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LOCAL REPRESENTATION THEORY AND MÖBIUS INVERSION

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Abstract: Various representation-theoretic parameters of a finite group are shown to satisfy formulas similar to formulas appearing in reformulations by Külshammer, Robinson, and Thévenaz of Alperin's Conjecture. The conjecture itself is reformulated again, now as a statement not mentioning characters or conjugacy classes.

The term “local representation theory” is a pun, because it invokes two senses of the word “local”: one from ring theory, and one from group theory. Indeed, local representation theory is an exploration of the interplay between these two senses by considering group representations over a commutative ring with suitable localization properties, and examining how representation-theoretic and other group-theoretic properties and parameters of a given finite group G are related to similar properties and parameters of local subgroups of G . In the familiar realm of p -local representation theory, the representations are over a local noetherian commutative ring with residue field of prime characteristic p , and a local group is deemed to be a group with a non-trivial normal p -subgroup. Recently, Robinson [17], [18], and then Külshammer–Robinson [15], [16] have shown that, in some way, local representation theory becomes easier when we instead take the local subgroups to be, say, those with a non-trivial normal solvable

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subgroup. As in [15], [16], [17], [18], however, our objective is not so much to replace p -local problems with easier local problems, but rather to approach p -local problems via a general local theory with easier specializations.

Discussion of the local ring-theoretic scenario and the rationale of this paper is deferred to Section 2. The local group-theoretic scenario is much as in Külshammer–Robinson [15]: let π be a set of rational primes, with $p \in \pi$, and let \mathcal{F} be a class of finite groups which contains all the solvable finite π -groups, and which is closed under isomorphism, subquotients, and extension. We write $O_{\mathcal{F}}(G)$ for the unique maximal normal subgroup of G belonging to \mathcal{F} . When $O_{\mathcal{F}}(G)$ is non-trivial, we say that G is \mathcal{F} -local.

Let f be a function taking values in some abelian group, and defined on the isomorphism classes of finite groups. We say that f is \mathcal{F} -centrally determined provided

$$f(G) - f(1) = \sum_{\underline{Q}} (-1)^{n(\underline{Q})} (f(C_G(Q)) - f(1))$$

for all finite groups G ; here, the index \underline{Q} runs over all the non-empty chains $(Q_1, \dots, Q_{n(\underline{Q})})$ of subgroups of G belonging to \mathcal{F} , and $Q := Q_{n(\underline{Q})}$. We say that f is \mathcal{F} -normally determined provided

$$f(G) - f(1) = \sum_{\underline{Q}} (-1)^{n(\underline{Q})} (f(N_G(Q)) - f(1))$$

for all G . More general definitions of \mathcal{F} -centrally and \mathcal{F} -normally determined functions will be given in Section 2, and we shall see other characterizations of such functions in Section 3. Our objective is to find functions f which are of concern in p -local representation theory, and which are \mathcal{F} -centrally determined or \mathcal{F} -normally determined, or at least satisfy a formula closely resembling some characterization of \mathcal{F} -centrally or \mathcal{F} -normally determined functions.

Given a rational integer r , we define a p -local integer

$$f_r^p(G) := \sum_{\chi} (|G|/\chi(1))^{2r}$$

where χ runs over the (absolutely) irreducible (ordinary) characters of G whose p -defect $\log_p(|G|/\chi(1))_p$ is zero. As cases of particular interest, note that $f_0^p(G)$ is the number of defect-zero p -blocks of G , and that $|G|^2 f_{-1}^p(G)$ is the sum of the dimensions of the defect-zero p -block algebras of G .

We shall reformulate the weaker version of Alperin’s Conjecture [1] as a statement not involving characters or conjugacy classes. Let τ^p be the function such that

$$|G|f_0^p(G) = \sum_{H \leq G} \tau^p(H).$$

Given $s \geq 0$, put $m := 2 + p^s(p - 1)$, and let $t_s^p(G)$ be the number of generating m -tuples (x_1, \dots, x_m) of p -elements of G such that

$$x_1 x_2 \dots x_m = x_m x_{m-1} \dots x_1.$$

Theorem 5.4 simplifies as:

Theorem 1: *Suppose that \mathcal{F} contains all the solvable finite groups, and that Alperin's conjecture holds for all the groups in \mathcal{F} . Then the following conditions are equivalent:*

- (a) *Alperin's Conjecture is correct.*
- (b) *(After Külshammer, Robinson) The function on finite groups given by $G \mapsto |G|f_0^p(G)$ is \mathcal{F} -normally determined.*
- (c) *(After Thévenaz) Whenever G is non-trivial and non- \mathcal{F} -local, $\tau^p(G) = 0$.*
- (d) *Whenever G is non-trivial and non- \mathcal{F} -local, then for all $u \geq 0$, there exist infinitely many $s \geq 0$ such that p^u divides $t_s^p(G)$.*

Theorem 2.5 and Corollary 4.5 immediately imply:

Proposition 2: *Let $K \trianglelefteq G$, and let r be a rational integer. Then we have a congruence of p -local integers*

$$|K|f_r^p(K) - 1 \equiv_{|G|_p} \sum_{\underline{Q}} (-1)^{n(\underline{Q})} (|C_K(Q)|f_r^p(C_K(Q)) - 1)$$

where the index \underline{Q} runs as above.

Let $\tilde{\chi}(\tilde{\mathcal{S}}_p(G))$ denote the reduced Euler characteristic of the poset of non-trivial p -subgroups of G . Corollary 4.3 specializes to:

Proposition 3: *The function $G \mapsto \tilde{\chi}(\tilde{\mathcal{S}}_p(G))$ is \mathcal{F} -centrally and \mathcal{F} -normally determined.*

1 : Rationale

If the reasoning in subsequent sections appears to follow many twists and turns, this is only because we sometimes need to detour around obstacles lying across the quite straight track. Most of the commentary on the results and arguments below has been placed together in this single section to make clear the ley of the track. First, let us describe the terrain.

We make frequent use of a few techniques in the topological combinatorics of G -posets as developed by Bouc [6], Thévenaz [21], Webb [24], and others. All the prerequisite background on G -posets may be found in Benson [5, Chapter 6]. Let $\mathcal{F}(G)$ denote the set of subgroups of G belonging to \mathcal{F} , let $\tilde{\mathcal{F}}(G)$ denote the G -poset consisting of the non-trivial elements of $\mathcal{F}(G)$, and let $\text{sd}(\tilde{\mathcal{F}}(G))$ denote the G -poset consisting of the non-empty chains of elements of $\tilde{\mathcal{F}}(G)$.

The p -local case is where $\pi = \{p\}$, and \mathcal{F} is the class \mathcal{S}_p of finite p -groups. Let us mention one striking indication that there are other cases which are in some way easier: Külshammer–Robinson [15, Theorem 2] may be interpreted as saying that an \mathcal{F} -local analogue of Alperin's Conjecture holds when, for instance, \mathcal{F} is the class of finite p -solvable groups. (The p -solvable case of Alperin's Conjecture was proved by T. Okuyama. I take this opportunity to rectify a historical paragraph in [2]. Overlooked

in [2], a theorem much stronger than the p -solvable case of the blockwise version of Alperin's Conjecture was proved in Ellers [9]. Also, contrary to a comment in [2], no omission occurs in the sketch proof in Robinson–Staszewski [19] of the p -solvable case of the blockwise version; the Külshammer–Puig improvement of Dade's theorem was cited in [19] precisely because that improvement is adequate to show that Dade's isomorphism is compatible with the Brauer correspondence.)

Given an element $\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(G))$ as in the preamble, then the normalizer $N_G(\underline{Q})$ (the stabilizer of \underline{Q} in G) is the intersection of all the normalizers $N_G(Q_i)$. The intersection of the centralizers $C_G(Q_i)$ is simply $C_G(\underline{Q})$. In p -local theory, reductions to normalizers of chains of p -subgroups (see Knörr–Robinson [13]) are of importance, and so too (often via the theory of Brauer pairs) are reductions to centralizers of p -subgroups. These traditions can be adapted to general local theory. The normalizer $N_G(\underline{Q})$ is always \mathcal{F} -local. The centralizer $C_G(\underline{Q})$ need not be \mathcal{F} -local. However, we can speak correctly of local reduction to centralizers of \mathcal{F} -subgroups; $C_G(\underline{Q})$ is, as a subgroup of G , local with respect to \mathcal{F} in the sense that $C_G(\underline{Q})$ is contained in the \mathcal{F} -local subgroup $N_G(\underline{Q})$.

Let f be a function whose domain is the set of subgroups of G , and whose codomain is some abelian group. After Bouc [6], we say that f is **\mathcal{F} -centrally determined** provided

$$f(H) - f(1) = \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(H))} (-1)^{n(\underline{Q})} (f(C_H(\underline{Q})) - f(1))$$

for all $H \leq G$. This condition will be reinterpreted in Remark 2.2, and then the connection with [6] will be clear. In a special case, Proposition 2.6 will again reinterpret the condition. After Külshammer–Robinson [15], [16], Robinson [17], [18], Thévenaz [22] we say that f is **\mathcal{F} -normally determined** provided

$$f(H) - f(1) = \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(H))} (-1)^{n(\underline{Q})} (f(N_H(\underline{Q})) - f(1))$$

for all $H \leq G$. This condition will be reinterpreted in Proposition 2.3, and then the connection with [22] will be clear. In a special case, Proposition 2.6 will again reinterpret the condition, and then the (already visible) connection with [15], [16], [17], [18] will be clear.

Let Θ be a commutative unital ring of characteristic zero such that no element of π has an inverse in Θ . Let Π be the set of prime ideals in Θ . For $\mathcal{P} \in \Pi$, we write $\Theta(\mathcal{P})$ for the localization of Θ at \mathcal{P} . All ΘG -modules are to be free over Θ , and finitely generated. Given a ΘG -module M , we define $M(\mathcal{P}) := \Theta(\mathcal{P}) \otimes_{\Theta} M$ as a $\Theta(\mathcal{P})G$ -module. Many of the fundamental principles of p -local theory fail for ΘG -modules. (For some purposes, advantage may be gained by insisting that Θ is a Dedekind domain and that π and Π are finite, but even then, the category of ΘG -modules need not possess the Krull–Schmidt property.) Nevertheless, standard proofs of Higman's Criterion are easily adapted to show that a given ΘG -module M is projective if and only if the G -algebra $\text{End}_{\Theta}(M)$ is projective. We then deduce the well-known fact:

Lemma 1.1: *Let M be a permutation ΘG -module. Then M is projective if and only if $M(\mathcal{P})$ is projective for all $\mathcal{P} \in \Pi$. If these conditions hold, and furthermore, π contains all the prime divisors of $|G|$, then M is free.*

A crucial difficulty in p -local theory is that, in attempts to reduce to local subgroups, there often arise intractable terms associated, in some way or another, with projective modules. Külshammer pointed out to me that if π contains all the prime divisors of $|G|$, then by Lemma 1.1, any projective permutation ΘG -module M is uniquely determined, up to isomorphism, by a single integer, namely, the Θ -rank of M . Given two distinct equations involving M , then the Θ -rank of M can be solved. Theorem 2.4, the engine of this paper, is based on this idea. It must be confessed, however, that Lemma 1.1 was recorded only because its conceptual clarity so facilitated our discussion. Actually, we shall prove Theorem 2.4 using the more technical version of the idea as it originally appeared in Robinson [17, Section 5].

A G -selection will be defined to be a certain kind of G -set. A Θ -weighted G -selection will be defined to be a G -selection equipped, in a certain way, with coefficients in Θ . In the context of Θ -weighted G -selections, Theorem 2.4 specialises to Theorem 3.2, whence derive all our subsequent results relating to \mathcal{F} -centrally determined functions. In the same context, ideas in Thévenaz [23], (see also Bouc's part of Thévenaz [22, 6.3]) generalize to Theorem 3.3, whence derive all our subsequent results relating to \mathcal{F} -normally determined functions. Theorems 3.2 and 3.3 are easy-to-follow recipes for producing local corollaries; the art of using them seems to lie in carefully selecting the main ingredient: the Θ -weighted G -selection. Towards the end of Section 3, we are ready to give some quick enumerative applications.

The sense of the word "local", in Sections 2 and 3, is always group-theoretic. A ring-theoretic, or more precisely, an arithmetic sense comes into play in Section 4. The idea — introduced in [3] — is to realise numerical representation-theoretic parameters of G as elements of the localizations $\Theta(\mathcal{P})$ with $\mathcal{P} \in \Pi$. (We would like to take advantage of allowing \mathcal{P} to vary, as in the above-mentioned idea of Robinson used in the proof of Theorem 2.4. See the paragraph at the end of this section.)

Corollary 4.3 is a generalization of Proposition 3. If we could find a G -selection whose order (as a set) is always $\tilde{\chi}(\tilde{\mathcal{S}}_p(G))$, or always $-\tilde{\chi}(\tilde{\mathcal{S}}_p(G))$, then we could hope that Theorems 3.2 and 3.3 would give Corollary 4.3 immediately. There is no possibility of finding such a G -selection (in general), because $\tilde{\chi}(\tilde{\mathcal{S}}_p(G))$ is sometimes strictly positive and sometimes strictly negative. Instead, we find an infinite sequence of G -selections whose orders p -adically converge to $-\tilde{\chi}(\tilde{\mathcal{S}}_p(G))$.

Corollary 4.5 is a variant of Proposition 2. If we could find a G -selection whose order is always $|G|f_p^r(G)$, then we could hope that Theorem 3.2 would give Corollary 4.5 immediately. There is no possibility of finding such a G -selection (in general), because if r is negative, then $f_p^r(G)$ need not be a rational integer. Instead, we find an infinite sequence of G -selections whose orders p -adically converge to $|G|f_p^r(G)$.

The reformulation of (the weaker version of) Alperin's Conjecture in Section 5 was inspired by Külshammer–Robinson [15] and Thévenaz [22]. We shall quickly see

that the version of Alperin's Conjecture considered by Külshammer and Robinson is equivalent to the (weaker) original version. However, we make crucial use of the p' -central extensions appearing in their version.

It is possible that, by using the monomial Burnside ring as in Külshammer–Robinson [16], the results in Section 5 could be proved without having to introduce a weighting on our G -selections. Weighting is certainly used crucially, though, in one further application of the methods discussed above. The p -adic approximation in [4] for the dimension of a trivial-source module resides entirely within the classical case $\mathcal{F} = \mathcal{S}_p$ (and is presented in a self-contained way), but should be viewed as part of the programme pursued here. There, Θ is replaced with a discrete valuation ring \mathcal{O} of characteristic zero, with $p \in J(\mathcal{O})$ (we can take \mathcal{O} to be the \mathcal{P} -adic completion of $\Theta(\mathcal{P})$, where $p \in \mathcal{P} \in \Pi$). In application of \mathcal{O} -weighted G -sets and a $J(\mathcal{O})$ -adic limiting argument, [4] shows that a certain combinatorial observation is compatible with “blockwise” decomposition. Admittedly, despite some further partial results of a “blockwise” nature (confined to the case $\mathcal{F} = \mathcal{S}_p$, and based on Lemma 4.4), it is not at all clear how “blockwise” generalization of the material in this paper could be achieved. However, enriching combinatorial entities by introducing coefficients seems to be a reasonable approach to this problem.

In studying \mathcal{F} -local theory as well as p -local theory, part of the classical role of p is now performed by the set of primes π . We introduced the ring Θ to try to make the local ring-theoretic aspects of local representation theory match the local group-theoretic aspects. The material in this paper could be presented without mentioning a ring such as Θ . Nevertheless, Lemma 1.1 and the proof of Theorem 2.4 strongly indicate that the congruence properties appearing in several of the results below have a deeper algebraic analogue as a projectivity property of certain virtual permutation modules over Θ . We suggest that Θ is worth regarding as something more than mere scaffolding. For distinct prime ideals $\mathcal{P}, \mathcal{Q} \in \Pi$, does \mathcal{F} -local theory enable us to find relationships between blocks of $\theta(\mathcal{P})G$ and blocks of $\theta(\mathcal{Q})G$?

2 : Centrally and normally determined functions

An element of the Burnside ring $B(G)$ of G is called a **virtual G -set**. Given a G -set Δ , let $[\Delta]$ denote the virtual G -set associated with Δ . Let $\Theta\Delta$ denote the permutation ΘG -module associated with Δ . To begin the definition of the **Green ring** $A(\Theta G)$ of ΘG , we specify that $A(\Theta G)$ is generated by the isomorphism classes of ΘG -modules. Given a ΘG -module X , let $[X]$ denote the element of $A(\Theta G)$ associated with X . The relations in $A(\Theta G)$ express precisely the condition that $[X] + [X'] = [X \oplus X']$. Tensor product provides $A(\Theta G)$ with a multiplication, and thus $A(\Theta G)$ becomes a ring. Our now completed definition of the Green ring coincides with the usual one when the Krull–Schmidt property holds for ΘG -modules. We define a ring homomorphism $B(G) \rightarrow A(\Theta G)$, written $\beta \mapsto \Theta\beta$, such that $[\Delta] \mapsto [\Theta\Delta]$. A virtual G -set β is said to

be Θ -projective provided $\Theta\beta$ corresponds to a linear combination of projective ΘG -modules. It is easy to check that the Θ -projective virtual G -sets comprise an ideal of $B(G)$. Given $H \leq G$, we write $\text{Ind}_H^G : B(H) \rightarrow B(G)$ and $\text{Res}_H^G : B(G) \rightarrow B(H)$ for the induction and restriction maps. Let $\Delta(H)$ denote the $N_G(H)$ -set consisting of the H -fixed elements of Δ . Let $k_H(\Delta)$ denote the number of H -orbits in Δ . The H -relative Brauer map $B(G) \rightarrow B(N_G(H))$, written $\beta \mapsto \beta(H)$, is defined to be the linear map such that $[\Delta](H) = [\Delta(H)]$. We write $k_H[\Delta] := k_H(\Delta)$, and extend k_H linearly to $B(G)$. Thus $k_H(\beta)$ is the multiplicity of the trivial character in the restriction to H of the virtual character of G afforded by $\Theta\beta$. In particular, k_H factorizes through $A(\Theta G)$. Frobenius reciprocity gives $k_H(\alpha) = k_G(\text{Ind}_H^G(\alpha))$ for $\alpha \in B(H)$. We write $|\beta| := k_1(\beta)$. Thus $|\Delta| = |\Delta|$.

Let Γ be a finite G -poset. We write $\text{sd}(\Gamma)$ for the G -poset consisting of the non-empty chains of elements of Γ , and write Γ/G for the poset consisting of the G -orbits of chains in $\text{sd}(\Gamma)$. (The subchain relation is the partial ordering in $\text{sd}(\Gamma)$, and induces the partial ordering in Γ/G .) Let $|\Gamma|$ denote the G -polyhedron associated with the G -simplicial complex whose vertices and simplexes are the elements of Γ and $\text{sd}(\Gamma)$, respectively. By considering an appropriate barycentric subdivision, we see that $|\Gamma|$ is G -homeomorphic to $|\text{sd}(\Gamma)|$, and that the G -orbit space $|\Gamma|/G$ is homeomorphic to $|\Gamma/G|$. We define the Euler characteristic of Γ to be $\chi(\Gamma) := \chi(|\Gamma|)$, and define the reduced Euler characteristic $\tilde{\chi}(\Gamma)$ similarly. Given $H \leq G$, let $\Gamma(H)$ denote the $N_G(H)$ -poset consisting of the H -fixed elements of Γ .

Whenever an element of $\text{sd}(\Gamma)$ is written in the form $\underline{\gamma}$, it is to be understood that $\underline{\gamma} = (\gamma_0 < \dots < \gamma_{n(\underline{\gamma})})$. For each natural number n , let $\text{sd}_n(\Gamma)$ denote the G -set consisting of the $\underline{\gamma} \in \text{sd}(\Gamma)$ with $n(\underline{\gamma}) = n$. The virtual G -set

$$\lambda_G(\Gamma) := \sum_{n \geq 0} (-1)^n [\text{sd}_n(\Gamma)] = \sum_{\underline{\gamma} \in \text{sd}(\Gamma)} (-1)^{n(\underline{\gamma})} \text{Ind}_{N_G(\underline{\gamma})}^G(1)$$

is called the Lefschetz invariant of Γ . (The notation indicates that the index $\underline{\gamma}$ in the sum runs over representatives of the G -orbits in $\text{sd}(\Gamma)$.) It is well-known that $\lambda_G(\Gamma)$ is a G -homotopy invariant of $|\Gamma|$. The virtual G -set $\tilde{\lambda}_G(\Gamma) := \lambda_G(\Gamma) - 1$ is called the reduced Lefschetz invariant of Γ . Thus

$$\chi(\Gamma) = \sum_{\underline{\gamma} \in \text{sd}(\Gamma)} (-1)^{n(\underline{\gamma})} = |\lambda_G(\Gamma)|,$$

in other words, $\tilde{\chi}(\Gamma) = |\tilde{\lambda}_G(\Gamma)|$. The following more general result, explained to me by Thévenaz, is implicit in Thévenaz [23, 2.2], and is not difficult to deduce from Curtis-Reiner [8, 66.15]. For convenience, we give an elementary proof.

Lemma 2.1: (Thévenaz) *Given a finite G -poset Γ , then*

$$\chi(|\Gamma|/G) = \chi(\Gamma/G) = \sum_{\underline{\gamma} \in \text{sd}(\Gamma)} (-1)^{n(\underline{\gamma})} = k_G(\lambda_G(\Gamma)).$$

Proof: Only the middle equality requires explanation. For $n \geq 1$, the poset of proper subsets of a set of order n has Euler characteristic χ_n satisfying

$$\chi_n = \sum_{0 < m < n} \binom{n}{m} (1 - \chi_m)$$

hence $\chi_n = 1 + (-1)^n$. Given an element $\underline{\gamma} \in \text{sd}_n(\Gamma)$, the elements of Γ/G whose maximal term is G -conjugate to $\underline{\gamma}$ contribute $1 - \chi_{n-1} = (-1)^n$ to $\chi(\Gamma/G)$. \square

Recall that the (group-theoretic) **Möbius function** μ is defined such that $\mu(G, G) = 1$, and if $H < G$, then $\mu(H, G)$ is the reduced Euler characteristic of the poset of subgroups L such that $H < L < G$. We write $\mu(G) := \mu(1, G)$. Obviously:

Remark 2.2: *Let f be a function defined on the subgroups of G , and taking values in some abelian group. Then*

$$\sum_{Q \in \text{esd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} f(C_G(Q)) + \sum_{Q \in \tilde{\mathcal{F}}(G)} \mu(Q) f(C_G(Q)) = 0.$$

In particular, f is \mathcal{F} -centrally determined if and only if, given any $H \leq G$, then

$$f(H) - f(1) + \sum_{Q \in \tilde{\mathcal{F}}(G)} \mu(Q) (f(C_G(Q)) - f(1)) = 0.$$

\square

Concerning the case where \mathcal{F} is the class of solvable groups, we mention that Kratzer–Thévenaz [14, 2.6] give an explicit formula for the Möbius function $\mu(Q)$ of a finite solvable group Q . As for the case $\mathcal{F} = \mathcal{S}_p$, recall that, for a group P of order p^r , we have $\mu(P) = (-1)^r p^{r(r-1)/2}$ when P is elementary abelian, otherwise $\mu(P) = 0$.

Given f as in Remark 2.2, we write

$$\phi f(H) := \sum_{L \leq H} \mu(L, H) f(L)$$

for $H \leq G$. Möbius inversion tells us that ϕf is also characterized by

$$f(H) = \sum_{L \leq H} \phi f(L).$$

(If G is cyclic, and $f(H) = |H|$, then $\phi f(H)$ is the arithmetic Euler function of $|H|$.) The following result is related to Thévenaz [22, 3.1] intuitively, but not related, in any obvious way, deductively; if the function $H \mapsto f(H)/|H|$ is p -locally determined, then f is \mathcal{S}_p -normally determined, but the converse is false.

Proposition 2.3: *Given a function f defined on the subgroups of G , and taking values in some abelian group, then f is \mathcal{F} -normally determined if and only if $\phi f(H) = 0$ for all non-trivial non- \mathcal{F} -local subgroups H of G .*

Proof: We have

$$-f(G) + \sum_{Q \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} f(N_G(Q)) = \sum_{L \leq G} \phi f(L) \tilde{\chi}(\tilde{\mathcal{F}}(G)(L))$$

because $Q \in \text{sd}(\tilde{\mathcal{F}}(G)(L))$ if and only if $L \leq N_G(Q)$. If $O_{\mathcal{F}}(L) \neq 1$, then $\tilde{\mathcal{F}}(G)(L)$ is conically contractible via the composite map $Q \mapsto QO_{\mathcal{F}}(L) \mapsto O_{\mathcal{F}}(L)$, and in particular, $\tilde{\chi}(\tilde{\mathcal{F}}(G)(L)) = 0$. Since $\phi f(1) = f(1)$, we have

$$f(G) - f(1) + \sum_{1 < L \leq G: O_{\mathcal{F}}(L)=1} \phi f(L) \tilde{\chi}(\tilde{\mathcal{F}}(G)(L)) = \sum_{Q \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} (f(N_G(Q)) - f(1)).$$

If $O_{\mathcal{F}}(G) = 1$, then $\tilde{\chi}(\tilde{\mathcal{F}}(G)(G)) = -1$. Induction on $|G|$ finishes the demonstration. \square

Let β be a virtual G -set. Given $Q \in \text{sd}(\tilde{\mathcal{F}}(G))$, we define

$$\beta(Q) := \text{Res}_{N_G(Q)}^{N_G(Q)}(\beta(Q)).$$

The map $B(G) \rightarrow B(N_G(Q))$ such that $\beta \mapsto \beta(Q)$ is called the Q -relative Brauer map. Following Bouc [6], we define the Lefschetz map $\lambda_G^{\mathcal{F}} : B(G) \rightarrow B(G)$ such that

$$\lambda_G^{\mathcal{F}}(\beta) := \sum_{Q \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} \text{Ind}_{N_G(Q)}^G(\beta(Q)).$$

The reduced Lefschetz map $\tilde{\lambda}_G^{\mathcal{F}}$ is defined by $\tilde{\lambda}_G^{\mathcal{F}}(\beta) := \lambda_G^{\mathcal{F}}(\beta) - \beta$. Thus $\lambda_G^{\mathcal{F}}(1) = \lambda_G(\tilde{\mathcal{F}}(G))$ and $\tilde{\lambda}_G^{\mathcal{F}}(1) = \tilde{\lambda}_G(\tilde{\mathcal{F}}(G))$.

To illuminate the connection between Remark 2.2 and the Lefschetz map, let us explain how Bouc [6] rewrites the Lefschetz map using Möbius invariants. Let $\tilde{U}(G)$ denote the G -poset consisting of the proper subgroups of G . Given $K \trianglelefteq G$, the Möbius invariant of K in G is defined to be the virtual G -set

$$\mu_G(K) := \tilde{\lambda}_G(\tilde{U}(K)).$$

Note that $|\mu_G(K)| = \mu(K)$. It is easy to check that

$$\lambda_G^{\mathcal{F}}(\beta) + \sum_{Q \in \tilde{\mathcal{F}}(G)} \text{Ind}_{N_G(Q)}^G(\mu_{N_G(Q)}(Q) \cdot \beta(Q)) = 0.$$

From this, or alternatively, from Remark 2.2, we obtain

$$|\lambda_G^{\mathcal{F}}(\beta)| + \sum_{Q \in \tilde{\mathcal{F}}(G)} \mu(Q) |\beta(Q)| = 0.$$

Theorem 2.4: *Let $H \leq G$, and $\beta \in B(G)$. Then*

$$k_H(\tilde{\lambda}_G^{\mathcal{F}}(\beta)) = -k_H(\beta) + \sum_{Q \in_H \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} k_H(\beta(Q))$$

If, furthermore, H is a π -subgroup of G , then

$$k_H(\tilde{\lambda}_G^{\mathcal{F}}(\beta))|H| = -|\beta| - \sum_{Q \in \tilde{\mathcal{F}}(G)} \mu(Q)|\beta(Q)|$$

which is divisible by $|G|_\pi$.

Proof: A routine Mackey decomposition argument shows that

$$\text{Res}_H^G(\lambda_G^{\mathcal{F}}(\beta)) = \sum_{Q \in_H \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} \text{Ind}_{N_H(Q)}^H \text{Res}_{N_H(Q)}^{N_G(Q)}(\beta(Q)).$$

The first asserted equality follows. Assume now that H is a π -subgroup. Let χ be the character of the ΘG -module $\Theta \tilde{\lambda}_G^{\mathcal{F}}(\beta)$. It suffices to show that $\text{Res}_H^G(\chi)$ is a multiple of the regular character. In fact, it is enough to show that χ vanishes on all the non-trivial π -elements of G . We shall adapt arguments from Knörr–Robinson [13] and Robinson [17]. Let $P \in \tilde{\mathcal{S}}_p(G)$. Let $\mathcal{P} \in \Pi$ with \mathcal{P} dividing (p) , and let \mathcal{O} be the \mathcal{P} -adic completion of $\Theta(\mathcal{P})$. The P -relative Brauer correspondent of the virtual $\mathcal{O}G$ -module $\mathcal{O} \tilde{\lambda}_G^{\mathcal{F}}(\beta)$ is the virtual $\mathcal{O}N_G(P)$ -module $\mathcal{O}(\tilde{\lambda}_G^{\mathcal{F}}(\beta)(P))$; see Broué [7]. We have

$$\begin{aligned} \lambda_G^{\mathcal{F}}(\beta)(P) &= (\text{Res}_{N(P)}^G(\lambda_G^{\mathcal{F}}(\beta)))(P) = \\ &= \sum_{Q \in_{N(P)} \text{sd}(\tilde{\mathcal{F}}(G)(P))} (-1)^{n(Q)} \text{Ind}_{N(P,Q)}^{N(P)} \text{Res}_{N(P,Q)}^{N(Q)}(\beta(Q)(P)) \end{aligned}$$

because P cannot fix any element of a G -set induced from a subgroup not containing P . The proof of [13, 4.1] shows that, excepting the singleton chain (P) , the $N_G(P)$ -orbits in $\text{sd}(\tilde{\mathcal{F}}(G)(P))$ occur in pairs, each orbit and its partner making opposite contributions to $\tilde{\lambda}_G^{\mathcal{F}}(\beta)(P)$. The chain (P) contributes $\beta(P)$, so $\tilde{\lambda}_G^{\mathcal{F}}(\beta)(P) = 0$. For trivial-source modules with vertex P , the P -relative Brauer correspondence and the Green correspondence coincide. Therefore $\mathcal{O} \tilde{\lambda}_G^{\mathcal{F}}(\beta)$ has no indecomposable direct factor with vertex P . (Alternatively, as in [13, 4.2], we could apply the Burry–Carlson–Puig Theorem.) Since P is arbitrary, $\tilde{\lambda}_G^{\mathcal{F}}(\beta)$ is \mathcal{O} -projective, and χ must vanish on the p -singular elements of G . But p can be any element of π , so χ vanishes on all the non-trivial π -elements. \square

In view of Lemma 1.1, we ask: for any π -subgroup H of G , must the virtual H -set $\text{Res}_H^G(\tilde{\lambda}_G^{\mathcal{F}}(\beta))$ be Θ -projective? An affirmative answer would seem to give a more conceptual rationale for the congruence condition in Theorem 2.4. To appreciate the main point of Theorem 2.4, see Theorem 3.2 and the comment following the proof of Theorem 3.2. A secondary point of Theorem 2.4 is that, when H is a π -subgroup, $k_H(\tilde{\lambda}_G^{\mathcal{F}}(\beta))|H|$ is independent of H .

The following result was obtained by Thévenaz (unpublished) using methods from [21].

Theorem 2.5: (Thévenaz) *Given a π -subgroup H of G , then*

$$\tilde{\chi}(\tilde{\mathcal{F}}(G)/H) = \tilde{\chi}(\tilde{\mathcal{F}}(G))/|H|.$$

Proof: This follows from Lemma 2.1 and Theorem 2.4. Indeed

$$|H|(-1 + \sum_{\underline{Q} \in_H \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})}) = -1 + \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})}.$$

□

Proposition 2.6: *Let f be a function defined on the subgroups of G , invariant on each conjugacy class of subgroups, and taking values in some abelian group. Suppose that π contains all the prime divisors of $|G|$. Then:*

(1) *The function $H \mapsto |H|f(H)$ is \mathcal{F} -centrally determined if and only if each*

$$f(H) - f(1) = \sum_{\underline{Q} \in_H \text{sd}(\tilde{\mathcal{F}}(H))} (-1)^{n(\underline{Q})} (f(C_H(\underline{Q}))|C_H(\underline{Q})|/|N_H(\underline{Q})| - f(1)).$$

(2) *The function $H \mapsto |H|f(H)$ is \mathcal{F} -normally determined if and only if each*

$$f(H) - f(1) = \sum_{\underline{Q} \in_H \text{sd}(\tilde{\mathcal{F}}(H))} (-1)^{n(\underline{Q})} (f(N_H(\underline{Q})) - f(1)).$$

Proof: This is immediate from Theorem 2.5. □

Given a function f as in Proposition 2.6, we write f for the linear function on $B(G)$ such that $f(\text{Ind}_H^G(1)) = f(H)$ for all $H \leq G$. The identity in Proposition 2.6(2) may be rewritten as $f(\tilde{\lambda}_H(\tilde{\mathcal{F}}(H))) = f(1)\tilde{\chi}(\tilde{\mathcal{F}}(H)/H)$.

3 : Weighted selections

Let $\Theta B(G)$ denote the Burnside ring of G with coefficients in Θ . We define a Θ -weighted G -set to be a finite set Δ equipped with a function $\delta : \Delta \rightarrow \Theta$ called the **density function** of Δ . We define the **weight** of Δ to be the element $\sum_{d \in \Delta} \delta(d)$ of Θ . If G acts on Δ , and the action commutes with δ , we call Δ a Θ -weighted G -set, and define

$$[\Delta] := \sum_{d \in_G \Delta} \delta(d) \text{Ind}_{N_G(d)}^G(1)$$

as an element of $\Theta B(G)$. (Note that $\text{Ind}_{N_G(d)}^G(1)$ is the virtual G -set associated with the G -orbit of d .) The weight of any Θ -weighted G -set Δ is $||[\Delta]||$. Any G -set may be regarded as a Θ -weighted G -set whose density function has constant value unity.

Let m be a positive integer. We let G act by termwise conjugation on the m -fold direct product G^m . Whenever an element of G^m is written in the form \underline{x} , it is to be understood that $\underline{x} = (x_1, \dots, x_m)$. Let $\langle \underline{x} \rangle$ denote the subgroup of G generated by $\{x_1, \dots, x_m\}$. We define a G -selection of degree m to be a G -subset of G^m . We define a Θ -weighted G -selection to be a Θ -weighted G -set whose underlying G -set is a G -selection. Let $\Omega(G)$ be a Θ -weighted G -selection of degree m with density function δ . Given $H \leq G$, we write $\Omega(H)$ for the Θ -weighted H -selection of degree m such that $\Omega(H) = H^m \cap \Omega(G)$ and the density function of $\Omega(H)$ is a restriction of δ . We write $\Omega(H)^* := \{\underline{x} \in \Omega(H) : \langle \underline{x} \rangle = H\}$ as a Θ -weighted H -set whose density function, again, is a restriction of δ . The **weight function** ω and the **Euler function** ϕ associated with $\Omega(G)$ are defined by $\omega(H) := |\Omega(H)|$ and $\phi(H) := |\Omega(H)^*|$. Whenever the expression $\Omega(G)$ is used to denote a Θ -weighted G -selection, it is to be understood that δ and ω and ϕ are as here. More generally, whenever subscripts (or superscripts) are used to indicate a particular Θ -weighted G -selection $\Omega_{i,j,\dots}(G)$, it is to be understood that $\delta_{i,j,\dots}$ and $\omega_{i,j,\dots}$ and $\phi_{i,j,\dots}$ denote the associated density function and weight function and Euler function. Any G -selection may be regarded as a Θ -weighted G -selection whose density function has constant value unity. Thus, if $\Omega(G)$ is a G -selection, then $\phi(G)$ is the number of generating m -tuples contained in $\Omega(G)$.

Remark 3.1: Given $H \leq G$, and a Θ -weighted G -selection $\Omega(G)$, then

- (1) $\omega(H) = \sum_{L \leq H} \phi(L)$,
 (2) $\phi(H) = \sum_{L \leq H} \mu(L, H)\omega(L)$.

Proof: This is a case of Möbius inversion. \square

Theorem 3.2: Given $H \leq G$, and a Θ -weighted G -selection $\Omega(G)$, then

$$\begin{aligned} k_H(\tilde{\chi}_{\mathcal{F}}[\Omega(G)]) &= -k_H[\Omega(G)] + \sum_{Q \in_H \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} k_H[\Omega(C_G(Q))] \\ &= \sum_{L \leq_H G: O_{\mathcal{F}}(C_G(L))=1} \phi(L) \tilde{\chi}(\tilde{\mathcal{F}}(C_G(L))/C_H(L)) |C_H(L)| / |N_H(L)|. \end{aligned}$$

If, furthermore, H is a π -subgroup of G , then

$$k_H(\tilde{\chi}_{\mathcal{F}}[\Omega(G)])|H| = -\omega(G) - \sum_{Q \in \tilde{\mathcal{F}}(G)} \mu(Q)\omega(C_G(Q))$$

which is divisible by $|G|_{\pi}$.

Proof: Theorem 2.4 given the first asserted equality and the rider. Then

$$\begin{aligned} &\sum_{Q \in \text{sd}(\tilde{\mathcal{F}}(G)), \underline{x} \in \Omega(C_G(Q))} (-1)^{n(Q)} |N_H(\underline{x}) \cap N_H(Q)| / |H| \\ &= \sum_{\underline{x} \in \Omega(G), Q \in \text{sd}(\tilde{\mathcal{F}}(C_G(\langle \underline{x} \rangle)))} (-1)^{n(Q)} |C_H(\langle \underline{x} \rangle) \cap N_H(Q)| / |H| \end{aligned}$$

$$= \sum_{\underline{x} \in_H \Omega(G)} \delta(\underline{x}) \chi(\tilde{\mathcal{F}}(C_G(\langle \underline{x} \rangle)) / C_H(\langle \underline{x} \rangle))$$

using Theorem 2.5. Subtracting $k_H[\Omega(G)]$ gives

$$k_H(\tilde{\lambda}_G^{\mathcal{F}}[\Omega(G)]) = \sum_{L \leq_H G} \phi(L) \tilde{\chi}(\tilde{\mathcal{F}}(C_G(L)) / C_H(L)) |C_H(L)| / |N_H(L)|.$$

If $O_{\mathcal{F}}(C_G(L)) \neq 1$, then $Q \mapsto QO_{\mathcal{F}}(C_G(L)) \mapsto O_{\mathcal{F}}(C_G(L))$ is a conical $C_H(L)$ -contraction of $\tilde{\mathcal{F}}(C_G(L))$, whereupon $\tilde{\chi}(\tilde{\mathcal{F}}(C_G(L)) / C_H(L)) = 0$. \square

Putting $H = 1$ in Theorem 3.2, and noting that $\phi(1) = \omega(1)$, we have

$$-\omega(G) + \omega(1) + \sum_{Q \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} (\omega(C_G(Q)) - \omega(1)) = \sum_{1 < L \leq G: O_{\mathcal{F}}(C_G(L))=1} \phi(L) \tilde{\chi}(\tilde{\mathcal{F}}(C_G(L)))$$

which is divisible by $|G|_{\pi}$. In other words, $\omega(G) - \omega(1)$ may be “approximated” as an alternating sum of terms $\omega(C_G(Q)) - \omega(1)$ with $Q \in \text{sd}(\tilde{\mathcal{F}}(G))$. The “error” is divisible by $|G|_{\pi}$, and is a measure of the failure of ω to be \mathcal{F} -centrally determined. Moreover, the “error” may be written as a sum of contributions coming from the non-trivial subgroups L of G such that $C_G(L)$ is non- \mathcal{F} -local.

Let $\Omega(G)$ be a Θ -weighted G -selection. Given a G -set Δ , we define

$$\Omega(\Delta) := \{(\underline{x}, d) : d \in \Delta, \underline{x} \in \Omega(N_G(d))\}$$

as a Θ -weighted G -set, where $g(\underline{x}, d) = (g\underline{x}, gd)$ for $g \in G$, and the density function is $(\underline{x}, d) \mapsto \delta(\underline{x})$. By linear extension, we obtain a map $\Omega : \Theta B(G) \rightarrow \Theta B(G)$. Note that $\Omega(1) = [\Omega(G)]$. Given $H \leq G$ and $\beta \in \Theta B(H)$, then

$$\Omega(\text{Ind}_H^G(\beta)) = \text{Ind}_H^G(\Omega(\beta)).$$

(This makes sense, because $\Omega(H)$ is a Θ -weighted G -selection.) Hence, for a finite G -poset Γ , we have

$$\Omega(\lambda_G(\Gamma)) = \sum_{\gamma \in_G \text{sd}(\Gamma)} (-1)^{n(\gamma)} \text{Ind}_{N_G(\gamma)}^G[\Omega(N_G(\gamma))].$$

Theorem 3.3: *Given $H \leq G$, a finite G -poset Γ , and a Θ -weighted G -selection $\omega(G)$, then*

$$\begin{aligned} k_H(\Omega(\tilde{\lambda}_G(\Gamma))) &= -k_H[\Omega(G)] + \sum_{\gamma \in_H \text{sd}(\Gamma)} (-1)^{n(\gamma)} k_{N_H(\gamma)}[\Omega(N_G(\gamma))] \\ &= \sum_{L \leq_H G} \phi(L) \tilde{\chi}(\Gamma(L) / C_H(L)) |C_H(L)| / |N_H(L)|. \end{aligned}$$

Proof: The demonstration is similar to part of the proof of Theorem 3.2. Note that, given $\gamma \in \text{sd}(\Gamma)$ and $\underline{x} \in \Omega(G)$, we have $\underline{\gamma} \in \text{sd}(\Gamma(\langle \underline{x} \rangle))$ if and only if $\underline{x} \in \Omega(N_G(\underline{\gamma}))$. Also, the conclusion can be rewritten as

$$\begin{aligned} k_H(\Omega(\lambda_G(\Gamma))) &= \sum_{\underline{\gamma} \in_H \text{sd}(\Gamma)} (-1)^{n(\underline{\gamma})} k_{N_H(\underline{\gamma})}[\Omega(N_G(\underline{\gamma}))] \\ &= \sum_{L \leq_H G} \phi(L) \chi(\Gamma(L)/C_H(L)) |C_H(L)/|N_H(L)|. \quad \square \end{aligned}$$

If L is an \mathcal{F} -local subgroup of G , then $Q \mapsto QO_{\mathcal{F}}(L) \mapsto O_{\mathcal{F}}(L)$ is a conical $C_H(L)$ -contraction of $\tilde{\mathcal{F}}(G)(L)$, whereupon $\tilde{\chi}(\tilde{\mathcal{F}}(G)(L)) = 0$. So, putting $H = 1$ and $\Gamma = \tilde{\mathcal{F}}(G)$ in Theorem 3.3, we have

$$\begin{aligned} -\omega(G) + \omega(1) + \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} (\omega(N_G(\underline{Q})) - \omega(1)) = \\ \sum_{1 < L \leq G: O_{\mathcal{F}}(L)=1} \phi(L) \tilde{\chi}(\tilde{\mathcal{F}}(G)(L)) \end{aligned}$$

In other words, $\omega(G) - \omega(1)$ may be “approximated” as an alternating sum of terms $\omega(N_G(\underline{Q})) - \omega(1)$. We are not aware of any general guarantee for the (p -adic) accuracy of this “approximation”. The “error” is a measure of the failure of ω to be \mathcal{F} -normally determined. Again, the “error” may be written as a sum of contributions coming from the non-trivial non- \mathcal{F} -local subgroups of G .

Comparing Theorem 3.3 with Thévenaz [23, 2.1, 2.2], and writing $\Omega_1(G) := G$ as a G -selection of degree unity, we see that $k_G(\Omega_1(\Lambda_G(\Gamma)))$ is the equivariant Euler characteristic $\chi_G(\Gamma)$ of Γ . As observed in [23], Alperin’s Conjecture is equivalent to the assertion that $k_G(\Omega_1(\Lambda_G(\tilde{\mathcal{S}}_p(G))))$ is the number of irreducible characters of G with positive p -defect.

The second equality in the following result is essentially Külshammer–Robinson [16, Theorem 1], and is a special case of Isaacs [12, Theorem 3]. The first equality, too, can be proved scarcely straying from the techniques in [16]. But let us present the argument in a form which shall serve as a template for our subsequent applications of Theorems 3.2 and 3.3. We allow G to act by conjugation on the set G_π consisting of the π -elements of G .

Corollary 3.4: (Isaacs, Külshammer, Robinson) *Given $\nu \subseteq \pi$ and $H \leq G \triangleright K$, then*

$$\begin{aligned} k_H(K_\nu) - 1 &= \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} (k_{N_H(\underline{Q})}(C_K(\underline{Q})_\nu) - 1) \\ &= \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} (k_{N_H(\underline{Q})}(N_K(\underline{Q})_\nu) - 1). \end{aligned}$$

Proof: If L is a non-trivial cyclic ν -subgroup of K , then $C_G(L)$ is \mathcal{F} -local. So by Theorem 3.2,

$$\tilde{\chi}(\tilde{\mathcal{F}}(G)/H) = -k_H(K_\nu) + \sum_{Q \in \text{Hsd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} k_{N_H(Q)}(N_K(Q)_\nu).$$

The first asserted equality now follows from Theorem 2.5. Similarly, Theorems 3.3 and 2.5 give the second asserted equality. \square

As in Külshammer–Robinson [15], Corollary 3.4 pertains to character theory in that, if $K \leq H$, then $k_H(K)$ is the number of H -classes of irreducible (ordinary) characters of K , while $k_H(K_p)$ is the number of H -classes of irreducible p -modular characters of K . When π contains all the prime divisors of $|G|$, Corollary 3.4 and Proposition 2.6 tell us that the function $H \mapsto |H|k_H(K_\nu)$ is \mathcal{F} -normally determined.

For a positive integer m , let $\Omega_m(G) := G^m$ as a G -selection of degree m . Then $\phi_m(G)$ is the number of generating m -tuples of elements of G . After Hall [10], we note that, by Remark 3.1,

$$\phi_m(G) = \sum_{H \leq G} \mu(H, G) |H|^m.$$

Supposing that G is non-abelian and simple, then the number $\phi_m(G)$ is of especial interest because, by [10, 1.6], $\phi_m(G)/|\text{Aut}(G)|$ is the maximum number d such that G^d can be generated by m elements. In this case, the next result gives another formula for $\phi_m(G)$.

Corollary 3.5. *Suppose that G is non-abelian and simple. Let Γ be a G -poset G -homotopic to $\tilde{U}(G)$, and for each $H \leq G$, let $f(H) := 1 - |H_\nu|^m/|H|$. Then the number of generating m -tuples of ν -elements of G is $|G|f(\tilde{\lambda}_G(\Lambda))$.*

Proof: Take \mathcal{F} to be the class of finite groups whose simple composition factors are all proper subquotients of G . Then $\mathcal{F}(G) = \mathcal{U}(G)$. Put $\Omega(G) := G_\nu^m$ as a G -selection. Then $\phi(G)$ is the number of generating m -tuples of ν -elements of G . Also, $\omega(1) = 1$. Any proper subgroup of G is \mathcal{F} -local, but $\tilde{\chi}(\tilde{U}(G)(G)) = -1$. By Theorem 3.3,

$$\phi(G) - \tilde{\chi}(\tilde{U}(G)) = |G_\nu|^m - \sum_{Q \in \text{sd}(\tilde{U}(G))} (-1)^{n(Q)} |N_G(Q)_\nu|^m.$$

By Theorem 2.5, $\phi(G) = |G|f(\tilde{\lambda}_G(\tilde{U}(G)))$, and the assertion follows from the homotopy invariance property of the Lefschetz invariant. \square

4 : Local arithmetic

Given $m \geq 1$ and $1 \leq n \leq \infty$ (meaning that m is a positive integer and that n is either a positive integer or infinity), let $\Omega_m^{p,n}(G)$ denote the G -selection consisting of the m -tuples of mutually commuting p -elements of G .

Lemma 4.1: *Let $m \geq 1$ and $1 \leq n \leq \infty$. Suppose that G is a finite p -group. Then we have congruences $\phi_m^{p,n}(G) \equiv_{p^m} \phi_m(G) \equiv_{p^m} \mu(G)$.*

Proof: Suppose that G is not elementary abelian. Then $\mu(G) = 0$. Let R be a non-trivial elementary abelian central subgroup of G contained in the Frattini subgroup of G . Then $\Omega_m^{p,n}(G)$ is a union of cosets of R^m , and so is the set of generating m -tuples of elements of G . So $|R|^m$ divides both $\phi_m^{p,n}(G)$ and $\phi_m(G)$. The assertion is proved in this case.

Now suppose that G is elementary abelian. Then

$$\phi_m(G) + \sum_{H < G} \phi_m(H) = |G|^m \quad \text{and} \quad \mu_m(G) + \sum_{H < G} \mu_m(H) = -\tilde{\chi}(\tilde{\mathcal{S}}_p(G)) = 0$$

and $\phi_m^{p,n}(G) = \phi_m(G)$. Induction on $|G|$ finishes the demonstration. \square

Proposition 4.2: *Given $1 \leq n \leq \infty$, then we have a p -adic limit*

$$\lim_{m \rightarrow \infty} \omega_m^{p,n}(G) = -\tilde{\chi}(\tilde{\mathcal{S}}_p(G)).$$

Proof: By Lemma 4.1, $\lim_{m \rightarrow \infty} \sum_{P \in \tilde{\mathcal{S}}_p(G)} \phi_m^{p,n}(P) = -\tilde{\chi}(\tilde{\mathcal{S}}_p(G))$. \square

Corollary 4.3: *Given $K \trianglelefteq G$, then*

$$\chi(\tilde{\mathcal{S}}_p(K)) = \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} \chi(\tilde{\mathcal{S}}_p(C_K(\underline{Q}))) = \sum_{\underline{Q} \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} \chi(\tilde{\mathcal{S}}_p(N_K(\underline{Q}))).$$

In other words, the function $H \mapsto \tilde{\chi}(\tilde{\mathcal{S}}_p(K \cap H))$, defined for $H \leq G$, is \mathcal{F} -centrally and \mathcal{F} -normally determined.

Proof: Given a non-trivial p -subgroup L of K , then $Q \mapsto QZ(L) \mapsto Z(L)$ is a conical contraction of $\tilde{\mathcal{F}}(C_G(L))$, so $\tilde{\chi}(\tilde{\mathcal{F}}(C_G(L))) = 0$. Similarly, $\tilde{\chi}(\tilde{\mathcal{F}}(G)(L)) = 0$. Proposition 4.2 and Theorems 3.2 and 3.3 now give the assertion. \square

We briefly record a generalization of Corollary 4.3: for $H \leq G \triangleright K$, let

$$\chi(K, h) := \sum_{h \in H} \chi(\tilde{\mathcal{S}}_p(C_K(h)))/|H|.$$

The proofs of Proposition 4.2 and Thévenaz [23, 2.2] can be adapted to show that $k_H[\Omega_m^{p,n}(G)]$ p -adically converges to $1 - \chi(K, H)$ as $m \rightarrow \infty$. A straightforward extension of the proof of Corollary 4.3 then gives

$$\begin{aligned} \chi(K, H) &= \sum_{\underline{Q} \in_H \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} \chi(C_K(\underline{Q}), N_H(\underline{Q})) = \\ &= \sum_{\underline{Q} \in_H \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} \chi(N_K(\underline{Q}), N_H(\underline{Q})). \end{aligned}$$

Since n plays no role in the proof of Corollary 4.3, our notation above could be simplified by omitting all mention of n . We introduced n in order to emphasise an

analogy between $\Omega_m^{p,n}(G)$ and the G -selections we are about to discuss. First, consider the G -selection $\Omega_{m,1}^{p,n}(G)$ consisting of the m -tuples $\underline{x} \in \Omega_m^{p,n}(G)$ such that $x_1 \dots x_m = 1$. In view of the bijection $\Omega_{m+1,1}^{p,n}(G) \rightarrow \Omega_m^{p,n}(G)$ given by $(x_1, \dots, x_{m+1}) \mapsto (x_1, \dots, x_m)$, we may replace $\Omega_m^{p,n}(G)$ with $\omega_{m,1}^{p,n}(G)$ in the statement of Proposition 4.2, and then the proof of Corollary 4.3 will still be valid. An interesting feature of the proof is the fact that, if we make a single change in the definition of $\Omega_{m,1}^{p,n}(G)$, then instead of obtaining local conclusions about $\tilde{\chi}(\tilde{\mathcal{S}}_p(G))$ (whose p -local nature is manifest in its definition), we obtain local conclusions about defect-zero characters (whose believed p -local properties are mostly conjectural). Indeed, relaxing the commutativity condition in the definition of $\Omega_{m,1}^{p,n}(G)$, let $\Omega_{0,m,1}^{p,n}(G)$ denote the G -selection consisting of the m -tuples \underline{x} such that $x_1 \dots x_m = 1$, and each x_i is a p -element of order at most p^n . Then Lemma 4.4 below tells us that $\omega_{0,m,1}^{p,n}(G) = |G|f_{-1}^p(G)$.

Let \mathcal{P} be a prime divisor of (p) in Θ , let \mathcal{O} be the \mathcal{P} -adic completion of $\Theta(\mathcal{P})$, and let κ be the algebraic closure of the field of fractions of \mathcal{O} . Given a rational integer r , and an idempotent e of $Z\mathcal{O}G$, we define

$$f_{r,e}^p(G) := \sum_{\chi} (|G|/\chi(1))^{2r}$$

where χ runs over the irreducible κG -characters which lie in e and which have p -defect zero. Let $\langle \cdot, \cdot \rangle$ be the bilinear form $\kappa G \times \kappa G \rightarrow \kappa$ such that, given $g, h \in G$, then $\langle g, h \rangle = 1$ if $gh = 1$, otherwise $\langle g, h \rangle = 0$. For any irreducible κG -character χ , we write e_{χ} for the primitive idempotent of $Z\kappa G$ such that $\chi(e_{\chi}) = \chi(1)$, and write ω_{χ} for the algebra map $Z\kappa G \rightarrow \kappa$ such that $\omega_{\chi}(e_{\chi}) = 1$. Note that $\langle e_{\chi}, e_{\chi} \rangle = \chi(1)^2/|G|$, and $\langle e_{\chi}, e_{\chi'} \rangle = 0$ for $\chi' \neq \chi$.

Let $1 \leq n \leq \infty$. Let u and v be natural numbers, not both zero. We define an \mathcal{O} -weighted G -selection $\Omega_{u,v,e}^{p,n}(G)$ consisting of the $(u+v)$ -tuples \underline{x} such that $x_{u+1}, x_{u+2}, \dots, x_{u+v}$ are all p -elements of order at most p^n . The density function is given by

$$\delta_{u,v,e}^{p,n}(\underline{x}) := \langle x_1 x_2 \dots x_u x_1^{-1} x_2^{-1} \dots x_u^{-1} x_{u+1} x_{u+2} \dots x_{u+v}, e \rangle.$$

Lemma 4.4: *Let r be a rational integer, and e an idempotent of $Z\mathcal{O}G$. Choose $1 \leq n \leq \infty$, and sequences u_1, u_2, \dots and v_1, v_2, \dots of natural numbers such that $u_s + v_s \rightarrow \infty$ as $s \rightarrow \infty$, and any integer of the form $p^t(p-1)/2$ divides $u_s - r - 1$ for sufficiently large s . Then we have \mathcal{P} -adic limits*

$$|G|f_{r,e}^p(G) = \lim_{s \rightarrow \infty} \omega_{2u,v,e}^{p,n}(G) = \lim_{s \rightarrow \infty} \omega_{2u+1,v,e}^{p,n}(G)/|G|.$$

Proof: We draw some inspiration from Iizuka-Watanabe [11] and Strunkov [20]. For $w \geq 1$, let

$$\zeta_w := \sum_{x_1, \dots, x_w \in G} x_1 \dots x_w x_1^{-1} \dots x_w^{-1}$$

as an element of $Z\mathcal{O}G$, and let $\zeta_0 = 1$. We claim that

$$\sum_{\chi \in \text{Irr}(\kappa G)} (|G|/\chi(1))^{2u} e_\chi = \zeta_{2u} = \zeta_{2u+1}/|G|.$$

The case $u = 0$ of the claim is trivial. Let tr_1^G denote the 1-relative trace map $\kappa G \rightarrow Z\kappa G$. Then

$$\sum_{\chi \in \text{Irr}(\kappa G)} \omega_\chi(\zeta_2) e_\chi = \zeta_2 = \sum_{g \in G} g \text{tr}_1^G(g^{-1}) = \sum_{g \in G} \text{tr}_1^G(g) \text{tr}_1^G(g^{-1})/|G|.$$

So $\omega_\chi(\zeta_2) = |G| \sum_{g \in G} \chi(g) \chi(g^{-1})/\chi(1)^2 = (|G|/\chi(1))^2$. Since

$$\zeta_3 = \sum_{x,y,z \in G} (xz^{-1})zy(xz^{-1})^{-1}z^{-1}y^{-1} = \sum_{x,y,z \in G} x(yz)x^{-1}(yz)^{-1} = |G|\zeta_2$$

the case $u = 1$ of the claim holds. For $w \geq 4$, we have

$$\begin{aligned} \zeta_w &= \sum_{x_1, \dots, x_{w-1} \in G} x_1 \dots x_{w-1} \text{tr}_1^G(x_1^{-1} \dots x_{w-1}^{-1}) \\ &= \sum_{x_1, \dots, x_{w-3}, y \in G} x_1 \dots x_{w-3} y \text{tr}_1^G(x_1^{-1} \dots x_{w-3}^{-1}) \text{tr}_1^G(y^{-1}) = \zeta_{w-2} \zeta_2. \end{aligned}$$

An inductive argument on u now establishes the claim.

Let η be the sum of those p -elements which have order at most p^n . We have $\omega_{2u,v,e}^{p,n}(G) = \omega_{2u+1,v,e}^{p,n}(G)/|G|$ because

$$\omega_{w,v,e}^{p,n}(G) = \langle \zeta_w \eta^v, e \rangle.$$

It remains to prove the first asserted equality. Now

$$\begin{aligned} \langle \zeta_{2u} \eta^v, e \rangle &= \sum_{\chi \in \text{Irr}(\kappa G)} (|G|/\chi(1))^{2u} \langle \omega_\chi(\eta)^v e_\chi, e \rangle \\ &= |G| \sum_{\chi \in \text{Irr}(\kappa G): \chi(e) = \chi(1)} (|G|/\chi(1))^{2u-2} \omega_\chi(\eta)^v. \end{aligned}$$

Let $\chi \in \text{Irr}(\kappa G)$. If χ has p -defect zero, then $\omega_\chi(\eta) = 1$, and $(|G|/\chi(1))^{2u-2}$ \mathcal{P} -adically converges to $(|G|/\chi(1))^{2r}$. Now assume that χ has positive p -defect. It suffices to show that $\omega_\chi(\eta) \in \mathcal{P}$. Supposing otherwise, let b be the block idempotent in ZOG such that $be_\chi = e_\chi$, and let P be a defect group of b . The residue field $k := O/P$ has characteristic p . Write Br_P for the P -relative Brauer map $ZOG \rightarrow ZkC_G(P)$. Let A be the maximal elementary abelian p -subgroup of $Z(P)$. By our assumptions, $b \in ZOG.\eta$ and $A \neq 1$. But $\text{Br}_P(b) \in ZkC_G(P).\text{Br}_P(\eta)$. Also, $\text{Br}_P(\eta)$ is in the principal ideal of $ZkC_G(P)$ generated by the sum of the elements in A , hence $\text{Br}_P(\eta)^2 = 0$. So the block idempotent $\text{Br}_P(b)$ of $kN_G(P)$ belongs to $J(ZkC_G(P))$. This is absurd. \square

Given a rational integer r , and a natural number s such that the integer $m(r, s) := 2r + 2 + p^s(p-1)$ is strictly positive, let $\Omega_{r,s}^p(G)$ be the G -selection consisting of all

the $m(r, s)$ -tuples \underline{x} such that

$$x_1 x_2 \dots x_{m(r,s)} x_1^{-1} x_2^{-1} \dots x_{m(r,s)}^{-1} = 1.$$

By Lemma 4.4, we have a p -adic limit

$$|G|f_r^p(G) = \lim_{s \rightarrow \infty} \omega_{r,s}^p(G).$$

Corollary 4.5: *Let $K \trianglelefteq G$, and let r be a rational integer. Then the p -local integers $|K|f_r^p(K)$ and*

$$\sum_{Q \in \text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(Q)} |C_K(Q)| f_r^p(C_K(Q)) = - \sum_{Q \in \tilde{\mathcal{F}}(G)} \mu(Q) |C_K(Q)| f_r^p(C_K(Q))$$

are congruent modulo $|G|_p$.

Proof: This follows from Theorem 3.2 and Lemma 4.4 by considering the G -selections of the form $\Omega_{r,s}^p(K)$. \square

Note that the residue class of $|K|f_r^p(K)$ modulo $|G|_p$ is determined by the residue classes of $|C_K(P)|f_r^p(C_K(P))$ modulo $|C_G(P)|_p$ for $P \in \tilde{\mathcal{S}}_p(G)$. Indeed, as p -local integers, $|N_G(P)|$ divides $|C_G(P)|\mu(P)$.

Corollary 4.6: *Suppose that G is simple. Let r be a rational integer. Then we have a congruence*

$$f_r^p(G) - 1 \equiv_{| \text{Out}(G) |_p} \sum_{Q \in \text{sd}(\tilde{\mathcal{U}}(G))} (-1)^{n(Q)} (f_r^p(N_G(Q)) - 1).$$

Proof: We may assume that G is non-abelian, because otherwise the assertion is very easy. By Theorem 3.3 and Lemma 4.4, we have a p -adic limit

$$\lim_{s \rightarrow \infty} \phi_{r,s}^p(G) = -|G|f_r^p(G) + 1 - \sum_{Q \in \text{sd}(\tilde{\mathcal{U}}(G))} (-1)^{n(Q)} (|N_G(Q)|f_r^p(N_G(Q)) - 1).$$

This limit is divisible by $| \text{Aut}(G) |_p$ because each $\Omega_{r,s}^p(G)^*$ is a union of regular $| \text{Aut}(G) |$ -sets. Theorem 2.5 finishes the argument. \square

5 : Central extensions and Alperin's conjecture

To ensure the existence of the Θ -weighted G -selections we wish to consider, we assume throughout this section that Θ is integrally closed. Let $1 \rightarrow E \rightarrow \hat{G} \rightarrow G \rightarrow 1$ be a finite p' -central extension of G , and let ϵ be a linear character of E . The primitive idempotent of ΘE corresponding to ϵ will also be denoted by ϵ (this will cause no confusion). Given $H \leq G$, we write \hat{H} for the preimage of H in \hat{G} . Given a Θ -weighted G -selection $\hat{\Omega}(\hat{G})$, we write $\hat{\omega}$ and $\hat{\phi}$ for the associated weight function and Euler function, and we define

$$\widehat{\phi}(H) := \sum_{L \leq G: LE = \widehat{H}} \widehat{\phi}(L).$$

Via the conjugation action of G on \widehat{G} , we regard $\widehat{\Omega}(\widehat{G})$ as a Θ -weighted G -set. Straightforward adaptations of Theorems 3.2 and 3.3 demonstrate the following two results.

Theorem 5.1: *Given \widehat{G} as above, and a Θ -weighted \widehat{G} -selection $\widehat{\Omega}(\widehat{G})$, then*

$$\begin{aligned} -\widehat{\omega}(\widehat{G}) + \widehat{\omega}(E) + \sum_{Q \in \text{sd}(\widehat{\mathcal{F}}(G))} (-1)^{n(Q)} (\widehat{\omega}(\widehat{C}_G(Q)) - \widehat{\omega}(E)) \\ = \sum_{1 < H \leq G: O_{\mathcal{F}}(C_G(H))=1} \widehat{\phi}(H) \widetilde{\chi}(\widehat{\mathcal{F}}(C_G(H))). \end{aligned}$$

which is divisible by $|G|_{\pi}$. \square

Theorem 5.2: *Given \widehat{G} as above, and a Θ -weighted \widehat{G} -selection $\widehat{\Omega}(\widehat{G})$, then*

$$\begin{aligned} -\widehat{\omega}(\widehat{G}) + \widehat{\omega}(E) + \sum_{Q \in \text{sd}(\widehat{\mathcal{F}}(G))} (-1)^{n(Q)} (\widehat{\omega}(\widehat{N}_G(Q)) - \widehat{\omega}(E)) \\ = \sum_{1 < H \leq G: O_{\mathcal{F}}(H)=1} \widehat{\phi}(H) \widetilde{\chi}(\widehat{\mathcal{F}}(G)(H)). \end{aligned}$$

\square

Let $k_{\epsilon}(G)$ denote the number of irreducible characters of G lying over ϵ .

Corollary 5.3: *Given ϵ as above, and $K \trianglelefteq G$, then the function $L \mapsto |I|k_{\epsilon}(\widehat{I})$ where $I := L \cap K$ is \mathcal{F} -centrally and \mathcal{F} -normally determined.*

Proof: Let $\widehat{\Omega}(\widehat{G}) = \widehat{K} \times \widehat{K}$ as a Θ -weighted \widehat{G} -selection of degree 2 with density function $(x, y) \mapsto \langle [x, y], \epsilon \rangle$. Choosing a prime q not dividing $|G|$, then $\widehat{\omega}(\widehat{G}) = \omega_{2,s,\epsilon}^{q,\infty}(\widehat{K})$ for all $s \geq 0$. By Lemma 4.4, $\widehat{\omega}(\widehat{G}) = |\widehat{K}|k_{\epsilon}(\widehat{K})$. (Without much difficulty, this deduction can also be obtained directly.) If $1 < H \leq G$ with $\widehat{\phi}(H) \neq 0$, then E contains the derived subgroup of some subgroup L of \widehat{H} with $LE = \widehat{H}$. Then H must be abelian, and in particular, $C_G(H)$ and $N_G(H)$ are \mathcal{F} -local. Theorems 5.1 and 5.2 now give the assertion. \square

When K is a normal π -subgroup of G , Theorem 2.5 allows us to rewrite the conclusion of Corollary 5.3 as

$$\begin{aligned} \sum_{Q \in K \text{sd}(\widehat{\mathcal{F}}(G))} (-1)^{n(Q)} (k_{\epsilon}(\widehat{C}_K(Q)) |C_K(Q)| / |N_K(Q)| - 1) \\ = k_{\epsilon}(\widehat{K}) - 1 = \sum_{Q \in K \text{sd}(\widehat{\mathcal{F}}(G))} (-1)^{n(Q)} (k_{\epsilon}(\widehat{N}_K(Q)) - 1). \end{aligned}$$

A variant of the second of these two equalities appears in Robinson [18, Section 1].

Given $K \trianglelefteq G$, then a congruence condition for $|\widehat{K}|f_{r,\epsilon}^p(\widehat{K})$ analogous to Corollary 4.5 follows easily from Theorem 5.1 and Lemma 4.4.

Recall that $f_{0,\epsilon}^p(\widehat{G})$ denotes the number of defect-zero p -blocks of \widehat{G} lying over ϵ . If $\widehat{G} = G$, then of course, ϵ is trivial, and $f_{0,\epsilon}^p(\widehat{G})$ is simply $f_0^p(G)$, the number of defect-zero p -blocks of G . Let $\ell^p(G)$ be the number of (absolutely) irreducible p -modular characters of G , and let $\ell_\epsilon^p(\widehat{G})$ be the number of irreducible p -modular characters of \widehat{G} lying over ϵ . We define

$$\alpha^p(G) := \ell^p(G) - \sum_{P \in \mathcal{G}\tilde{S}_p(G)} f_0^p(N_G(P)/P),$$

$$\alpha_\epsilon^p(\widehat{G}) := \ell_\epsilon^p(\widehat{G}) - \sum_{P \in \mathcal{G}\tilde{S}_p(G)} f_{0,\epsilon}^p(\widehat{N}_G(P)/P).$$

(The p -subgroups of \widehat{G} may be identified with the p -subgroups of G .) Clearly,

$$\alpha^p(\widehat{G}) = \sum_{\epsilon \in \text{Irr}(E)} \alpha_\epsilon^p(\widehat{G}).$$

Theorem 5.4: *Suppose that π is the set of all primes, and that $\alpha^p(Q) = f_0^p(Q)$ for all $Q \in \mathcal{F}$. If any of the following statements hold for all finite groups G , all central extensions \widehat{G} of G by a finite p' -group E , and all linear characters ϵ of E , then all eight of them do.*

- (a1) (Alperin) $f_0^p(G) = \alpha^p(G)$.
- (a2) (Alperin) $f_{0,\epsilon}^p(\widehat{G}) = \alpha_\epsilon^p(\widehat{G})$.
- (b1) (After Külshammer, Robinson)

$$f_0^p(G) - 1 = \sum_{\underline{Q} \in \mathcal{G}\text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} (f_0^p(N_G(\underline{Q})) - 1).$$

- (b2) (Külshammer, Robinson)

$$f_{0,\epsilon}^p(\widehat{G}) - 1 = \sum_{\underline{Q} \in \mathcal{G}\text{sd}(\tilde{\mathcal{F}}(G))} (-1)^{n(\underline{Q})} (f_{0,\epsilon}^p(\widehat{N}_G(\underline{Q})) - 1).$$

(c0) (Thévenaz) *If $\tau^p(G) \neq 0$, then $O_p(G)$ is elementary abelian, and $G/O_p(G)$ is an abelian p' -group of rank at most 2.*

(c1) (After Thévenaz) *If $\tau^p(G) \neq 0$, then G is \mathcal{F} -local or trivial.*

(d0) *If $O_p(G)$ is not elementary abelian, or $G/O_p(G)$ is not an abelian p' -group of rank at most 2, then any power of p divides $t_s^p(G)$ for infinitely many s .*

(d1) *If G is non-trivial and non- \mathcal{F} -local, then any power of p divides $t_s^p(G)$ for infinitely many s .*

Proof: Clearly, (a2) implies (a1). Conversely, suppose that (a1) holds (for all finite groups G). Then

$$\sum_{\epsilon \in \text{Irr}(E)} (\alpha_\epsilon^p(\widehat{G}) - f_{0,\epsilon}^p(\widehat{G})) = 0.$$

If ϵ is not faithful, then $(\alpha_\epsilon^p(\widehat{G}) - f_{0,\epsilon}^p(\widehat{G})) = 0$ because we can replace \widehat{G} with $\widehat{G}/\text{Ker}(\epsilon)$. On the other hand, when E is cyclic, the faithful linear characters of E are permuted transitively by the full Galois group of the cyclotomic field of $|E|$ -th roots of unity. Therefore, for faithful ϵ , the value of $(\alpha_\epsilon^p(\widehat{G}) - f_{0,\epsilon}^p(\widehat{G}))$ is constant, and must be zero. We have shown that (a1) and (a2) are equivalent.

Since $\alpha_\epsilon^p(\widehat{Q}) = f_{0,\epsilon}^p(\widehat{Q})$ for all $Q \in \mathcal{F}(G)$, Külshammer–Robinson [15, Theorem 6] (and its proof), shows that (a2) is equivalent to (b2). By Thévenaz [22, 6.3(6)], (a1) is equivalent to (c0). In the notation of Section 4, $t_s^p(G) = \phi_{0,s}^p(G)$, so by Lemma 4.4, we have a p -adic limit $\lim_{s \rightarrow \infty} t_s^p(G) = \tau^p(G)$. Hence (c0) is equivalent to (d0), also (c1) is equivalent to (d1). Propositions 2.3 and 2.6(2) imply that (b1) and (c1) are equivalent. Clearly, (b2) implies (b1).

We have shown that the conditions (a1), (a2), (b2), (c0), (d0) are mutually equivalent, as are the conditions (b1), (c1), (d1), and moreover, the former imply the latter. If G is non- \mathcal{F} -local and (c0) holds for G , then (c1) holds for G . On the other hand, if G is \mathcal{F} -local, then (b2) must hold for G (and for all \widehat{G} and ϵ). Indeed, let f be the linear map on $B(G)$ such that $f(\text{Ind}_H^G(1)) := f_{0,\epsilon}^p(\widehat{H})$ for all $H \leq G$. Since $\widetilde{\mathcal{F}}(G)$ is conically G -contractible, $f(\widetilde{\lambda}_G(\widetilde{\mathcal{F}}(G))) = 0$, that is to say, (b2) holds. \square

Lemma 4.4 provides various other functions which may take the place of $t_s^p(\widehat{G})$ or $t_{s,\epsilon}^p(\widehat{G})$ in conditions (d0) and (d1). For example, $t_s^p(\widehat{G})$ may be replaced with the number of $(2+s)$ -tuples $(y_1, y_2, x_1, \dots, x_s)$ of generators of \widehat{G} such that $[y_1, y_2]x_1 \dots x_s = 1$ and each $x_i^p = 1$.

Via Proposition 2.6(2), condition (b1) says precisely that the function $G \mapsto |G|f_0^G(G)$ is \mathcal{F} -normally determined. Similarly, for fixed \widehat{G} and ϵ as above, condition (b2) says precisely that the function sending each subgroup H of G to $|H|f_{0,\epsilon}^p(\widehat{H})$ is \mathcal{F} -normally determined.

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References

1. J. Alperin, Weights for finite groups, Proc. Sympos. Pure Math. **47** (1987), 369-379.
2. L. Barker, On p -soluble groups and the number of simple modules associated with a given Brauer pair, Quart. J. Math. (Ser. 2) **48** (1997), 133-160.
3. L. Barker, Counting positive defect irreducible characters of a finite group, New Zealand J. Math. (to appear).
4. L. Barker, Projective alternating sums: variations on a theme of Steinberg, (preprint).
5. D. J. Benson, Representations and cohomology II: cohomology of groups and modules, Cambridge Univ. Press, 1991.

6. S. Bouc, Projecteurs dans l'anneau de Burnside, projecteurs dans l'anneau de Green, modules de Steinberg généralisés, *J. Alg.* **139** (1991), 395-445.
7. M. Broué, On Scott modules and p -permutation modules: an approach through the Brauer morphism, *Proc. American Math. Soc.* **93** (1985), 401-408.
8. C. W. Curtis, I. Reiner, *Methods of representation theory*, Vol. II, Wiley, New York, 1987.
9. H. Ellers, The defect groups of a clique, p -solvable groups, and Alperin's conjecture, *J. reine angewandte Math.* **468** (1995), 1-48.
10. P. Hall, The Eulerian functions of a group, *Quart. J. Math. (Ser. 2)* **7** (1936), 134-151.
11. K. Iizuka, A. Watanabe, On the number of blocks of irreducible characters of a finite group with a given defect group, *Kumamoto J. Sci. (Math.)* **9** (1973), 55-61.
12. M. Isaacs, Numbers of classes and chains of subgroups in finite groups, (preprint).
13. R. Knörr, G. R. Robinson, Some remarks on a conjecture of Alperin, *J. London Math. Soc. (2)* **39** (1989), 48-60.
14. C. Kratzer, J. Thévenaz, Fonction de Möbius d'un group fini et anneau de Burnside, *Comment. Math. Helvetici* **59** (1984), 425-438.
15. B. Külshammer, G. R. Robinson, On Alperin's conjecture and certain subgroup complexes, *Algebras and Representation Theory* (to appear).
16. B. Külshammer, G. R. Robinson, On the number of conjugacy classes of π -elements in a finite group, *J. Algebra* (to appear).
17. G. R. Robinson, Further consequences of conjectures like Alperin's, *J. Group Theory* (to appear).
18. G. R. Robinson, On local control of the number of conjugacy classes, *J. Algebra* (to appear).
19. G. R. Robinson, R. Staszewski, More on Alperin's conjecture, *Astérisque* **181-182** (1990), 237-255.
20. S. P. Strunkov, Existence and the number of p -blocks of defect zero in finite groups (translation), *Algebra and Logic* **30** (1991), 231-241.
21. J. Thévenaz, Permutation representations arising from simplicial complexes, *J. Combinatorial Theory (Ser. A)* **46** (1987), 121-155.
22. J. Thévenaz, Locally determined functions and Alperin's conjecture, *J. London Math. Soc. (Ser. 2)* **45** (1992), 446-468.
23. J. Thévenaz, Equivariant K -theory and Alperin's conjecture, *J. Pure Applied Algebra* **85** (1993), 185-202.
24. P. J. Webb, Subgroup complexes, *Proc. Symp. Pure Math.* **47** (1987), 349-365.

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