that $f^!(w) = \psi_{\mathbb{R}}^p(w)$, the second-series is of the form $\sum_{k \geq p^{n-2}} b_k [\psi_{\mathbb{R}}^p(w)]^k$ which by Proposition 2.2.2 can be written as $\sum_{k \geq p^{n-2}} a_k w^k$ where $v_p(a_k) \geq \Phi(k)$. A lower-bound for the exponents of the coefficients of the first series is given by $\psi(p^{n-2}, k)$ which is attained for $k = p^{n-2}$. $\Phi(j) \geq \psi(j,j)$, in general and using the special case of this for $j = p^{n-2}$, we can subtract a scalar-multiple of the first series from the second to eliminate the term $w^{p^{n-2}}$ and the resulting series starts with the term $w^{p^{n-2}+1}$. If m is the exponent of the multiplying factor then $m + \psi(j,k) \geq \Phi(k)$ and an immediate consequence of the special-case of this inequality for $j = p^{n-2}$ is that the p^{n-1}-th coefficient of the resulting series is prime to p. We continue this process inductively until we knock off the terms $w^{p^{n-2}+1}, w^{p^{n-2}+2}, \ldots, w^{p^{n-1}-1}$ and in the end, obtain a series in $\mathrm{Ker}[(\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$ that starts with the term $w^{p^{n-1}}$ and whose p^{n-1}-th coefficient is prime to p.

Lemma 2.2.3. *Let p be an odd prime, $m \in \mathbb{Z}^+$ and $u_m(p)$ the even binomial coefficient defined in 1.1.5. Then*

$$v_p(u_m(p)) = \begin{cases} 2 & \text{if } 1 \leq m \leq \frac{p-1}{2} \\ 1 & \text{if } \frac{p+1}{2} \leq m \leq p-1. \end{cases}$$

Observation 2.2.4. Let p be an odd prime, $n \in \mathbb{Z}^+$ and let $[\psi_{\mathbb{R}}^p(w)]^{p^{n-1}} = \sum_{k=p^{n-1}}^{p^n} a_k w^k$ in $\tilde{K}_{\mathbb{R}}(P_\infty(\mathbb{C}))$.Then

$$a_k = \sum_{\substack{s_1+\ldots+s_p=p^{n-1} \\ s_1+2s_2+\ldots+ps_p=k}} \frac{(p^{n-1})!}{s_1!s_2!\ldots s_p!} \prod_{m=1}^{p-1} [u_m(p)]^{s_m}.$$

Proof. It is an immediate consequence of Proposition 1.1.6. □

Definition 2.2.5. Let p be an odd prime, $n \in \mathbb{Z}^+$ and $p^{n-1} \leq k \leq p^n - 1$. We let S_k denote the set of all sequences $s = (s_1, ..., s_p)$ of non-negative integers such that $s_1 + ... + s_p = p^{n-1}$ and $s_1 + 2s_2 + ... + ps_p = k$. For $s \in S_k$, define $T(s) = \frac{(p^{n-1})!}{s_1!s_2!...s_p!}$ and $\theta(s) = T(s) \prod_{m=1}^{p-1} [u_m(p)]^{s_m}$. Observation 2.2.4 can be stated in an equivalent form, i.e.,

Observation 2.2.6. Under the hypothesis of Observation 2.2.4, $a_k = \sum_{s\in S_k} \theta(s)$.

Definition 2.2.7. Let p be an odd prime and $s \in S_k$. Define $e_p(s) = 2s_1 + \cdots + 2s_{\frac{p-1}{2}} + s_{\frac{p+1}{2}} + \cdots + s_{p-1}$.

We state the following Corollary to Lemma 2.2.3.

Corollary 2.2.8. *$v_p(\theta(s)) = v_p(T(s)) + e_p(s)$.*

Definition 2.2.9. For $k \in \mathbb{Z}^+$, define a number N_k by $v_p(N_k) = \sup_{1\le r\le [\frac{k}{p-1}]}(1 + v_p(r))$. Let $N_{k,p}$ denote its p-component.

We now observe that [**5**, Lemma 6.1] can be proved under more general hypothesis; i.e.,

Lemma 2.2.10. *Let p be a prime, $n, k \in \mathbb{Z}^+$. If $v_p(n) \ge v_p(N_{k-1})$ then $v_p(\binom{n}{k}) = v_p(n) - v_p(k)$.*

Proof. Identical with that of [**5**, Lemma 6.1]. □

In the following, p is an odd prime and $n \in \mathbb{Z}^+$.

Definition 2.2.11. Let I_i be the closed interval, $I_i = [p^n - p^i + 1, p^n - p^{i-1}]$ in $\mathbb{Z}^+$ $(1 \le i \le n-1)$ and let $I_n = [p^{n-1}, p^n - p^{n-1}]$. Then $[p^{n-1}, p^n - 1] = \cup_{i=1}^n I_i$.

Lemma 2.2.12. *Let $s \in S_k$. Then $s_p \ge k - (p-1)p^{n-1}$. If $k \in I_i$ then $s_p \ge p^{n-1} - p^i + 1$.*

Proof. $s_1+s_2+\cdots+s_p = p^{n-1}$ and $s_1+2s_2+\cdots+ps_p = k$ and substracting the first equation from the second yields 1. $s_2+2s_3+\cdots+(p-2)s_{p-1}+(p-1)s_p = k - p^{n-1}$, or equivalently $s_2 + 2s_3 + \cdots + (p-2)s_{p-1} + (p-2)s_p + p^{n-1} = k - s_p$. $LHS = (s_2 + \cdots + s_p) + (s_3 + \cdots + s_p) + \cdots + (s_{p-1} + s_p) + p^{n-1} \le (p-2)p^{n-1} + p^{n-1} = (p-1)p^{n-1}$. Thus, $k - s_p \le (p-1)p^{n-1}$ or, equivalently, $s_p \ge k - (p-1)p^n$. If $k \in I_i$ then $k \ge p^n - p^i + 1$ and hence $s_p \ge k - (p-1)p^{n-1} \ge p^n - p^i + 1 - (p-1)p^{n-1} = p^{n-1} - p^i + 1$. □

Corollary 2.2.13. *Let $s \in S_k$ and $k \in I_i$. Then $v_p(T(s)) \ge n - i$.*

Proof. It follows from the second part of Lemma 2.2.12 that $v_p(s_p) \le i - 1$. $T(s) = \binom{p^{n-1}}{s_p}\frac{(s_1+\cdots+s_{p-1})!}{s_1!\ldots s_{p-1}!}$ and it follows from Lemma 2.2.10 that $v_p(\binom{p^{n-1}}{s_p})$ $= n - 1 - v_p(s_p) \ge n - 1 - (i-1) = n - i$. □

Definition 2.2.14. For each $p^{n-1} \le k \le p^n - 1$, we define a unique special sequence $s^0(k)$ by $(s^0(k))_p = [\frac{k-p^{n-1}}{p-1}]$. Let $r = k - p^{n-1} - (p-1)(s^0(k))_p$. Then $0 \le r \le p-2$. The remaining (possibly) non-zero indices of $s^0(k)$ are $(s^0(k))_{r+1} = 1$ if $r \ge 1$ and $(s^0(k))_1 = p^{n-1} - (s^0(k))_p - 1 + \delta_{r0}$ where δ_{r0} is the Kronecker-delta. If we arrange the integers in decreasing fashion from

$k = p^n - 1$ to $k = p^{n-1}$ in p^{n-1} blocks B_j of $(p-1)$ consecutive integers, then $(s^0(k))_p = p^{n-1} - j$ is constant on each block. If $B_j = (k_1, \dots, k_{p-1})$, $k_i = k_{i-1} + 1$, $k_i = p^n - j(p-1) + i - 1$ then the non-zero indices of $s^0(k_i)$ apart from $(s^0(k_i))_p$ are : $(s^0(k_i))_i = 1$ and

$$(s^0(k_i))_1 = \begin{cases} j & \text{if } i = 1 \\ j-1 & \text{if } 2 \le i \le p-1 \end{cases}.$$

Observation 2.2.15. If we arrange the integers in decreasing fashion from $k = p^n - 1$ to $k = p^{n-1}$ in $2p^{n-1}$ blocks of $\frac{p-1}{2}$ consecutive integers then $e_p(s^0(k))$ is constant on each block and increases by 1 with each increasing block and takes the value 1 on the first block.

Proof. Let $B_j^1 = (k_{\frac{p+1}{2}}, \dots, k_{p-1})$ and $B_j^2 = (k_1, \dots, k_{\frac{p-1}{2}})$. Then it is clear from the above and the definition of $e_p(s^0(k))$ that $e_p(s^0(k))$ is constant on B_i^j $(i = 1, 2)$ and increases by 1 in passing from B_j^1 to B_j^2 and from B_j^2 to B_{j+1}^1 and takes the value 1 on B_1^1. □

Lemma 2.2.16. *If $p^{n-1} \le k \le p^n - 1$ and $s \in S_k$ then $e_p(s) \ge e_p(s^0(k))$.*

Proof. Define $u(s) = \sum_{i=1}^{\frac{p-1}{2}} s_i$ and $v(s) = \sum_{i=\frac{p+1}{2}}^{p-1} s_i$. Then by definition, $e_p(s) = 2u + v = 2(u + v + s_p) - v - 2s_p = 2p^{n-1} - v - 2s_p$. Hence:

1. $e_p(s) - e_p(s^0(k)) = [v(s^0(k)) - v(s)] + 2((s^0(k))_p - s_p)$ $s_2 + 2s_3 + \cdots + (p-1)s_p = k - p^{n-1} = r + (p-1)(s^0(k))_p$ where $0 \le r \le p-2$ and thus;
2. $s_2 + 2s_3 + \cdots + (\frac{p-1}{2})s_{\frac{p+1}{2}} + \cdots + (p-2)s_{p-1} = (p-1)((s^0(k))_p - s_p) + r$ $LHS \ge \sum_{i=\frac{p+1}{2}}^{p-1} (i+1)s_i \ge (\frac{p-1}{2}) \sum_{i=\frac{p+1}{2}}^{p-1} s_i = (\frac{p-1}{2})v(s)$ gives $v(s) \le 2((s^0(k))_p - s_p) + \frac{2r}{p-1}$ and hence $v(s) \le 2((s^0(k))_p - s_p) + [\frac{2r}{p-1}]$.

(i) If $r \ge \frac{p-1}{2}$, $(s^0(k))_{r+1} = 1$ and $v(s^0(k)) = 1$ and thus $v(s) \le 2((s^0(k))_p - s_p) + 1$.

(ii) If $r \le \frac{p-1}{2}$, $v(s^0(k)) = 0$ and thus $v(s) \le 2((s^0(k))_p - s_p)$ and in either case, $v(s) \le 2((s^0(k))_p - s_p) + v(s^0(k))$ and the result follows from 1 above. □

Lemma 2.2.17. *For $k \in I_i$, $n - i + e_p(s^0(k)) \ge \Phi(k)$.*

Proof. It follows from Definition 2.2.1 in a straightforward way. □

Corollary 2.2.18. *Let $p^{n-1} \le k \le p^n - 1$ and $s \in S_k$. Then $v_p(\theta(s)) \ge \Phi(k)$.*

Proof. It is an immediate consequence of Corollaries 2.2.8, 2.2.13 and Lemma 2.2.17. □

Proof of Proposition 2.2.2. It suffices to prove that if $[\psi_{\mathbb{R}}^p(w)]^{p^{n-1}+j} = \sum_{k \ge p^{n-1}+j} a_k^j w^k$ then $v_p(a_k^j) \ge \Phi(k)$. $[\psi_{\mathbb{R}}^p(w)]^{p^{n-1}+j} = [\psi_{\mathbb{R}}^p(w)]^{p^{n-1}} [\psi_{\mathbb{R}}^p(w)]^j$

$= [\sum_{i\geq p^{n-1}} a_i w^i][\sum_{l\geq j} b_l w^l] = \sum_{k\geq p^{n-1}+j} a_k^j w^k$. $a_j^k = \sum_{i+l=k} a_i b_l$. By Observation 2.2.4 and Corollary 2.2.17, $v_p(a_i) \geq \Phi(i) \geq \Phi(k)$ for $i \leq k$ and thus $v_p(a_i b_l) \geq \Phi(k)$. Hence $v_p(a_k^j) \geq \Phi(k)$. □

2.3. Kernel of $(\psi_{\mathbb{R}}^t - 1)$. In what follows t will be an integer not divisible by the prime p.

Proposition 2.3.1. *Let p be an odd prime and $j, n, t, v \in \mathbb{Z}^+$ such that $(p,t) = 1$. For $k \geq j$, define*

$$\psi(j,k) = \begin{cases} n - 1 - v_p(j) - \left[\frac{k-j}{p}\right] & \text{if } j \leq k \leq j + p(n-1-v_p(j)) - 1 \\ 0 & \text{if } k \geq j + p(n-1-v_p(j)). \end{cases}$$

Then there exist $a_{j,k} \in \mathbb{Z}$ such that:

(i) $v_p(a_{j,j}) = \psi(j,j) = n - 1 - v_p(j)$.
(ii) $v_p(a_{j,k}) \geq \psi(j,k)$.
(iii) $\sum_{k\geq j} a_{j,k} w^k \in \operatorname{Ker}[(\Psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$.

Proof. By induction on n.

For $n = 1$ it follows from Lemma 1.1.4.

Let $n > 1$ and assume it to be true for $n-1$. Let $K_i = \operatorname{Ker}[(\Psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^i))$.

For $0 \leq v_p(j) \leq n-2$, the result can be obtained by applying the transfer map $\tau_{n-1} : \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^{n-1})) \to \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$ and by using the induction-hypothesis and Proposition 2.1.3 and by noting that τ_{n-1} maps K_{n-1} to K_n which is a consequence of (ii) of Proposition 2.1.1. For $j = p^{n-1}$, the p-th power map, $g : L_0^{2v}(p^n) \to L_0^{2v}(p^{n-1})$ factors through $L_0^{2v}(p^{n-1})$, i.e., there exists a map $f : L_0^{2v}(p^n) \to L_0^{2v}(p^{n-1})$ such that the following diagram commutes:

$$\begin{tikzcd} L_0^{2v}(p^n) \arrow[r, "f"] \arrow[dr, "g"'] & L_0^{2v}(p^{n-1}) \arrow[d] \\ & L_0^{2v}(p^n) \end{tikzcd}$$

Thus, $f^!(w) = \psi_{\mathbb{R}}^p(w)$. By the induction-hypothesis, there exist $\sum_{s\geq p^{n-2}} b_{p^{n-2},s} w^s \in K_{n-1}$. Applying $f^!$ and by noting that $f^!$ maps K_{n-1} to K_n, we obtain $\sum_{s\geq p^{n-2}} b_{p^{n-2},s} [\psi_{\mathbb{R}}^p(w)]^s \in K_n$. By Proposition 2.2.2, there exist $c_k \in \mathbb{Z}\,(k \geq p^{n-2})$ with $v_p(c_k) \geq \Phi(k)$ such that:

1. $\sum_{k\geq p^{n-2}} c_k w^k \in K_n$.

We now claim the following statement. For $p^{n-2} \leq j \leq p^{n-1}$, there exist $c_{j,k} \in \mathbb{Z}\,(k \geq 1)$ with $(p, c_{j,p^{n-1}}) = 1$, $v_p(c_{j,k}) \geq \Phi(k)$ such that $\sum_{k\geq j} c_{j,k} w^k \in K_n$.

Proof. By induction on j. For $j = p^{n-2}$ this follows from 1 above.

Let $p^{n-2} < j \leq p^{n-1}$ and assume it to be true for $(j-1)$. Since $0 \leq v_p(j-1) \leq n-2$, by the first part of the proof, there exist coefficients $a_{j-1,k} \in \mathbb{Z}\,(k \geq j-1)$ with $v_p(a_{j-1,j-1}) = \psi(j-1, j-1) = n-1-v_p(j-1)$ and $v_p(a_{j-1,k}) \geq \psi(j-1,k)$ such that:

2. $\sum_{k\geq j-1} a_{j-1,k} w^k \in K_n$.

By the induction-hypothesis, there exist coefficients $c_{j-1,k} \in \mathbb{Z}\,(k \geq j-1)$ with $(p, c_{j-1,p^{n-1}}) = 1$, $v_p(c_{j-1,k}) \geq \Phi(k)$ such that

3. $\sum_{k\geq j-1} c_{j-1,k} w^k \in K_n$.

Define $m = v_p(c_{j-1,j-1}) - v_p(a_{j-1,j-1}) \geq \Phi(j-1) - \psi(j-1, j-1) \geq 0$ $a_{j-1,j-1} = p^{v_p(a_{j-1,j-1})}\alpha_{j-1}$ and $c_{j-1,j-1} = p^{v_p(c_{j-1,j-1})}\gamma_{j-1}$ where (p, α_{j-1}) $= (p, \gamma_{j-1}) = 1$. Multiply Equation 2 by $p^m\gamma_{j-1}$ and 3 by $-\alpha_{j-1}$ and add up the resulting equations to obtain:

5. $\sum_{k\geq j} c_{j,k} w^k \in K_n$.

Let $\Delta\psi(j-1,k)$ and $\Delta\Phi(k)$ be the respective increases in $\psi(j-1,k)$ and $\Phi(k)$ from $j-1$ to k. $\psi(j-1,k)$ and $\Phi(k)$ are constant on each p-block of consecutive (increasing) integers starting with $j-1$ and p^{n-2} respectively and decrease by 1 with each increasing block. Thus $\Delta\psi(j-1,k) \geq \Delta\Phi(k)$ $m + \psi(j-1, j-1) \geq \Phi(j-1)$ and $m + \psi(j-1,k) = m + \psi(j-1, j-1) + \Delta\psi(j-1,k) \geq \Phi(j-1) + \Delta\Phi(k) = \Phi(k)$. Hence $v_p(p^m\gamma_{j-1}a_{j-1,k}) \geq m + \psi(j-1,k) \geq \Phi(k)$ and also $v_p(-\alpha_{j-1}c_{j-1,k}) = v_p(c_{j-1,k}) \geq \Phi(k)$ and thus $v_p(c_{j,k}) \geq \Phi(k)$.

(i) By the induction-hypothesis, $(p, \alpha_{j-1}c_{j-1,p^{n-1}}) = 1$ and if:
  a) $j-1 > p^{n-1}-p$ then $\Phi(j-1) \geq n+1$ and $\psi(j-1,j-1) \leq n$ and thus $m \geq \Phi(j-1) - \psi(j-1,j-1) \geq 1$;
  b) $j-1 = p^{n-1}-p$, then $\Phi(j-1) = n$ and $\psi(j-1,j-1) = n-1$ and thus $m \geq \Phi(j-1) - \psi(j-1,j-1) = n-(n-1) = 1$;
  c) $j-1 \leq p^{n-1}-p-1$ then $v_p(a_{j-1,p^{n-1}}) \geq \psi(j-1,p^{n-1}) \geq 1$.

  In all three cases,

(ii) $p/p^m\gamma_{j-1}a_{j-1,p^{n-1}}$.

We deduce from (i) and (ii) above that $(p, a_{j,p^n}) = 1$ and this proves the statement.

We deduce from the special case of the statement for $j = p^{n-1}$ that there exist coefficients $a_{p^{n-1},k}$ $(k \geq p^{n-1})$ with $(p, a_{p^{n-1},p^{n-1}}) = 1$ such that $\sum_{k\geq p^{n-1}} a_{p^{n-1},k} w^k \in K_n$.

More generally, for $v_p(j) \geq n-1$, let $j = p^{n-1}j'$. Then by what we have already proved and since K_n is an ideal in $\tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$,

$$\left(\sum_{k \geq p^{n-1}} a_{p^{n-1},k} w^k\right)\left(\sum_{l \geq j'} a_{j',l} w^l\right) \in K_n$$

and hence the result. □

We now extend this to $p = 2$. We replace $L_0^k(p^n)$ for odd p by $L^k(2^n)$ for $p = 2$. Let t be an odd integer. Here $L^k(4)$ plays the role of $L_0^k(p)$, p odd, and the analogous result to Lemma 1.1.4 is that $\psi_{\mathbb{R}}^t - 1 = 0$ in $\tilde{K}_{\mathbb{R}}(L^k(4))$. We, necessarily, assume $n \geq 2$ and consider the sequence of transfer-maps, $\tilde{K}_{\mathbb{R}}(L^k(4)) \to \cdots \to \tilde{K}_{\mathbb{R}}(L^k(2^n))$. The analogue of Proposition 2.3.1 is:

Proposition 2.3.2. *Let $t, j, v, n \in \mathbb{Z}^+$, t odd, $n \geq 2$ and define*

$$\psi(j,k) = \begin{cases} n - 2 - v_2(j) - [\frac{k-j}{2}] & \text{if } j \leq k \leq j + 2(n-2-v_2(j)) - 1 \\ 0 & \text{if } k \geq j + 2(n-2-v_2(j)). \end{cases}$$

Then there exist $a_{j,k} \in \mathbb{Z}$ such that:

(i) $v_2(a_{j,j}) = \psi(j,j) = n - 2 - v_2(j)$;
(ii) $v_2(a_{j,k}) \geq \psi(j,k)$ *for* $k \geq j$;
(iii) $\sum_{k \geq j} a_{j,k} w^k \in \operatorname{Ker}(\psi_{\mathbb{R}}^t - 1) \subseteq \tilde{K}_{\mathbb{R}}(L^{2v}(2^n))$.

Proof. Almost identical with that of Proposition 2.3.1. □

Let p be a prime, $n \in \mathbb{Z}^+$ and let G_{p^n} be the multiplicative group of units in $\mathbb{Z}_{p^n}$.

$$G_{p^n} = \begin{cases} \mathbb{Z}_{p^{n-1}(p-1)} & \text{if } p \text{ is odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} & \text{if } p = 2, \text{ where the first summand is generated by } -1. \end{cases}$$

Proposition 2.3.3. *Let p be a prime, $t \in \mathbb{Z}^+$ such that $(t,p) = 1$ and that t is a generator of G_{p^2} if p is odd and a generator of $G_8/\{\mp 1\}$ if $p = 2$. Define*

$$n_p = \begin{cases} 1 & \text{if } p = 2 \\ \frac{p-1}{2} & \text{if } p \text{ is odd} \end{cases} \quad \text{and} \quad \epsilon_p = \frac{3 + (-1)^p}{2}.$$

Let $n, v, j \in \mathbb{Z}^+$ and assume that $j \equiv 0 \pmod{n_p}$. For $k \geq j$, define

$$\psi_p(j,k) = \begin{cases} n - \epsilon_p - v_p(j) - [\frac{k-j}{p}] & \text{if } j \leq k \leq j + p(n - \epsilon_p - v_p(j)) - 1 \\ 0 & \text{if } k \geq j + p(n - \epsilon_p - v_p(j)). \end{cases}$$

Then there exist $a_{j,k} \in \mathbb{Z}$ such that:

(i) $v_p(a_{j,j}) = \psi_p(j,j) = n - \epsilon_p - v_p(j)$;
(ii) $v_p(a_{j,k}) \geq \psi_p(j,k)$;
(iii) $\sum_{k \geq j} a_{j,k} w^k \in \operatorname{Ker}(\psi_{\mathbb{R}}^t - 1) \subseteq \tilde{K}_{\mathbb{R}}(\overline{L}^{2v}(p^n))$.

Proof. For $p = 2$ it reduces to the statement of Proposition 2.3.2. For p odd, it follows from Proposition 2.3.1 that there exist $b_{j,k} \in \mathbb{Z}$ such that:

(i) $v_p(b_{j,j}) = \psi_p(j,j) = n - 1 - v_p(j)$.
(ii) $v_p(b_{j,k}) \geq \psi_p(j,k)$.
(iii)

$$\begin{aligned} 0 &= \left[\left(\psi_{\mathbb{R}}^t\right)^{\frac{p-1}{2}} - 1\right]\left(\sum_{k\geq j} b_{j,k}w^k\right) \\ &= \left(\psi_{\mathbb{R}}^t - 1\right)\left(1 + \psi_{\mathbb{R}}^t + \cdots + \left(\psi_{\mathbb{R}}^t\right)^{\frac{p-5}{2}} + \left(\psi_{\mathbb{R}}^t\right)^{\frac{p-3}{2}}\right)\left(\sum_{k\geq j} b_{j,k}w^k\right) \\ &= \left(\psi_{\mathbb{R}}^t - 1\right)\left(\sum_{k\geq j} a_{j,k}w^k\right) \qquad \text{i.e.,} \end{aligned}$$

$$\sum_{k\geq j} a_{j,k}w^k = \left[1 + \psi_{\mathbb{R}}^t + \cdots + \left(\psi_{\mathbb{R}}^t\right)^{\frac{p-5}{2}} + \left(\psi_{\mathbb{R}}^t\right)^{\frac{p-3}{2}}\right]\left(\sum_{k\geq j} b_{j,k}w^k\right)$$

$$a_{j,j} = \left[1 + (t^{2j} - 1) + (t^{4j} - 1) + \cdots + (t^{(p-3)j} - 1)\right] b_{j,j}.$$

Since $2j \equiv 0 \pmod{(p-1)}$, it follows from [1, Lemma 2.12] that

$$v_p\left(t^{2mj} - 1\right) = 1 + v_p(2mj) \geq 1 \qquad \left(1 \leq m \leq \frac{p-3}{2}\right)$$

i.e., p divides all the terms inside the bracket except the first one and thus the bracket is not divisible by p. Hence

$$v_p(a_{j,j}) = v_p(b_{j,j}) = \psi_p(j,j) = n - 1 - v_p(j).$$

If $[\psi_{\mathbb{R}}^{t^m}(w^s)]_k$ denotes the coefficient of w^k in the expansion of $\psi_{\mathbb{R}}^{t^m}(w^s)$ then

$$a_{j,k} = b_{j,k} + \sum_{1\leq m\leq \frac{p-3}{2}} \sum_{s\geq j} b_{j,s}\left[\psi_{\mathbb{R}}^{t^m}(w^s)\right]_k.$$

$v_p(b_{j,s}) \geq \psi_p(j,s) \geq \psi_p(j,k)$ and hence $v_p(a_{j,k}) \geq \psi_p(j,k)$. □

3. J-Groups of Lens spaces.

3.1. J-triviality.

Lemma 3.1.1. *Let $k, n \in \mathbb{Z}^+$, p and q be distinct primes such that q is a generator of G_{p^n} if p is odd and of the summand $\mathbb{Z}_{2^{n-2}}$ if $p = 2$ and $u \in \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^n))$. Then $J(u) = 0$ in $J(\overline{L}^k(p^n))$ iff there exists $x \in \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^n))$ such that $u = (\psi_{\mathbb{R}}^q - 1)x$.*

Proof. It follows from the Adams conjecture (for an elementary proof see [**9**]), [**2**, Theorem 1.1] and the fact that $\tilde{K}_{\mathbb{R}}(\overline{L}^k(p^n))$ is a p-group that J-trivial bundles over $\overline{L}^k(p^n)$ are finite linear combinations of the form:
1. $\sum_{(k,p)=1}(\psi^k_{\mathbb{R}}-1)y$.
If $k = p_1 \cdots p_r$ for prime, p_i $(1 \leq i \leq r)$ then:
2. $(\psi^k_{\mathbb{R}}-1)x = (\psi^{p_1}_{\mathbb{R}}-1)\psi^{p_2\cdots p_r}(x)+(\psi^{p_2}_{\mathbb{R}}-1)\psi^{p_3\cdots p_r}(x)+\cdots+(\psi^{p_{r-1}}_{\mathbb{R}}-1)\psi^{p_r}_{\mathbb{R}}(x)$
and hence we may, without loss of generality, assume that in 1, k runs over the set of complementary primes to p. Let $k = q'$ be such a prime. Thus $q' \equiv \pm q^m \pmod{p^n}$ for some $m \in \mathbb{Z}^+$. Hence if η is the Hopf-bundle over $\overline{L}^k(p^n)$, $\eta^{q'} = \eta^{\pm q^m}$ i.e., $\psi^{q'}_{\mathbb{C}}(\mu) = \psi^{\pm q^m}_{\mathbb{C}}(\mu)$ and taking realifications yields $\psi^{q'}_{\mathbb{R}}(w) = \psi^{q^m}_{\mathbb{R}}(w)$ and thus, $(\psi^{q'}_{\mathbb{R}}-1)w^i = (\psi^{q^m}_{\mathbb{R}}-1)w^i = (\psi^q_{\mathbb{R}}-1)x$ by 2 above.

Also for $p = 2$, $n \geq 2$ and if λ is the reduction of the canonical line-bundle over $\overline{L}^k(p^n)$ as defined in Section 1.1, then $(\psi^q_{\mathbb{R}}-1)\lambda = 0$ for q odd. □

In his solution of the vector-field problem, Adams has (essentialy) proved that $J(P^n) = \tilde{K}_{\mathbb{R}}((P^n))$. We now extend his result.

Corollary 3.1.2. $J(L^k(4)) = \tilde{K}_{\mathbb{R}}(L^k(4))$.

Proof. Assume that $n = 2k$ is even. (i) If $q = 4m+1$, $\eta^q = \eta$ and hence $(\psi^q_{\mathbb{C}}-1)\mu = 0$. (ii) If $q = 4m-1$, $\eta^q = \eta^{-1}$ and hence $(\psi^q_{\mathbb{C}}-\psi^{-1}_{\mathbb{C}})\mu = 0$. $r[(\psi^q_{\mathbb{C}}-1)\mu] = r[(\psi^q_{\mathbb{C}}-\psi^{-1}_{\mathbb{C}})\mu] = (\psi^q_{\mathbb{R}}-1)w$ and hence $(\psi^q_{\mathbb{R}}-1)w = 0$ in either case. Also $(\psi^q_{\mathbb{R}}-1)\lambda = 0$. Thus $(\psi^q_{\mathbb{R}}-1) = 0$ for q odd and the result follows from Lemma 3.1.1. □

3.2. Injectivity of the map, $c^! : J(\overline{L}^{2v}(p^n))/\overline{L}^{2v-2}(p^n) \to J(\overline{L}^{2v}(p^n))$.

Lemma 3.2.1. *Let p be a prime; $i, n, s, t, v \in \mathbb{Z}^+$ such that $(p,t) = 1$ and $sw^v = (\psi^t_{\mathbb{R}}-1)(\sum_{j=i}^{v} m_j w^j)$ in $\tilde{K}_{\mathbb{R}}(\overline{L}^{2v}(p^n))$ for $1 \leq i \leq v$ and $m_j \in \mathbb{Z}$ $(i \leq j \leq v)$. Then there exist $n_j \in \mathbb{Z}$ $(i+1 \leq j \leq v)$ such that $sw^v = (\psi^t_{\mathbb{R}}-1)(\sum_{j=i+1}^{v} n_j w^j)$.*

Proof. $sw^v = m_i(t^{2i}-1)w^i + \sum_{j=i+1}^{v} m'_j w^j$.

(i) Let p be odd and $2i \not\equiv 0 \pmod{(p-1)}$. It follows from Section 1.1 that $p^n \mathbin{/} m_i(t^{2i}-1)$ and from [**1**, Lemma 2.12] that p does not divide $(t^{2i}-1)$. Hence p^n/m_i. We deduce from Section 1.1 that $m_i w^i = \sum_{j=i+1}^{v} \alpha_j w^j$ and we put $n_j = m_j + \alpha_j$ $(i+1 \leq j \leq v)$.

(ii) Let p be odd and $2i \equiv 0 \pmod{(p-1)}$. It follows from Section 1.1 that $v_p(m_i(t^{2i}-1)) \geq n$ and from [**1**, Lemma 2.12] that $v_p(t^{2i}-1) = 1 + v_p(2i)$. Thus, $v_p(m_i) \geq n-1-v_p(2i) = n-1-v_p(i) = \psi_p(i,i)$ where $\psi_p(i,j)$ is as defined in Proposition 2.3.3.

(iii) Let $p = 2$. It follows from Section 1.1 that $v_2(m_i(t^{2i}-1)) \geq n+1$ and from [**1**, Lemma 2.12] that $v_2(t^{2i}-1) = 2 + v_2(2i)$. Thus $v_2(m_i) \geq n+1-$

$2 - v_2(2i) = n - 2 - v_2(i) = \psi_2(i,i)$. It follows from Proposition 2.3.3 that in both cases (ii) and (iii), there exist $\beta_j \in \mathbb{Z}$ $(i \le j \le v)$ with $\beta_i = m_i$ such that $\sum_{j=i}^{v} \beta_j w^j \in \operatorname{Ker}(\psi_{\mathbb{R}}^t - 1)$. Hence $(\psi_{\mathbb{R}}^t - 1)m_i w^i = -(\psi_{\mathbb{R}}^t - 1)(\sum_{j=i+1}^{v} \beta_j w^j)$ and we put $n_j = m_j - \beta_j$. □

Proposition 3.2.2. *Let p be a prime and $n, v \in \mathbb{Z}^+$ and $c : \overline{L}^{2v}(p^n) \to \overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)$. Then the induced homomorphism*

$$c^! : J\left(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)\right) \to J\left(\overline{L}^{2v}(p^n)\right)$$

is injective.

Proof. Let $c^! J(sw^v) = 0$ in $J(\overline{L}^{2v}(p^n))$. Let q be a prime which is a generator of G_{p^n}. We claim the following:

Statement. For each $1 \le i \le v$, there exist $m_j \in \mathbb{Z}$ $(i \le j \le v)$ such that $sw^v = (\psi_{\mathbb{R}}^q - 1)(\sum_{j=i}^{v} m_j w^j)$.

Proof. By induction on i.

For $i = 1$, it follows from Lemma 3.1.1, Section 1.1 and the fact that for $p = 2$, $(\psi_{\mathbb{R}}^q - 1)\lambda = 0$. Let $i > 1$ and assume it to be true for $i - 1$. Then it is true for i by Lemma 3.2.1. This proves the Statement and the Proposition follows from the special case of the statement for $i = v$. □

Corollary 3.2.3. *We have an exact sequence,*

$$0 \to J(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)) \xrightarrow{c^!} J(\overline{L}^{2v}(p^n)) \xrightarrow{i^!} J(\overline{L}^{2v-2}(p^n)) \to 0.$$

Proof. The exactness of the four terms on the right follows from [**1**, Theorem 3.12], [**2**, Theorem 1.1] and the Adams conjecture. The injectivity of $c^!$ follows from Proposition 3.2.2. □

Lemma 3.2.4. $c^! : J(L^{4v+1}(p^n)/L^{4v}(p^n)) \to J(L^{4v+1}(p^n))$ *is injective.*

Proof. By Lemma 1.1.2, $\tilde{K}_{\mathbb{R}}(L^{4v+1}(2^n)/L^{4v}(2^n)) = \mathbb{Z}_2$ and generator maps to w^{2v+1}. The proof is identical with that of Proposition 3.2.2. □

Corollary 3.2.5. *The following sequence is exact,*

$$0 \to J(L^{4v+1}(p^n)/L^{4v}(p^n)) \xrightarrow{c^!} J(L^{4v+1}(p^n)) \xrightarrow{i^!} J(L^{4v}(p^n)) \to 0.$$

Proof. Identical with that of Corollary 3.2.3. □

3.3. Order of $J(\overline{L}^k(p^n))$.

Definition 3.3.1. We define as in [**1**, Section 2] numbers $m(t)$ by: For p odd,

$$v_p(m(t)) = \begin{cases} 0 & \text{if } t \not\equiv 0 \pmod{p-1} \\ 1 + v_p(t) & \text{if } t \equiv 0 \pmod{p-1}. \end{cases}$$

For $p = 2$,

$$v_2(m(t)) = \begin{cases} 1 & \text{if } t \not\equiv 0 \pmod 2 \\ 2 + v_2(t) & \text{if } t \equiv 0 \pmod 2. \end{cases}$$

Definition 3.3.2. Let p be a prime and $v, n \in \mathbb{Z}^+$. Define

$$e(p, v, n) = \begin{cases} p^{\min\,(n, v_p(m(2v)))} & \text{if } p \text{ is odd} \\ 2^{\min\,(n+1, v_2(m(2v)))} & \text{if } p = 2. \end{cases}$$

Note that $e(p, v, n) = 1$ if $v \not\equiv 0 \pmod{(p-1)}$.

For $v \equiv 0 \pmod{(p-1)}$, $e(p, v, n) = p^{\epsilon_p + (\min\,(n, 1+v_p(2v)))}$ where

$$\epsilon_p = \begin{cases} 0 & \text{if } p \text{ is odd} \\ 1 & \text{if } p = 2. \end{cases}$$

Lemma 3.3.3. *Let p be a prime; $v, n \in \mathbb{Z}^+$, $n \geq 2$ if $p = 2$. Then $J(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)) = \mathbb{Z}_{e(p,v,n)}$.*

Proof. By Lemma 1.1.1,

$$\tilde{K}_{\mathbb{R}}(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)) = \begin{cases} \mathbb{Z}_{p^n} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{2^{n+1}} & \text{if } p = 2 \end{cases}$$

and is generated by w^v. By [2, Theorem 1.1] and the Adams conjecture, $J(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)) = \tilde{K}_{\mathbb{R}}(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n))/W$ where $W = \cap_f W_f$ where W_f is the subgroup generated by

$$\sum_{k \in \mathbb{Z}^+} a_k k^{f(k)}(\psi_{\mathbb{R}}^k - 1)w^v = \sum_{k \in \mathbb{Z}^+} a_k k^{f(k)}(k^{2v} - 1)w^v.$$

Let K_p be the principal ideal in $\mathbb{Z}$ generated by p^n if p is odd and by 2^{n+1} if $p = 2$. Let $\phi_p : \mathbb{Z} \to \mathbb{Z}/K_p = \tilde{K}_{\mathbb{R}}(B_{4v}(\mathbb{Z}_{p^n})/B_{4v-4}(\mathbb{Z}_{p^n}))$ be the surjection. Define $W'_f = \phi_p^{-1}(W_f)$ and $W' = \cap_f W'_f = \phi_p^{-1}(W)$. Let $h(f, 2v)$ be the highest common divisor of the integers $k^{f(k)}(k^{2v} - 1)$. Then W'_f is the principal ideal generated by $h(f, 2v)$ and by [1, Theorem 2.7], W_f is the principal ideal generated by $m(2v)$.

$$\begin{aligned} J(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)) &= (\mathbb{Z}/K_p)/W = (\mathbb{Z}/K_p)/(W'/W' \cap K_p) \\ &= (\mathbb{Z}/K_p)/((W' + K_p)/K_p) = \mathbb{Z}/(W' + K_p) \end{aligned}$$

and $W' + K_p$ is the principal ideal generated by $e(p, v, n)$. $\square$

Proposition 3.3.4. *Let p be a prime and $v, n \in \mathbb{Z}^+$ and $n \geq 2$ if $p = 2$. Then*

$$\left| J\left(\overline{L}^{2v}(p^n)\right) \right| = \begin{cases} \prod_{v'=1}^{v} e(p, v', n) & \text{if } p \text{ is odd} \\ 2\prod_{v'=1}^{v} e(2, v', n) & \text{if } p = 2. \end{cases}$$

Proof. It follows by induction from Corollary 3.2.3. □

Definition 3.3.5. Let p be a prime and $k, n \in \mathbb{Z}$. Define $G(p,k,n)$ and $G_0(p,k,n)$ to be subgroups of $J(L^k(p^n))$ and $J(L_0^k(p^n))$ generated by the powers of w respectively.

Lemma 3.3.6. *For p odd,*

$$J(L^k(p^n)) = \begin{cases} \mathbb{Z}_2 \oplus J(L_0^k(p^n)) & \text{if } k \equiv 0 \pmod 4 \\ J(L_0^k(p^n)) & \text{otherwise.} \end{cases}$$

Proof. This is [**13**, Proposition 1.3]. □

Corollary 3.3.7. *For p odd, $G(p,k,n) = G_0(p,k,n)$.*

Corollary 3.3.8. *$G(p,k,n) = J(L^k(p^n))$ for p odd and $k \not\equiv 0 (\mathrm{mod}\, 4)$ and is a subgroup of index 2 if either p is odd and $k \equiv 0 (\mathrm{mod}\, 4)$ or $p = 2$.*

We now state the following Corollary to Proposition 3.3.4.

Corollary 3.3.9. $|G(p, 2v, n)| = \prod_{v'=1}^{v} e(p, v', n)$.

Proposition 3.3.10. *Let p be a prime; $v, n \in \mathbb{Z}^+$. Then*

$$\left|J\left(L^{4v+1}(p^n)\right)\right| = \begin{cases} \left|J(L^{4v}(p^n))\right| & \text{if } p \text{ is odd} \\ 2\left|J(L^{4v}(2^n))\right| & \text{if } p = 2. \end{cases}$$

Proof. It follows from Lemma 1.1.2 and the fact that

$$J\left(L^{4v+1}(\mathbb{Z}_2)/L^{4v}(\mathbb{Z}_2)\right) = J\left(P^{8v+2}/P^{8v}\right) = \mathbb{Z}_2 \quad \text{that}$$

$$J\left(L^{4v+1}(p^n)/L^{4v}(p^n)\right) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \mathbb{Z}_2 & \text{if } p = 2. \end{cases}$$

The result follows from this and Corollary 3.2.5. □

Proposition 3.3.11. *$J(L^{4v+3}(p^n)) = J(L^{4v+2}(p^n))$.*

Proof. It follows from [**1**, Theorem 3.12] that there is an exact sequence,

$$J(L^{4v+3}(p^n)/L^{4v+2}(p^n)) \xrightarrow{c^!} J(L^{4v+3}(p^n)) \xrightarrow{i^!} J(L^{4v+2}(p^n)) \to 0.$$

By Lemma 1.1.3, $\tilde{K}_{\mathbb{R}}(L^{4v+3}(p^n)/L^{4v+2}(p^n)) = 0$ and hence

$$J\left(L^{4v+3}(p^n)/L^{4v+2}(p^n)\right) = 0.$$

□

3.4. Approximation to complex projective spaces by lens spaces. Let $i : L^k(p^n) \to P_k(\mathbb{C})$ be the inclusion. Let $J_p(P_k(\mathbb{C}))$ denote the p-summand of $J(P_k(\mathbb{C}))$.

Observation 3.4.1. *$i^!$ maps $J_p(P_k(\mathbb{C}))$ onto $G(p,k,n)$.*

Theorem 3.4.2. *$i^!$ maps $J_p(P_k(\mathbb{C}))$ isomorphically onto $G(p,k,n)$ iff $n \geq v_p(N_k)$.*

Proof. (i) Let $k = 2v$ be even. By Corollary 3.3.9,

$$|G(p,k,n)| = \prod_{v'=1}^{v} e(p,v',n).$$

It follows from the proof of [4, Lemma 6.1] that there is a short exact sequence,

$$0 \to J\left(P_{2v}(\mathbb{C})/P_{2v-2}(\mathbb{C})\right) \to J\left(P_{2v}(\mathbb{C})\right) \to J\left(P_{2v-2}(\mathbb{C})\right) \to 0,$$

and from [4, Lemma 5.3] that $J(P_{2v}(\mathbb{C})/P_{2v-2}(\mathbb{C})) = \mathbb{Z}_{m(2v)}$ and hence by induction that $|J_p(P_{2v}(\mathbb{C}))| = \prod_{v'=1}^{v} m_p(2v')$ where $m_p(2v')$ is the p-component of $m(2v')$. Thus $|G(p,k,n)| = |J_p(P_k(\mathbb{C}))|$ iff $e(p,v',n) = m(2v')$ for all $1 \leq v' \leq v$ iff $e(p,v',n) = m_p(2v')$ for all $1 \leq v' \leq v$ and $2v' \equiv 0 \pmod{p-1}$ iff $n \geq 1+v_p(2v')$ for all $1 \leq v' \leq v$ and $2v' \equiv 0 \pmod{p-1}$, and putting $2v' = r(p-1)$, iff $n \geq 1 + v_p(r)$ for all $1 \leq r \leq [\frac{2v}{p-1}]$, i.e., iff $n \geq v_p(N_k)$.

(ii) $k = 4v+1$.

$$|J_p(P_k(\mathbb{C}))| = \begin{cases} |J_p(P_{4v}(\mathbb{C}))| & \text{if } p \text{ is odd} \\ 2\,|J_2(P_{4v}(\mathbb{C}))| & \text{if } p = 2 \end{cases}$$

by [4, Lemma 6.2] and

$$|G(p,k,n)| = \begin{cases} |G(p,4v,n)| & \text{if } p \text{ is odd} \\ 2|G(2,4v,n)| & \text{if } p = 2 \end{cases}$$

by Proposition 3.3.10. The result follows from (i) above and the fact that $N_k = N_{4v}$.

(iii) $k = 4v+3$.

$$|J_p(P_k(\mathbb{C}))| = |J_p(P_{4v+2}(\mathbb{C}))|$$

by [4, Lemma 6.2] and $|G(p,k,n)| = |G(p,4v+2,n)|$ by Proposition 3.3.11 and the result follows from (i) above and the fact that $N_k = N_{4v+2}$. □

Corollary 3.4.3. *Let $i : L^k(m) \to P_k(\mathbb{C})$ be the inclusion. Then $i^!$ maps $J(P_k(\mathbb{C}))$ isomorphically onto the subgroup of $J(L^k(m))$ generated by w iff N_k/m.*

Stable co-degrees of vector-bundles enables us as in [8, Section 4] or [9, Definition 1.1.4] to define a degree-function q on $J(X)$; e.g., a function $q : J(X) \to \mathbb{Z}^+$ such that $u = 0$ in $J(X)$ iff $q(u) = 1$. The degree-function imposes on $J(X)$ an additional structure other than the usual algebraic structure. We now conjecture a stronger version of Theorem 3.4.5.

Conjecture 3.4.4. *Let p be a prime. $n \in \mathbb{Z}^+$ and $n \geq 2$ if $p = 2$. Then the map $i^! : J_p(P_k(\mathbb{C})) \to J(L^k(p^n))$ is a q-isometry iff $n \geq v_p(N_k)$.*

3.5. The transfer map on the J-groups.

Let $\tau : \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^n)) \to \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^n))$ be the transfer-map defined on the $\tilde{K}_{\mathbb{R}}$-groups.

Lemma 3.5.1. *τ passes to the quotient and defines $\tau : J(\overline{L}^k(p^n)) \to J(\overline{L}^k(p^{n+1}))$.*

Proof. Let q be a prime which is a generator of both G_{p^n} and $G_{p^{n+1}}$ if p is odd and of the summands $\mathbb{Z}_{2^{n-2}}$ and $\mathbb{Z}_{2^{n-1}}$ if $p = 2$. By Lemma 3.1.1, J-trivial bundles on $\overline{L}^k(p^n)$ are of the form $(\psi^q_{\mathbb{R}} - 1)x$, $x \in \tilde{K}(\overline{L}^k(p^n))$. By (ii) of Proposition 2.1.1, $\tau \circ (\psi^q_{\mathbb{R}} - 1)x = (\psi^q_{\mathbb{R}} - 1) \circ \tau(x)$ is J-trivial on $\overline{L}^k(p^{n+1})$. □

Corollary 3.5.2. *The transfer map $\tau : \tilde{K}_{\mathbb{R}}(L^k(p^n)) \to \tilde{K}_{\mathbb{R}}(L^k(p^{n+1}))$ passes to the quotient and defines $\tau : J(L^k(p^n)) \to J(L^k(p^{n+1}))$.*

Proof. The case $p = 2$ is already proved in Lemma 3.5.1. For p odd and $k \not\equiv 0 \,(\mathrm{mod}\, 4)$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \tilde{K}_{\mathbb{R}}(L^k_0(p^n))$ and it also follows from Lemma 3.5.1. For p odd and $k \equiv 0 \,(\mathrm{mod}\, 4)$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \mathbb{Z}_2 \oplus \tilde{K}_{\mathbb{R}}(L^k_0(p^n))$ where the first summand is generated by u and $\tau(u) = u$. By Lemma 3.3.6, $J(L^k(p^n)) = \mathbb{Z}_2 \oplus J(L^k_0(p^n))$ where the first summand is generated by $J(u)$. Hence J-trivial elements on $\tilde{K}_{\mathbb{R}}(L^k(p^n))$ are of the form x where $i^! J(x) = 0$ in $J(L^k_0(p^n))$. Hence $i^![\tau(J(x))] = \tau(i^! J(x)) = 0$ by Lemma 3.5.1. Since $i^!$ is an isomorphism on the 2nd-summand, $\tau(J(x)) = 0$ and hence $J(\tau(x)) = 0$. □

Proposition 3.5.3. *Let $i : L^k(p^n) \to L^k(p^{n+1})$ be the $\mathbb{Z}_p$-fibration and $\tau : J(L^k(p^n)) \to J(L^k(p^{n+1}))$ be the transfer-map. Then $\tau(i^!(x)) = px$ $\forall x \in G(p, k, n+1)$.*

Proof. By Corollary 3.3.7, $G(p, k, n) = G_0(p, k, n)$ for p odd and hence we shall assume without loss of generality that $\tau : \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^n)) \to \tilde{K}_{\mathbb{R}}(\overline{L}^k(p^{n+1}))$. We let

$$\overline{G}(p, k, n) = \begin{cases} G_0(p, k, n), & p \text{ odd} \\ G(p, k, n), & p \text{ even}. \end{cases}$$

By (i) and (v) of Proposition 2.1.1,

$$\begin{aligned}\tau(\psi_{\mathbb{R}}^{p^i}(w)) &= \sum_{s=0}^{p-1} \psi_{\mathbb{R}}^{p^i+sp^n}(w) \qquad (0 \le i \le n-1) \\ &= \sum_{s=0}^{p-1} \psi_{\mathbb{R}}^{p^i} \circ \psi_{\mathbb{R}}^{p^{1+sp^{n-i}}}(w).\end{aligned}$$

$J(\overline{L}^k(p^n))$ is a p-group and $(1+sp^{n-i})$ is prime to p and it follows from (ii) of [9, Proposition 2.3.3] that $\psi_{\mathbb{R}}^{1+sp^{n-i}}(w) = w$ and hence:

1. $\tau(\psi_{\mathbb{R}}^{p^i}(w)) = p\psi_{\mathbb{R}}^{p^i}(w)$ $(0 \le i \le n-1)$.

The group $i^!\overline{G}(p,k,n+1) = \overline{G}(p,k,n)$ is generated by $\{\psi_{\mathbb{R}}^m(w) : 0 \le m \le p^n - 1\}$. Let p^i $(1 \le i \le n-1)$ be the p-primary component of m. It follows from (ii) of [9, Proposition 2.3.3] that $\psi_{\mathbb{R}}^m(w) = \psi_{\mathbb{R}}^{p^i}(w)$ in $J(\overline{L}^k(p^n))$. Hence the group $\overline{G}(p,k,n)$ is generated by $\{\psi_{\mathbb{R}}^{p^i}(w) : 1 \le i \le n-1\}$. The result follows from this and Equation 1 above. □

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