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BOOLEAN NORMAL FORMS, SHELLABILITY, AND RELIABILITY COMPUTATIONS*

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Abstract. Orthogonal forms of positive Boolean functions play an important role in reliability theory, since the probability that they take value 1 can be easily computed. However, few classes of disjunctive normal forms are known for which orthogonalization can be efficiently performed. An interesting class with this property is the class of shellable disjunctive normal forms (DNFs). In this paper, we present some new results about shellability. We establish that every positive Boolean function can be represented by a shellable DNF, we propose a polynomial procedure to compute the dual of a shellable DNF, and we prove that testing the so-called lexico-exchange (LE) property (a strengthening of shellability) is NP-complete.

Key words. Boolean functions, orthogonal DNFs, dualization, shellability, reliability

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1. Introduction.

A classical problem of Boolean theory is to derive an orthogonal form, or disjoint products form, of a positive Boolean function given in DNF (see section 2 for definitions). In particular, this problem has been studied extensively in reliability theory, where it arises as follows. One of the fundamental issues in reliability is to compute the probability that a positive Boolean function (describing the state—operating or failed—of a complex system) take value 1 when each variable (representing the state of individual components) takes value 0 or 1 randomly and independently of the value of the other variables (see, for instance, [3, 25]). For functions in orthogonal form, this probability is very easily computed by summing the probabilities associated to all individual terms, since any two terms correspond to pairwise incompatible events. This observation has prompted the development of several reliability algorithms based on the computation of orthogonal forms (see, e.g., [18, 21]).

In general, however, orthogonal forms are difficult to compute and few classes of DNFs seem to be known for which orthogonalization can be efficiently performed. An interesting class with this property, namely, the class of *shellable* DNFs, has been introduced and investigated by Ball and Provan [2, 22]. As discussed by these authors, the DNFs describing several important classes of reliability problems (k-out-of-n systems, all-terminal connectedness, all-point reachability, etc.) are shellable. Moreover,

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besides its unifying role in reliability theory, shellability also provides a powerful theoretical and algorithmic tool in the study of simplicial polytopes, abstract simplicial complexes, and matroids. (This is actually where the shellability concept originates (see, e.g., [7, 8, 11, 16]); let us simply mention here, without further details, that abstract simplicial complexes are in a natural one-to-one relationship with positive Boolean functions.)

Shellability is the main topic of this paper. In section 2, we briefly review the basic concepts and notations to be used in this paper. In section 3, we establish that every positive Boolean function can be represented by a shellable DNF, and we characterize those orthogonal forms that arise from shellable DNFs by a classical orthogonalization procedure. In section 4, we prove that the dual (or, equivalently, the inverse) of a shellable DNF can be computed in polynomial time. Finally, in section 5, we define an important subclass of shellable DNFs, namely, the class of DNFs which satisfy the so-called LE property, and we prove that testing membership in this class is NP-complete.

2. Notations, definitions, and basic facts. Let $\mathbb{B} = \{0, 1\}$ and let n be a natural number. For any subset $S \subseteq \{1, 2, \ldots, n\}$, $\mathbf{1}_S$ is the characteristic vector of S, i.e., the vector of $\mathbb{B}^n$ whose jth coordinate is 1 if and only if $j \in S$. Similarly, $\mathbf{0}_S \in \mathbb{B}^n$ denotes the binary vector whose jth coordinate is 0 exactly when $j \in S$. The *lexicographic order* $\prec_L$ on subsets of $\{1, 2, \ldots, n\}$ is defined as usual: for all $S, T \subseteq \{1, 2, \ldots, n\}$, $S \prec_L T$ if and only if $\min\{j \in \{1, 2, \ldots, n\} \,|\, j \in S \setminus T\} < \min\{j \in \{1, 2, \ldots, n\} \,|\, j \in T \setminus S\}$.

We assume that the reader is familiar with the basic concepts of Boolean algebra and we introduce here only the notions that we explicitly use in the paper (see, e.g., [19, 20] for more information).

A *Boolean function* of n variables is a mapping $f : \mathbb{B}^n \longrightarrow \mathbb{B}$. We denote by $x_1, x_2, \ldots, x_n$ the variables of a Boolean function and we let $\mathbf{x} = (x_1, \ldots, x_n)$. The *complement* of variable x_j is $\overline{x}_j = 1 - x_j$. A DNF is a Boolean expression of the form

$$\Psi(x_1, \ldots, x_n) = \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j \bigwedge_{j \in J_k} \overline{x}_j, \tag{2.1}$$

where $I_k, J_k \subseteq \{1, 2, \ldots, n\}$ and $I_k \cap J_k = \emptyset$ for all $1 \leq k \leq m$. The *terms* of Ψ are the elementary conjunctions

$$T_k(x_1, \ldots, x_n) = T_{I_k, J_k}(x_1, \ldots, x_n) = \bigwedge_{j \in I_k} x_j \bigwedge_{j \in J_k} \overline{x}_j \quad (k = 1, 2, \ldots, m).$$

(By abuse of terminology, we sometimes call "terms" the pairs (I_k, J_k) themselves.)

It is customary to view any DNF Ψ (or, more generally, any Boolean expression) as defining a Boolean function: for any assignment of $0 - 1$ values to the variables $(x_1, \ldots, x_n)$, the value of $\Psi(x_1, \ldots, x_n)$ is simply computed according to the usual rules of Boolean algebra. With this in mind, we say that the DNF Ψ *represents* the Boolean function f (and we simply write $f = \Psi$) if $f(\mathbf{x}) = \Psi(\mathbf{x})$ for all binary vectors $\mathbf{x} \in \mathbb{B}^n$. It is well known that every Boolean function admits (many) DNF representations.

A Boolean function f is called *positive* if $f(\mathbf{x}) \geq f(\mathbf{y})$ whenever $\mathbf{x} \geq \mathbf{y}$, where the latter inequality is meant componentwise. For a positive Boolean function f, there is a unique minimal family of subsets of $\{1, 2, \ldots, n\}$, denoted $\mathcal{P}_f$, such that $f(\mathbf{1}_S) = 1$ if and only if $S \supseteq P$ for some $P \in \mathcal{P}_f$. A subset S for which $f(\mathbf{1}_S) = 1$ is called an *implicant* set (or *implicant*, for short) of f, and if $S \in \mathcal{P}_f$, then S is called a *prime implicant* (set) of f.

Prime implicants of positive Boolean functions have a natural interpretation in many applied contexts. For instance, in reliability theory, prime implicants of a *coherent structure function* are in one-to-one correspondence with the *minimal pathsets* of the system under study, i.e., with those minimal subsets of elements which, when working correctly, allow the whole system to work (see, e.g., [3, 25]).

Every positive Boolean function can be represented by at least one *positive DNF*, i.e., by a DNF of the form

$$\Phi(x_1, \ldots, x_n) = \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j. \tag{2.2}$$

Clearly, if $I_k \subseteq I_l$ for some $k \neq l$, then the Boolean function represented by (2.2) does not change when we drop the term corresponding to I_l. Hence, Φ represents f if and only if the (containment wise) minimal subsets of $\mathcal{I} = \{I_1, \ldots, I_m\}$ are exactly the prime implicants of f.

Besides its representations by positive DNFs, every positive Boolean function can also be represented by a variety of nonpositive DNFs. Let us record the following fact for further reference.

LEMMA 2.1. *If the DNF Ψ given by* (2.1) *represents a positive Boolean function f, then $f = \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j$ (and $f \equiv 1$ if $I_k = \emptyset$ for some $k \in \{1, 2, \ldots, m\}$).*

Proof. If Ψ represents f, then $f(\mathbf{x}) = \Psi(\mathbf{x}) \leq \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j$ for all $\mathbf{x} \in \mathbb{B}^n$ (since the inequality holds termwise).

To prove the reverse inequality, assume that $\bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j^* = 1$ for some $\mathbf{x}^* \in \mathbb{B}^n$. Then there is an index k, $1 \leq k \leq m$, such that $\bigwedge_{j \in I_k} x_j^* = 1$, or, equivalently, $\mathbf{1}_{I_k} \leq \mathbf{x}^*$. Now, $f(\mathbf{1}_{I_k}) = \Psi(\mathbf{1}_{I_k}) = 1$ and hence, since f is positive, $f(\mathbf{x}^*) = 1$. □

As explained in the Introduction, this paper pays special attention to orthogonal DNFs: the DNF (2.1) is said to be *orthogonal* (or is an *ODNF*, for short) if, for every pair of terms T_k, T_l ($k, l \in \{1, 2, \ldots, m\}, k \neq l$) and for every $\mathbf{x} \in \mathbb{B}^n$, $T_k(\mathbf{x})T_l(\mathbf{x}) = 0$. Equivalently, (2.1) is orthogonal if and only if $(I_k \cap J_l) \cup (I_l \cap J_k) \neq \emptyset$ for all $k \neq l$.

In subsequent sections, we use the following basic properties of ODNFs.

LEMMA 2.2. *Let us assume that* (2.1) *is an ODNF of a positive Boolean function, let $k \in \{1, 2, \ldots, m\}$, and let $\mathcal{A}_k = \{I_l \mid l \in \{1, 2, \ldots, m\}, \text{ and } I_l \cap J_k \neq \emptyset\}$. Then J_k is a minimal transversal of $\mathcal{A}_k$, and $S \cap I_k \neq \emptyset$ holds for all other minimal transversals $S \neq J_k$ of $\mathcal{A}_k$.*

Proof. Let us assume that S is a transversal of $\mathcal{A}_k$ for which $S \cap I_k = \emptyset$. Then $\mathbf{0}_S \geq \mathbf{1}_{I_k}$, and hence $\Psi(\mathbf{0}_S) \geq \Psi(\mathbf{1}_{I_k}) = 1$. Furthermore, for every term $T_l(\mathbf{x})$ of Ψ, $l \neq k$, we have $T_l(\mathbf{0}_S) = 0$, since either $I_k \cap J_l \neq \emptyset$ or $I_l \cap J_k \neq \emptyset$, i.e., $I_l \in \mathcal{A}_k$ and hence $I_l \cap S \neq \emptyset$, and in both cases the literals of T_l corresponding to these intersections have value 0 at the vector $\mathbf{0}_S$. Thus $T_k(\mathbf{0}_S) = 1$ must hold, and hence $J_k \subseteq S$ is implied.

On the other hand, J_k itself is a transversal of $\mathcal{A}_k$ (by definition of $\mathcal{A}_k$), which proves that J_k is the only minimal transversal of $\mathcal{A}_k$ which is disjoint from I_k. □

LEMMA 2.3. *Let us assume that* (2.1) *is an ODNF of a positive Boolean function, let $k \in \{1, 2, \ldots, m\}$, and let Ψ^k denote the disjunction of all terms of Ψ but term T_k. Then Ψ^k represents a positive Boolean function if and only if $J_k \cap I_l \neq \emptyset$ for all $l \in \{1, 2, \ldots, m\} \setminus k$.*

Proof. Assume first that Ψ^k represents a positive Boolean function and let $l \in \{1, 2, \ldots, m\}, l \neq k$. Then, by Lemma 2.1, $\Psi^k(\mathbf{1}_{I_l}) = 1$. On the other hand, $T_k(\mathbf{0}_{J_k}) =$

1 and, since Ψ is an ODNF, all terms other than T_k vanish at $\mathbf{0}_{J_k}$, so that $\Psi^k(\mathbf{0}_{J_k}) = 0$. Thus we conclude that $\mathbf{1}_{I_l} \not\leq \mathbf{0}_{J_k}$, or, equivalently, $J_k \cap I_l \neq \emptyset$.

Let us assume next that Ψ^k does not represent a positive Boolean function. In particular, Ψ^k does not represent $\bigvee_{\substack{l=1\\l\neq k}}^m \bigwedge_{j\in I_l} x_j$. Hence there exists $l \neq k$ and there exists a set S containing I_l such that $\Psi^k(\mathbf{1}_S) = 0$. On the other hand, since Ψ defines a positive Boolean function, $\Psi(\mathbf{1}_S) = 1$ must hold (by Lemma 2.1). This implies that $T_k(\mathbf{1}_S) = 1$, i.e., $J_k \cap S = \emptyset$. Therefore, $J_k \cap I_l = \emptyset$ follows. $\square$

3. Shellable DNFs. As mentioned earlier, any positive Boolean function can be represented by a variety of DNFs. We now introduce one particular way of generating such a DNF representation.

In what follows, the symbol $\mathcal{I}$ always denotes an arbitrary family of subsets of $\{1, 2, \ldots, n\}$, and π denotes a permutation of the sets in $\mathcal{I}$. Let us denote by $\pi(I)$ the rank of the set $I \in \mathcal{I}$ (i.e., its placement order) in the order of π.

DEFINITION 3.1. *For every family $\mathcal{I}$ of subsets of $\{1, 2, \ldots, n\}$, every permutation π of the sets in $\mathcal{I}$, and every set $I \in \mathcal{I}$, the $(\mathcal{I}, \pi)$-*shadow *$J_{\mathcal{I},\pi}(I)$ of I is the set*

$$J_{\mathcal{I},\pi}(I) = \{j \in \{1, 2, \ldots, n\} \mid \exists\ I' \in \mathcal{I},\ \pi(I') < \pi(I),\ I' \setminus I = \{j\}\}. \tag{3.1}$$

LEMMA 3.1. *For every permutation π of the sets of $\mathcal{I}$, the positive Boolean function $f = \bigvee_{I\in\mathcal{I}} \bigwedge_{j\in I} x_j$ is represented by the DNF*

$$\Psi_{\mathcal{I},\pi} = \bigvee_{I\in\mathcal{I}} \bigwedge_{j\in I} x_j \bigwedge_{j\in J_{\mathcal{I},\pi}(I)} \overline{x}_j. \tag{3.2}$$

Proof. Clearly, $f(\mathbf{x}) \geq \Psi_{\mathcal{I},\pi}(\mathbf{x})$ for every Boolean vector $\mathbf{x}$. In order to prove the reverse inequality, let us consider any Boolean vector $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = 1$. Denote by $I \in \mathcal{I}$ the first set (according to the permutation π) for which $\bigwedge_{j\in I} x_j^* = 1$. We claim that $x_j^* = 0$ for all $j \in J_{\mathcal{I},\pi}(I)$, from which there follows $\bigwedge_{j\in I} x_j^* \bigwedge_{j\in J_{\mathcal{I},\pi}(I)} \overline{x}_j^* = 1$ and $\Psi_{\mathcal{I},\pi}(\mathbf{x}^*) = 1$, as required. To establish the claim, notice that, for every $j \in J_{\mathcal{I},\pi}(I)$, there is a set $I' \in \mathcal{I}$ such that $I' \subseteq I \cup \{j\}$ and $\pi(I') < \pi(I)$. By choice of I, $\bigwedge_{k\in I'} x_k^* = 0$, and thus $x_j^* = 0$. $\square$

Example 3.1. Let us consider the family $\mathcal{I} = \{I_1 = \{1, 2\}, I_2 = \{2, 3\}, I_3 = \{3, 4\}\}$ and the permutation $\pi = (I_1, I_3, I_2)$. Then $J_{\mathcal{I},\pi}(I_1) = J_{\mathcal{I},\pi}(I_3) = \emptyset$, $J_{\mathcal{I},\pi}(I_2) = \{1, 4\}$, and thus the positive Boolean function $f = x_1x_2 \vee x_2x_3 \vee x_3x_4$ is also represented by the DNF

$$f = \Psi_{\mathcal{I},\pi} = x_1x_2 \vee x_3x_4 \vee \overline{x}_1 x_2 x_3 \overline{x}_4.$$

The notion of "shadow" has been put to systematic use by Ball and Provan [2] in their discussion of shellability and upper bounding procedures for reliability problems, and by Boros [9] in his work on "aligned" Boolean functions (a special class of shellable functions). Let us now recall one of the definitions of shellable DNFs.

DEFINITION 3.2. *A positive DNF $\Psi = \bigvee_{I\in\mathcal{I}} \bigwedge_{j\in I} x_j$ is called* shellable *if there exists a permutation π of $\mathcal{I}$ (called* shelling order *of $\mathcal{I}$, or of Ψ) with the following property: for every pair of sets $I_1, I_2 \in \mathcal{I}$ with $\pi(I_1) < \pi(I_2)$, there exists $j \in I_1 \cap J_{\mathcal{I},\pi}(I_2)$ (or equivalently: there exists $j \in I_1$ and $I_3 \in \mathcal{I}$ such that $\pi(I_3) < \pi(I_2)$ and $I_3 \setminus I_2 = \{j\}$).*

Definition 3.2 is due to Ball and Provan [2], who observe that it is essentially equivalent (up to complementation of all sets in $\mathcal{I}$) to the "classical" definition of shellability used, for instance, in [8, 11, 16]. The connection between Lemma 3.1 and the notion of shellability is clarified in the next lemma (this result is implicit in [2], where alternative characterizations of shellability can also be found).

LEMMA 3.2. *Permutation π is a shelling order of $\mathcal{I}$ if and only if the DNF $\Psi_{\mathcal{I},\pi}$ defined by* (3.2) *is orthogonal.*

Proof. Consider any two terms, e.g., $T_1 = \bigwedge_{j\in I_1} x_j \bigwedge_{j\in J_{\mathcal{I},\pi}(I_1)} \overline{x}_j$ and $T_2 = \bigwedge_{j\in I_2} x_j \bigwedge_{j\in J_{\mathcal{I},\pi}(I_2)} \overline{x}_j$ of $\Psi_{\mathcal{I},\pi}$, for which $\pi(I_1) < \pi(I_2)$.

Assume first that π is a shelling order of $\mathcal{I}$. By Definition 3.2, there is an index j in $I_1 \cap J_{\mathcal{I},\pi}(I_2)$. This shows that $\Psi_{\mathcal{I},\pi}$ is orthogonal.

Conversely, assume that $\Psi_{\mathcal{I},\pi}$ is orthogonal. If $I_1 \cap J_{\mathcal{I},\pi}(I_2)$ is nonempty, then I_1 and I_2 satisfy the condition in Definition 3.2. So, assume now that $I_1 \cap J_{\mathcal{I},\pi}(I_2) = \emptyset$, and assume further that $I \cap J_{\mathcal{I},\pi}(I_2) \neq \emptyset$ for all $I \in \mathcal{I}$ such that $\pi(I) < \pi(I_1)$ (if this is not the case, simply replace I_1 by I in the proof). Since $\Psi_{\mathcal{I},\pi}$ is orthogonal, there must be some index j in $I_2 \cap J_{\mathcal{I},\pi}(I_1)$. By Definition 3.1, there exists a set $I_3 \in \mathcal{I}$ with $\pi(I_3) < \pi(I_1)$ such that $I_3 \setminus I_1 = \{j\}$. Now, we derive the following contradiction: on the one hand, by our choice of I_1, $I_3 \cap J_{\mathcal{I},\pi}(I_2)$ may not be empty (since $\pi(I_3) < \pi(I_1)$); on the other hand, $I_3 \cap J_{\mathcal{I},\pi}(I_2)$ must be empty, since $j \notin J_{\mathcal{I},\pi}(I_2)$ and $I_1 \cap J_{\mathcal{I},\pi}(I_2) = \emptyset$. $\square$

Observe that the DNF $\Psi_{\mathcal{I},\pi}$ associated to a shelling order π of $\mathcal{I}$ is orthogonal in a rather special way: namely, for any two terms T_1 and T_2 such that $\pi(I_1) < \pi(I_2)$, the "positive part" $\bigwedge_{j\in I_1} x_j$ of the first term is orthogonal to the "negative part" $\bigwedge_{j\in J_{\mathcal{I},\pi}(I_2)} \overline{x}_j$ of the second term (this follows directly from Definition 3.2).

As one may expect, not every positive DNF is shellable: a minimal counterexample is provided by the DNF

$$\Phi(x_1, \dots, x_4) = x_1x_2 \vee x_3x_4.$$

On the other hand, it can be shown that every positive Boolean function can be represented by shellable DNFs (see also [9, Theorem 1]).

THEOREM 3.3. *Every positive Boolean function f can be represented by a shellable DNF.*

Proof. As a first proof, let us consider the DNF

$$\Phi = \bigvee_{I\in\mathcal{I}} \bigwedge_{j\in I} x_j,$$

where $\mathcal{I}$ denotes the family of all implicants of the function f, and let π be a permutation ordering these implicants in a nonincreasing order by their cardinality. Then Φ represents f, and it is easy to see by Definition 3.2 that π is a shelling order of Φ.

Since the above DNF can, in general, be very large compared to the number of prime implicants of f, let us show below another construction, using only a smaller subset of the implicants.

Call a *leftmost implicant* of f any implicant I of f for which $I \setminus \{h(I)\}$ is not an implicant of f, where $h(I)$ denotes the highest-index element of the subset I. Let $\mathcal{L}$ denote the family of leftmost implicants of f. Clearly, all prime implicants of f are in $\mathcal{L}$; therefore f is represented by the DNF $\Psi_{\mathcal{L}} = \bigvee_{I\in\mathcal{L}} \bigwedge_{j\in I} x_j$. Let us now consider the permutation π of $\mathcal{L}$ induced by the lexicographic order of these implicants. We claim that π is a shelling order of $\mathcal{L}$.

To prove the claim, let I_1 and I_2 be two leftmost implicants of f with $I_1 \prec_L I_2$, and let $j = \min\{i | i \in I_1 \setminus I_2\}$. If $j = h(I_1)$, then $j \in I_1 \cap J_{\mathcal{I},\pi}(I_2)$ (take $I_3 = I_1$ in Definition 3.2), and we are done. So, assume next that $j < h(I_1)$. Let $T = I_2 \cup \{j\}$ and denote by $j_1, \ldots, j_h$ the elements of $\{i \in I_2 | i \geq j\}$ ordered by increasing value: $j = j_1 < j_2 < \cdots < j_h = h(I_2)$. Clearly, T is an implicant of f, while $T \setminus \{j_2, j_3, \ldots, j_h\}$ is not (since the latter set is contained in $I_1 \setminus \{h(I_1)\}$). Consider now the last implicant in the sequence T, $T \setminus \{j_h\}$, $T \setminus \{j_{h-1}, j_h\}, \ldots$, $T \setminus \{j_2, j_3, \ldots, j_h\}$, and call it I_3. By definition, I_3 is a leftmost implicant of f. Moreover, $I_3 \prec_L I_2$ and $I_3 \setminus I_2 = \{j\}$. Thus, here again $j \in I_1 \cap J_{\mathcal{I},\pi}(I_2)$, and we conclude that π is a shelling order of $\mathcal{L}$. ∎

Example 3.2. The leftmost implicants of the function $f(x_1, \ldots, x_4) = x_1x_2 \vee x_3x_4$ are the sets $\{1, 2\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, and $\{3, 4\}$, listed here in lexicographic order. The corresponding DNF

$$\Psi_{\mathcal{L}} = x_1x_2 \vee x_1x_3x_4 \vee x_2x_3x_4 \vee x_3x_4$$

represents f and is shellable, since the DNF

$$\Psi_{\mathcal{L},\prec_L} = x_1x_2 \vee x_1\overline{x}_2x_3x_4 \vee \overline{x}_1x_2x_3x_4 \vee \overline{x}_1\overline{x}_2x_3x_4$$

(which also represents f, by Lemma 3.1) is orthogonal.

Observe that, as illustrated by the above example, the number of leftmost implicants of a positive function f is usually (much) larger than the number p_f of its prime implicants. As a matter of fact, one can construct functions with n variables and p_f prime implicants for which the smallest shellable DNF representation involves a number of terms that grows exponentially with n and p_f. A proof of this statement will be provided in the next section.

In the remainder of this section, we concentrate on characterizing those ODNFs that arise from shellable DNFs in the following sense. Let us consider an arbitrary DNF

$$\Psi(x_1, \ldots, x_n) = \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j \bigwedge_{j \in J_k} \overline{x}_j. \tag{3.3}$$

We say that DNF Ψ is a *shelled ODNF* if Ψ is orthogonal and Ψ is of the form $\Psi_{\mathcal{I},\pi}$ (see (3.2)), where $\mathcal{I} = \{I_1, \ldots, I_m\}$ and π is a shelling order of $\mathcal{I}$.

The *initial segments* of Ψ are the m DNFs $\Psi_1, \ldots, \Psi_m$ defined by

$$\Psi_l(x_1, \ldots, x_n) = \bigvee_{k=1}^{l} \bigwedge_{j \in I_k} x_j \bigwedge_{j \in J_k} \overline{x}_j \tag{3.4}$$

for $l = 1, 2, \ldots, m$. The next lemma provides a partial characterization of shelled ODNFs.

LEMMA 3.4. *For a DNF Ψ of the form* (3.3)*, the following statements are equivalent.*

(i) *Ψ is orthogonal and Ψ is of the form $\Psi_{\mathcal{I},id}$, where id denotes the identity permutation $(I_1, \ldots, I_m)$.*

(ii) *$I_l \cap J_k \neq \emptyset$ for all $1 \leq l < k \leq m$, and, for every $k \leq m$ and every index $j \in J_k$, there exists $l < k$ such that $I_l \setminus I_k = \{j\}$.*

(iii) *Ψ is orthogonal and each initial segment of Ψ represents a positive function.*

Proof. The equivalence of statements (i) and (ii) follows easily from Definition 3.1 and from the comments formulated after the proof of Lemma 3.2.

In view of Lemma 3.1, statement (i) implies statement (iii).

Finally, let us assume that statement (iii) holds, and let us establish statement (ii). Repeated use of Lemma 2.3 implies that $I_l \cap J_k \neq \emptyset$ for all $1 \leq l < k \leq$

m. Together with Lemma 2.2, this also implies that J_k is a minimal transversal of $\mathcal{A}_k = \{I_1, I_2, \ldots, I_{k-1}\}$ for every $1 < k \le m$. Fix $j \in J_k$ and consider the set $L(j) = \{l \,|\, 1 \le l < k, I_l \cap J_k = \{j\}\}$. Since J_k is a *minimal* transversal of $\mathcal{A}_k$, $L(j)$ is not empty. Moreover, for each $l \in L(j)$, $j \in I_l \setminus I_k$ (since I_k and J_k are disjoint). Now there are two cases.

• There is some $l \in L(j)$ such that $I_l \setminus I_k = \{j\}$: then statement (ii) holds, and we are done.

• For all $l \in L(j)$, there exists $i_l \in \{1, 2, \ldots, n\}, i_l \ne j$, such that $i_l \in I_l \setminus I_k$. In this case, the set $S := (J_k \setminus \{j\}) \cup \{i_l \mid l \in L(j)\}$ is a transversal of $\mathcal{A}_k$, is disjoint of I_k, and does not contain J_k. But this contradicts Lemma 2.2, and so the proof is complete. □

Notice that, for a positive DNF $\Psi = \bigvee_{I \in \mathcal{I}} \bigwedge_{j \in I} x_j$ and a permutation π of $\mathcal{I}$, Definition 3.2 provides a straightforward polynomial-time procedure to test whether π is a shelling order of Ψ. By contrast, the complexity of recognizing shellable DNFs is an important and intriguing open problem (mentioned, for instance, in [2, 11]). We shall return to this issue in section 5. For now, let us show that Lemma 3.4 allows for easy recognition of shelled ODNFs, even when π is not given.

THEOREM 3.5. *One can test in polynomial time whether a given DNF is a shelled ODNF.*

Proof. Consider a DNF $\Psi = \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j \bigwedge_{j \in J_k} \overline{x}_j$, as in (3.3). First, we can easily check in polynomial time whether Ψ is orthogonal. In the affirmative, then we test whether Ψ represents a positive Boolean function: in view of Lemma 2.1, it suffices to test whether Ψ represents the function $f = \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j$ or, equivalently, whether $\Psi(\mathbf{1}_{I_k}) \equiv 1$ for $k = 1, 2, \ldots, m$; this is again polynomial, since Ψ is orthogonal.

Now we try to find the last term in the shelling order of f (assuming that there is one) with the help of Lemma 3.4(iii) and Lemma 2.3. Indeed, in view of these lemmas, I_k ($k \in \{1, 2, \ldots, m\}$) is a *candidate* for being the last term in a shelling order of f if and only if $J_k \cap I_l \ne \emptyset$ for all $l \in \{1, 2, \ldots, m\} \setminus \{k\}$. This condition can be tested easily in polynomial time. Moreover, it is clear that every candidate remains a candidate after deletion of any other term. Thus by Lemma 2.3, we can choose an arbitrary candidate as the last term, delete it from Ψ, and repeat the procedure with the remaining terms.

We conclude that Ψ is a shelled ODNF if and only if this procedure terminates with a complete order of its terms. □

4. Dualization of shellable DNFs.

Let us start by recalling some definitions and facts about dualization (see, e.g., [12, 20] for more information). The *dual* of a Boolean function $f(\mathbf{x})$ is the Boolean function $f^d(\mathbf{x})$ defined by

$$f^d(x_1, x_2, \ldots, x_n) = \overline{f}\,(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n).$$

It is well known that the dual of a positive function is positive. If $f = \bigvee_{I \in \mathcal{I}} \bigwedge_{j \in I} x_j$, then $f^d = \bigwedge_{I \in \mathcal{I}} \bigvee_{j \in I} x_j$ (by De Morgan's laws), and a DNF representation of f^d can be obtained by applying the distributive laws to the latter expression. As a result, it is easy to see that the prime implicants of f^d are exactly the minimal transversals of the family of prime implicants of f, i.e., $\mathcal{P}_f$. In the context of reliability theory, the prime implicants of f^d represent the *minimal cutsets* of the system under study, namely, the minimal subsets of elements whose failure causes the whole system to fail (see [3, 25]).

The *dualization problem* can now be stated as follows: given the list of prime implicants of a positive Boolean function f (or, more generally, given a positive DNF

Ψ representing f), compute all prime implicants of f^d. Because of the fundamental role played by duality in many applications, the dualization problem has attracted some attention in the literature (sometimes under the name of "inversion" or "complementation" problem; see, e.g., [17, 26] and the thorough discussions in [6, 12]). The question of the algorithmic complexity of dualization, however, has not been completely settled yet. Observe that, in general, f^d may have many more prime implicants than f, as illustrated by the following example.

Example 4.1. For each $n \geq 1$, define the function

$$h_n(x_1, x_2, \ldots, x_{2n}) = \bigvee_{j=1}^{n} x_{2j-1}x_{2j}.$$

Then h_n has n prime implicants, but its dual is easily seen to have 2^n prime implicants (each prime implicant of h_n^d contains exactly one of the indices $2j-1$ and $2j$ for $j = 1, 2, \ldots, n$).

Recently, Fredman and Khachiyan [13] gave a dualization algorithm which runs in time $O(L^{o(\log L)})$ on an arbitrary positive function f, where L is the number of prime implicants of f and f^d. The existence of a dualization algorithm whose running time is bounded by a polynomial of L is, however, still an open problem. (It is generally assumed that such a dualization algorithm does not exist; see, e.g., [6, 12, 15] for further discussion.)

In this section, we are going to prove that shellable DNFs can be dualized in time polynomial in their input size. Notice that this implies, in particular, that the number of prime implicants of the dual is polynomially bounded in the number of prime implicants of the shellable DNF. These results generalize a sequence of previous results on regular and aligned DNFs, since these are special classes of shellable DNFs (see [5, 9, 10, 23, 24]).

We first state an easy lemma.

LEMMA 4.1. *If $I_1, I_2 \in \mathcal{I}$, and $I_1 \subset I_2$, then $\pi(I_2) < \pi(I_1)$ in any shelling order of $\mathcal{I}$.*

Proof. This is an immediate consequence of Definition 3.2. ∎

THEOREM 4.2. *If a positive Boolean function $f(x_1, x_2, \ldots, x_n)$ can be represented by a shellable DNF of m terms, then its dual f^d can be represented by a shellable DNF of at most nm terms.*

Proof. We prove the theorem by induction on m.

If f has a shellable DNF consisting of 1 term, i.e., if $f = \bigwedge_{j \in I} x_j$ is an elementary conjunction, then its dual is represented by $\bigvee_{j \in I} x_j$, which is a shellable DNF with at most n terms.

Let us now assume that the statement has been established for functions representable by shellable DNFs of at most $m-1$ terms. Let $\Psi = \bigvee_{I \in \mathcal{I}} \bigwedge_{j \in I} x_j$ be a shellable DNF of f and σ be a shelling order of $\mathcal{I}$.

Then, by Lemma 3.2,

$$\Psi_{\mathcal{I},\sigma} = g \vee \bigwedge_{j \in A} x_j \bigwedge_{j \in B} \overline{x}_j$$

is an ODNF of f, where A is the last element of $\mathcal{I}$ according to σ, B is the $(\mathcal{I}, \sigma)$-shadow of A, and g is the function represented by the disjunction of the first $m-1$ terms of Ψ.

Notice that g is a positive function (by Lemma 3.4(iii)) and that g has a shellable DNF of $m-1$ terms. So, according to the induction hypothesis, g^d has a shellable DNF of at most $n(m-1)$ terms. Let us write

$$g^d = \bigvee_{R\in\mathcal{R}} \bigwedge_{j\in R} x_j,$$

where $\mathcal{R}$ is a family of implicants of g^d containing all of its prime implicants, $|\mathcal{R}| \le n(m-1)$, and $\mathcal{R}$ admits a shelling order that we denote by π.

By Lemmas 2.2 and 2.3, B is a prime implicant of g^d (so that $B \in \mathcal{R}$), and, for all other sets $R \in \mathcal{R}\setminus\{B\}$, there holds $R\cap A \neq \emptyset$. Hence, for all $R \in \mathcal{R}\setminus\{B\}$, we have the identity

$$\left(\bigwedge_{j\in R} x_j\right)\left(\bigvee_{j\in A} x_j \bigvee_{j\in B} \overline{x}_j\right) = \bigwedge_{j\in R} x_j,$$

and for $R = B$ we have

$$\left(\bigwedge_{j\in B} x_j\right)\left(\bigvee_{j\in A} x_j \bigvee_{j\in B} \overline{x}_j\right) = \bigvee_{i\in A} \bigwedge_{j\in B\cup\{i\}} x_j.$$

Thus we can write

$$f^d = g^d\left(\bigvee_{j\in A} x_j \bigvee_{j\in B} \overline{x}_j\right) = \left(\bigvee_{R\in\mathcal{R}\setminus\{B\}} \bigwedge_{j\in R} x_j\right) \vee \left(\bigvee_{i\in A} \bigwedge_{j\in B\cup\{i\}} x_j\right). \tag{4.1}$$

Let $B_i = B\cup\{i\}$ for all $i\in A$, and define

$$\mathcal{R}' = (\mathcal{R}\setminus\{B\}) \cup \{B_i \,|\, i\in A \text{ and } \nexists R\in\mathcal{R} \text{ with } (R\subseteq B_i, \pi(R)<\pi(B))\}.$$

In view of (4.1), the DNF

$$\Phi = \bigvee_{R\in\mathcal{R}'} \bigwedge_{j\in R} x_j \tag{4.2}$$

represents f^d. We are going to show that $\mathcal{R}'$ admits a shelling order. Since $|\mathcal{R}'| < mn$, this will complete the proof.

Let us define a permutation π' of $\mathcal{R}'$ by inserting the sets B_i ($i \in A, B_i \in \mathcal{R}'$) in place of B in π. More precisely, for each pair of sets $R, S \in \mathcal{R}'$, $R \neq S$, we let $\pi'(R) < \pi'(S)$ if any of the following holds:

- $R, S \in \mathcal{R}$ and $\pi(R) < \pi(S)$, or
- $R \in \mathcal{R}$, $S = B_i$ for some $i \in A$, and $\pi(R) < \pi(B)$, or
- $R = B_i$ for some $i \in A$, $S \in \mathcal{R}$, and $\pi(B) < \pi(S)$, or
- $R = B_i$ and $S = B_j$ for some $i, j \in A$ and $i < j$.

We claim that π' is a shelling order of $\mathcal{R}'$. To prove the claim, let us show first that

$$J_{\mathcal{R}',\pi'}(R) \supseteq J_{\mathcal{R},\pi}(R) \tag{4.3}$$

for every $R \in \mathcal{R}\cap\mathcal{R}'$. If $\pi(R) < \pi(B)$, this is obvious. So let us assume that $\pi(R) > \pi(B)$, and let $j \in J_{\mathcal{R},\pi}(R)$. If $\{j\} = S\setminus R$ for some $S \in \mathcal{R}\cap\mathcal{R}'$, then clearly

$j \in J_{\mathcal{R}',\pi'}(R)$ too. On the other hand, if $\{j\} = B \setminus R$, let i be any element in $A \cap R$ (remember that $A \cap R \neq \emptyset$ by Lemma 2.2). Then $\{j\} = B_i \setminus R$. If $B_i \in \mathcal{R}'$, we conclude again that $j \in J_{\mathcal{R}',\pi'}(R)$. If $B_i \notin \mathcal{R}'$, there is a set $S \in \mathcal{R} \cap \mathcal{R}'$ such that $\pi(S) < \pi(B)$ and $S \subseteq B_i$. Moreover, $j \in S$ since otherwise $S \subset R$ would follow, contradicting Lemma 4.1. Thus $\{j\} = S \setminus R$, implying again $j \in J_{\mathcal{R}',\pi'}(R)$. This establishes (4.3).

Let us show next that

$$J_{\mathcal{R}',\pi'}(B_i) \supseteq J_{\mathcal{R},\pi}(B) \cup \{j \in A \,|\, B_j \in \mathcal{R}', j < i\} \tag{4.4}$$

for every $B_i \in \mathcal{R}'$. Clearly, if $B_j \in \mathcal{R}'$ and $j < i$, then $\{j\} = B_j \setminus B_i$, and thus $j \in J_{\mathcal{R}',\pi'}(B_i)$. If $j \in J_{\mathcal{R},\pi}(B)$, let $R \in \mathcal{R}$ be such that $\pi(R) < \pi(B)$ and $\{j\} = R \setminus B$. Since $R \subseteq B_j$ and $B_i \in \mathcal{R}'$, we deduce $j \neq i$, and thus $\{j\} = R \setminus B_i$, which implies $j \in J_{\mathcal{R}',\pi'}(B_i)$.

Relations (4.3) and (4.4), together with the hypothesis that π is a shelling order of $\mathcal{R}$, prove that the DNF $\Phi_{\mathcal{R}',\pi'}$ associated to (4.2) is orthogonal. Hence by Lemma 3.2, π' is a shelling order of $\mathcal{R}'$ and the proof is complete. ∎

As a side remark, we notice that equality actually holds in relations (4.3) and (4.4). More interestingly, we can now prove the following result.

COROLLARY 4.3. *Let $f(x_1, x_2, \ldots, x_n)$ be a positive Boolean function. Given Ψ, a positive DNF of f, and given π, a shelling order of Ψ, we can generate all prime implicants of f^d in $O(nm^2)$ time, where m is the number of terms of Ψ.*

Proof. Let $\Psi = \bigvee_{k=1}^{m} \bigwedge_{j \in I_k} x_j$ and assume that the identity permutation $(I_1, \ldots, I_m)$ is a shelling order of Ψ. The proof of Theorem 4.2 immediately suggests a recursive dualization procedure, whereby the initial segments $\Psi_1, \Psi_2, \ldots, \Psi_m = \Psi$ (see (3.4)) are sequentially dualized. Using relation (4.1), all prime implicants of Ψ_{i+1} can easily be generated in $O(nm)$ time once the prime implicants of Ψ_i are known for $i = 1, 2, \ldots, m-1$. The overall $O(nm^2)$ time bound follows. ∎

Let us mention here that in the special cases of aligned and regular functions, there are more efficient dualization algorithms known in the literature (see, e.g., [5, 9, 10, 24]), which run in $O(n^2 m)$ time. None of those procedures, however, seem to be extendable for the class of shellable functions.

In the previous section, we have established that every positive function can be represented by a shellable DNF (Theorem 3.3). This result, combined with Theorem 4.2, might raise the impression that every positive function can be dualized in polynomial time. This, however, is *not* the case. In fact, there exist positive Boolean functions in $2n$ variables which have only n prime implicants but for which every shellable DNF representation involves at least $2^n - 1$ terms. Consider, e.g., the family of functions h_n ($n = 1, 2, \ldots$) introduced in Example 4.1. It was shown in [1] that any ODNF, thus in particular any shellable DNF of h_n, must have at least $2^n - 1$ terms.

Observing the similarity between dualization and orthogonalization procedures, one might think that the main reason one needs so many terms in an ODNF of h_n is that its dual h_n^d has many prime implicants. (As we observed in Example 4.1, h_n^d has 2^n prime implicants.) While this might be true, such a direct relation between the size of an ODNF of a Boolean function f and the size of its dual f^d, as far as we know, has not been established yet.

5. The LE property for DNFs.

As mentioned before, the computational complexity of recognizing shellable DNFs is currently unknown (see [2, 11]). In this section, we consider a closely related problem, namely, the problem of recognizing DNFs with the so-called *LE property*, and we show that this problem is NP-complete.

DEFINITION 5.1. *A positive DNF* $\Psi(x_1, x_2, \dots, x_n) = \bigvee_{I \in \mathcal{I}} \bigwedge_{j \in I} x_j$ *has the* LE property with respect to $(x_1, x_2, \dots, x_n)$ *if, for every pair of terms* $I_1, I_2 \in \mathcal{I}$ *with* $I_1 \prec_L I_2$*, there exists* $I_3 \in \mathcal{I}$ *such that* $I_3 \prec_L I_2$ *and* $I_3 \setminus I_2 = \{j\}$*, where* $j = \min\{i \mid i \in I_1 \setminus I_2\}$.

We say that Ψ *has the* LE property with respect to *a permutation* σ *of* $(x_1, x_2, \dots, x_n)$*, or that* σ *is an* LE order for Ψ*, if*

$$\Psi^{\sigma}(x_1, \dots, x_n) = \bigvee_{I \in \mathcal{I}} \bigwedge_{j \in I} \sigma(x_j)$$

has the LE property with respect to $(x_1, x_2, \dots, x_n)$.

Finally, we simply say that Ψ *has the* LE property *if* Ψ *has the LE property with respect to some permutation of its variables.*

The LE property has been studied extensively by Ball and Provan [2, 22]. The interest in this concept is motivated by the simple observation that every DNF with the LE property is also shellable: more precisely, if σ is an LE order for Ψ, then the lexicographic order is a shelling order of Ψ^{σ} (just compare Definition 3.2 and Definition 5.1). As a matter of fact, most classes of shellable DNFs investigated in the literature do have the LE property (see [2, 9]).

In view of Definition 5.1, verifying whether a DNF Ψ has the LE property with respect to $(x_1, x_2, \dots, x_n)$ can easily be done in polynomial time. Provan and Ball [22] present an $O(n^2 m)$ procedure for this problem, where n is the number of variables and m is the number of terms of Ψ. However, these authors also point out that the existence of an efficient procedure to determine whether a given DNF has the LE property (with respect to *some* unknown order of its variables) is an "interesting open question." The remainder of this paper will be devoted to a proof that such an efficient procedure is unlikely to exist.

THEOREM 5.1. *It is NP-complete to decide whether a given DNF has the LE property.*

The proof of Theorem 5.1 involves a transformation from the following *balanced partition* problem.

Input: A finite set V and a family $\mathcal{H} = \{H_1, H_2, \dots, H_m\}$ of subsets of V such that $|H_i| = 4$ for $i = 1, 2, \dots, m$.

Question: Is there a *balanced partition* of $(V, \mathcal{H})$, i.e., a partition of V into $V_1 \cup V_2$ such that $|V_1 \cap H_i| = |V_2 \cap H_i| = 2$ for $i = 1, 2, \dots, m$?

LEMMA 5.2. *The balanced partition problem is NP-complete.*

Proof. We provide a transformation from *hypergraph* 2*-colorability* to balanced partition, where hypergraph 2-colorability is defined as follows.

Input: A finite set X and a family $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$ of subsets of X such that $|E_i| = 3$ for $i = 1, 2, \dots, m$.

Question: Is there a 2*-coloring* of $(X, \mathcal{E})$, i.e., a partition of X into $X_1 \cup X_2$ such that $X_1 \cap E_i \neq \emptyset$ and $X_2 \cap E_i \neq \emptyset$ for $i = 1, 2, \dots, m$?

Hypergraph 2-colorability is NP-complete (see [14]). Given an instance $(X, \mathcal{E})$ of this problem, we let $V = X \cup \{e_1, e_2, \dots, e_m\}$, where $e_1, e_2, \dots, e_m$ are m new elements, and we let $\mathcal{H} = \{H_1, H_2, \dots, H_m\}$, where $H_i = E_i \cup \{e_i\}$ for $i = 1, 2, \dots, m$. Then $(X, \mathcal{E})$ has a 2-coloring if and only if $(V, \mathcal{H})$ has a balanced partition. ◻

The proof of Theorem 5.1 also requires a series of technical lemmas (the proofs of which may be skipped in a first reading).

LEMMA 5.3. *If the DNF* $\Psi(x_1, x_2, \dots, x_n) = \bigvee_{I \in \mathcal{I}} \bigwedge_{j \in I} x_j$ *has the LE property with respect to* $(x_1, x_2, \dots, x_n)$*, then the DNF* $\Psi|_{x_i=0}$ *obtained by fixing* x_i *to* 0 *in* Ψ, *that is,*

$$\Psi|_{x_i=0}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = \bigvee_{\substack{I\in\mathcal{I}\\ i\notin I}} \bigwedge_{j\in I} x_j,$$

has the LE property with respect to $(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ *for all* $i=1,2,\ldots,n$.

Proof. This is a straightforward consequence of Definition 5.1. ∎

LEMMA 5.4. *The DNF*

$$\phi(a,b,c,d,y) = abcd \vee aby \vee acy \vee ady \vee bcy \vee bdy \vee cdy$$

does not have the LE property with respect to any permutation of $\{a,b,c,d,y\}$ *in which* y *has rank either* 4 *or* 5*. On the other hand,* ϕ *has the LE property with respect to all permutations in which* y *has rank* 3*.*

Proof. Consider first any permutation π of $\{a,b,c,d,y\}$ in which y has rank 5. By symmetry, we can assume without loss of generality that $\pi=(a,b,c,d,y)$. Then the first two terms of ϕ in lexicographic order are $I_1 = abcd$ and $I_2 = aby$, and these terms do not fulfill the condition in Definition 5.1.

An identical reasoning applies when y has rank 4 (say, as in $\pi=(a,b,c,y,d)$).

Assume next that y has rank 3 in π. By symmetry, we can assume that $\pi = (a,b,y,c,d)$. Then the terms of ϕ satisfy

$$aby \prec_L abcd \prec_L acy \prec_L ady \prec_L bcy \prec_L bdy \prec_L cdy,$$

and it is easy to verify that ϕ has the LE property with respect to π. ∎

LEMMA 5.5. *The DNF*

$$\theta(a,b,c,d,y) = aby \vee cdy \vee abc \vee abd \vee acd \vee bcd$$

does not have the LE property with respect to any permutation of $\{a,b,c,d,y\}$ *in which* y *has rank* 1*, nor with respect to any of the eight permutations* (a,y,c,d,b), (a,y,d,c,b), (b,y,c,d,a), (b,y,d,c,a), (c,y,a,b,d), (c,y,b,a,d), (d,y,a,b,c), (d,y,b,a,c) *(in words, these are all permutations* $\pi=(\pi_1,\ldots,\pi_5)$ *in which* y *has rank* 2 *and either* $\{\pi_3,\pi_4\}=\{a,b\}$ *or* $\{\pi_3,\pi_4\}=\{c,d\}$*). On the other hand,* θ *has the LE property with respect to all permutations in which* y *has rank* 3*.*

Proof. Consider any permutation π of $\{a,b,c,d,y\}$ in which y has rank 1. Then, $I_1 = aby$ and $I_2 = cdy$ are the first two terms of θ in lexicographic order, and these terms do not satisfy Definition 5.1.

Consider next any of the 8 permutations listed; without loss of generality say $\pi=(a,y,c,d,b)$ (the other cases are similar due to simple symmetries). Then the first two terms of θ are aby and acd, and they violate again Definition 5.1.

Now let π be an arbitrary permutation of $\{a,b,c,d,y\}$ in which y has rank 3. By symmetry, we only need to distinguish between the permutations $\pi^1=(a,b,y,c,d)$, $\pi^2=(a,c,y,b,d)$, and $\pi^3=(a,c,y,d,b)$. The lexicographic order of the terms of θ with respect to π^1 is

$$aby \prec_L abc \prec_L abd \prec_L acd \prec_L bcd \prec_L cdy.$$

Similarly, with respect to π^2 we get

$$abc \prec_L acd \prec_L aby \prec_L abd \prec_L cdy \prec_L bcd,$$

and with respect to π^3 we get

$$acd \prec_L abc \prec_L aby \prec_L abd \prec_L cdy \prec_L bcd.$$

In all three cases, θ has the LE property with respect to the corresponding permutation. $\square$

LEMMA 5.6. *Let $\psi_1, \psi_2, \ldots, \psi_k$ be positive DNFs in the same variables $x_1, \ldots, x_n$, and assume that $\psi_1, \psi_2, \ldots, \psi_k$ all have the LE property with respect to $(x_1, \ldots, x_n)$. Then, the DNF*

$$\Psi(t_1, \ldots, t_k, x_1, \ldots, x_n) = \bigvee_{1 \le i < j \le k} t_i t_j \vee \bigvee_{i=1}^{k} t_i \psi_i(x_1, \ldots, x_n)$$

has the LE property with respect to $(t_1, \ldots, t_k, x_1, \ldots, x_n)$.

Proof. Let I_1 and I_2 be two terms of Ψ with $I_1 \prec_L I_2$ in the lexicographic order induced by $\pi = (t_1, \ldots, t_k, x_1, \ldots, x_n)$. We consider four cases which together exhaust all possibilities.

Case 1: $I_1 = t_i t_j$ and $I_2 = t_i T$, where $i, j \in \{1, \ldots, k\}$ and T is either one of the variables $\{t_{j+1}, \ldots, t_k\}$ or a term of Ψ_i. In both cases, we can set $I_3 = I_1$ in Definition 5.1.

Case 2: $I_1 = t_i t_j$ and $I_2 = t_r T$, where $i, j, r \in \{1, \ldots, k\}$, $i < j$, $i < r$, and T is either one of the variables $\{t_{r+1}, \ldots, t_k\}$ or a term of Ψ_r. Here, we can set $I_3 = t_i t_r$.

Case 3: $I_1 = t_i T_1$ and $I_2 = t_j T_2$, where $i, j \in \{1, \ldots, k\}$, $i < j$, and T_1, T_2 are terms of Ψ_i and Ψ_j, respectively. Then $I_3 = t_i t_j$ satisfies Definition 5.1.

Case 4: $I_1 = t_i T_1$ and $I_2 = t_i T_2$, where $i \in \{1, \ldots, k\}$ and T_1, T_2 are terms of Ψ_i. Since $I_1 \prec_L I_2$ and Ψ_i has the LE property with respect to $(x_1, \ldots, x_n)$, there exists a term of Ψ_i, say, T_3, such that $T_3 \prec_L T_2$ and $T_3 \setminus T_2 = \{\min(j \mid j \in T_1 \setminus T_2)\}$. Then we can set $I_3 = t_i T_3$ in Definition 5.1. $\square$

We are now ready for a proof of Theorem 5.1.

Proof of Theorem 5.1. Let $V = \{1, 2, \ldots, n\}$ and $\mathcal{H} = \{H_1, \ldots, H_m\}$ define an instance of the balanced partition problem. With this instance, we associate $n+1+4m$ variables, denoted x_i $(i = 1, 2, \ldots, n)$, y, and t_{jk} $(j = 1, 2, \ldots, m; k = 1, 2, 3, 4)$. We also define $4m$ DNFs ψ_{jk} $(j = 1, 2, \ldots, m; k = 1, 2, 3, 4)$ as follows: if $H_j = \{i_1, i_2, i_3, i_4\}$, with $i_1 < i_2 < i_3 < i_4$, then we let

$$\begin{aligned}
\psi_{j1}(x_1, x_2, \ldots, x_n, y) &= \phi(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, y),\\
\psi_{j2}(x_1, x_2, \ldots, x_n, y) &= \theta(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, y),\\
\psi_{j3}(x_1, x_2, \ldots, x_n, y) &= \theta(x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}, y),\\
\psi_{j4}(x_1, x_2, \ldots, x_n, y) &= \theta(x_{i_1}, x_{i_4}, x_{i_2}, x_{i_3}, y),
\end{aligned}$$

where ϕ and θ are the functions introduced in Lemma 5.4 and Lemma 5.5, respectively. We look at $\psi_{j1}, \ldots \psi_{j4}$ as DNFs in the variables $(x_1, x_2, \ldots, x_n, y)$.

Now we define

$$\Psi(t_{11}, \ldots, t_{m4}, x_1, \ldots, x_n, y) = \bigvee_{\substack{i,j \in \{1,2,\ldots,m\} \\ l,k \in \{1,2,3,4\} \\ (i,l) \neq (j,k)}} t_{il} t_{jk} \vee \bigvee_{\substack{j \in \{1,2,\ldots,m\} \\ k \in \{1,2,3,4\}}} t_{jk} \psi_{jk}(x_1, \ldots, x_n, y).$$

We claim that Ψ has the LE property if and only if $(V, \mathcal{H})$ has a balanced partition.

Indeed, assume that Ψ has the LE property with respect to some permutation π. We associate with π the following partition of V into $V_1 \cup V_2$:

$$V_1 = \{i \in \{1, 2, \dots, n\} \,|\, x_i \text{ precedes } y \text{ in } \pi\},$$
$$V_2 = \{i \in \{1, 2, \dots, n\} \,|\, x_i \text{ follows } y \text{ in } \pi\}.$$

To show that this partition is balanced, consider an arbitrary subset in $\mathcal{H}$, say, H_1, and assume without loss of generality that $H_1 = \{1, 2, 3, 4\}$. Since Ψ has the LE property with respect to π, we deduce from Lemma 5.3 that each of $\psi_{11}, \psi_{12}, \psi_{13}$, and ψ_{14} has the LE property with respect to the permutation of $\{x_1, x_2, x_3, x_4, y\}$ induced by π (to see this, simply fix all variables t_{jk} to 0 except one of them, e.g., t_{11}).

Now, combining Lemma 5.4 and Lemma 5.5, we conclude that y must have rank 3 in the permutation of $\{x_1, x_2, x_3, x_4, y\}$ induced by π (notice that Lemma 5.5, applied simultaneously to ψ_{12}, ψ_{13}, and ψ_{14}, excludes all 24 permutations in which y has rank 2). Hence, $|V_1 \cap H_1| = |V_2 \cap H_1| = 2$, as required of a balanced partition.

Conversely, assume now that $(V, \mathcal{H})$ has a balanced partition (V_1, V_2). Say without loss of generality that $V_1 = \{1, 2, \dots, l\}$ and $V_2 = \{l + 1, \dots, n\}$, and define the permutation

$$\pi = (t_{11}, t_{12}, \dots, t_{m4}, x_1, x_2, \dots, x_l, y, x_{l+1}, \dots, x_n).$$

Consider any set $H_j \in \mathcal{H}$, say, $H_j = \{i_1, i_2, i_3, i_4\}$. Since (V_1, V_2) is balanced, y has rank 3 in the permutation of $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, y\}$ induced by π. Thus Lemma 5.4 and Lemma 5.5 imply that $\psi_{j1}, \psi_{j2}, \psi_{j3}$, and ψ_{j4} all have the LE property with respect to π. By Lemma 5.6, we conclude that Ψ also has the LE property with respect to π.

This concludes the proof. $\square$

The proof of Theorem 5.1 establishes that testing the LE property is already NP-complete for DNFs of degree 5 or more (if we call *degree* of a DNF the number of literals in its longest term). On the other hand, for a DNF $\Psi = \bigvee_{(i,j)\in E} x_i x_j$ of degree 2, it can be shown that Ψ has the LE property if and only if Ψ is shellable, or equivalently, if and only if the graph $G = (\{1, 2, \dots, n\}, E)$ is cotriangulated (see Theorem 2 in [4]). This implies, in particular, that the LE property can be tested in polynomial time for DNFs of degree 2. The complexity of this problem remains open for DNFs of degree 3 or 4.

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