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# Continuum quantum systems as limits of discrete quantum systems: II. State functions

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**Abstract**

In this second of four papers on the eponymous topic, pointwise convergence of a 'discrete' state function to a 'continuum' state function is shown to imply the algebraic criterion for convergence that was introduced in the prequel. As examples (and as a prerequisite for the sequels), the normal approximation theorem and the convergence of the Kravchuk functions to the Hermite–Gaussians are expressed in terms of the algebraic notion of convergence.

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## 1. Motive

The eventual aim of this paper, and its three companions [5–7] is to characterize, in terms of limits, a link between two algebraic topics, the theory of Wigner distributions, and the theory of the angular momentum algebra $su(2)$. That the two topics are indeed closely related was established long ago by Stratonovich [23], and consolidated by Várilly–Gracia-Bondía [24]. A strong indication that the connection can be expressed using limits appeared in Atakishiyev *et al* [1].

Via the Wigner distribution, or more precisely, via the Weyl–Wigner correspondence, an infinite-dimensional state space is related to a Euclidian phase space. The correspondence is covariant with respect to the group of affine canonical transforms, which is generated by the symplectic transforms and the Heisenberg translates. Littlejohn [15] reviewed, in lucid detail, the algebraic aspects of this covariance. Stratonovich, adapting this idea, showed how a finite-dimensional state space may be related to a spherical phase space. The underlying symmetries here are expressed by the group $SU(2)$. Atakishiyev *et al* observed that the three canonical bases of the irreducible representations of $SU(2)$ (the three sets of eigenvectors of the standard generators of $su(2)$) may be regarded as a basis of position vectors, a basis of momentum vectors, and a basis of harmonic oscillator energy eigenstates.

In the fourth paper of this series, some affine canonical transforms will be realized as limits of finite-dimensional actions of $SU(2)$. In this paper, and the other two, suitable notions of limits are defined and examined. Consider a 'continuum' Hilbert space $\mathcal{L}_\infty$ and 'discrete'

Hilbert spaces $\mathcal{L}_n$ indexed by some variable parameter $n$. For example, $\mathcal{L}_\infty$ might be $L^2(\mathbb{R}^m)$ for some positive integer $m$, while $\mathcal{L}_n$ might be $L^2(\mathbb{Z})$, or some finite-dimensional inner product space. How might objects associated with $\mathcal{L}_\infty$—such as vectors, operators, and quantum systems—be realized as limits of analogous objects associated with $\mathcal{L}_n$? One answer has been supplied by Digernes *et al* [9]. We shall give a different answer, one that is more concerned with preservation of algebraic structure. After all, if our link between the two topics mentioned above is to reflect the rich algebraic features of both, then preservation of algebraic structure must be a requirement.

The third paper concerns convergence of operators and convergence of quantum systems, and explains how, by transcription of differential operators to difference operators, some very simple continuum systems—the circular rotor, the one-dimensional box, and the harmonic oscillator, the fractional Fourier transform—are limits of finite-dimensional systems. All of these examples are well known, and the 'limits' can be recognized using sheer common sense, but a rigorous treatment is helpful preliminary for the more sophisticated application indicted above.

The purpose of this paper is to show that convergence of vectors, as defined generally in the first paper, is in accord with the heuristic pointwise criterion for convergence that is already in frequent use, as in, for instance Atakishiyev–Wolf [3], Atakishiyev *et al* [1], and some works cited (in this connection) in [5, section 1]. Theorem 5.1, below, asserts that the Kravchuk functions converge (in our sense) to the Hermite–Gaussians (the harmonic oscillator energy eigenstates). This will be needed to prove [6, theorem 4.1], which asserts that the Kravchuk function fractional Fourier transform converges (in our sense) to the continuum fractional Fourier transform. In connection with (heuristic versions of) these two results, Atakishiyev–Wolf [3, section 5] wrote

> 'A mathematically precise formulation of this limit in terms of Hilbert spaces should be made, but we leave this rather technical matter for further research'.

Our language and notation is taken from quantum mechanics, because this is the area in which most of our source material resides. We point out, however, that a 'quantum system' is a dynamical system on a Hilbert space; various one-parameter groups of signal transforms are just as much 'quantum systems' as any quantum *mechanical* system. The theory of Wigner distributions is of no less significance to signal analysis and optics than it is to quantum physics. See, for instance, the books by Mecklenbräuker–Hlawatsch [16] and Ozaktas *et al* [19]. The problem of discretizing phase space is of much concern in signal processing, in part because numerical data and numerical calculations are, by nature, discrete.

Let us note (in roughly ascending order of possible physical interest) five general motives for studying correspondences between 'continuum' and 'discrete' quantum systems:

(1) Numerical calculations pertaining to continuum models are often carried out using digital machines. It would be desirable to have a discrete theory that reflects the nature of the calculations. It would also be desirable to have a systematic way of relating the scenario of the calculations and the scenario of the continuum models. Leonhardt [14] has examined discrete Wigner distributions in connection with quantum state tomography. Discrete Wigner distributions and related discrete transforms are of topical interest in signal processing. See, for instance, the works by Richman *et al* [22] and Pei *et al* [20]. The senses in which these and similar discrete constructions *converge* to the continuum model, for large samples, is not mathematically clear.
(2) Finite-dimensional linear algebra is—in theoretical foundation, if not always in practical application—almost trivial in comparison with operator theory for infinite-dimensional

Hilbert spaces. As a theoretical technique, it may be desirable to have a way in which subtle problems concerning infinite-dimensional representations may be reduced to trite problems concerning finite-dimensional representations.

(3) Notwithstanding the premise of (2), the finite-dimensional scenario may sometimes be richer than the continuum scenario. For instance, difference equations often have more families of solutions than the corresponding differential equations. There is the possibility of being able to work in the discrete scenario, deriving results using objects that have no continuum analogues, and then passing to the continuum scenario by taking limits. Hakioğlu [10, 11] and Hakioğlu–Tependelenlioğlu [12] have approached the quantum phase problem in this way, constructing action and angle operators on finite-dimensional state spaces.

(4) Atakishiyev *et al* [1] proposed a physically realizable optical waveguide system with only finitely many pure states. As part of their analysis of the system, they examined, as a limiting case, a continuum system with infinitely many pure states. Any physically realizable optical system can have only finite resolution and finite extent, and hence (in some imprecise sense) can have only finitely many observable states. Yet, when studying such a system, one may wish to use an infinite-state system to investigate the limiting behaviour.

(5) The Weyl–Wigner correspondence, and the representation theory of $SU(2)$ are of considerable interest in fundamental mathematics and mathematical physics. Although connections between the two theories are already recognized, a mathematically precise correspondence would be, presumably, a useful theoretical tool.

## 2. Convergence must preserve inner products

Our 'continuum' and 'discrete' spaces are to be interfaced to each other by means of an *inductive resolution*. The term was defined generally in [5, section 2]. In this paper, we shall confine our attention to 'sample-point' inductive resolutions, as in [5, examples 2.A, 2.B, 2.C]. Let us reintroduce the idea in a more physically compelling manner. As a 'Gedankenexperiment', let us suppose that some function $\psi : \mathbb{R} \to \mathbb{C}$ is playing a role in a mathematical model of a physical system. It is known that $\psi$ is continuous and suitably well behaved, furthermore, approximate measurements of $\psi$ have been taken at sample points. For convenience of discussion, let us suppose that, for some positive integer $n$, and for all integers $X$ with $-n/2 < X \leqslant n/2$, we have a measurement $F(X) \approx \psi(X/\sqrt{n})$. How might we relate the known function $F$ to the unknown function $\psi$?

One answer, which goes back at least as far as Eudoxus (and almost as far back as kindergarten) is to treat the integer $n$ as a variable with no upper bound, and to demand that, for sufficiently large $n$, the errors of measurement are arbitrarily small. That is to say, we might demand that, given any $x \in \mathbb{R}$, and any sequence of integers $(X_n)_n$ satisfying $\lim_{n\to\infty} X_n/\sqrt{n} = x$, then

$$\psi(x) = \lim_{n\to\infty} \Psi_n(X_n).$$

When this condition holds, we say that $\Psi_n$ *pointwise converges* to $\psi$.

Let us now suppose that the 'physical system' is a quantum system (by which we mean a dynamical system on a Hilbert space, for instance: a quantum mechanical system, or a one-parameter group of signal transforms, or a one-parameter group of symmetries of some other quantum system). The structure of a Hilbert space is its linear space structure, together with its inner product. Whatever definition we eventually adopt for convergence of vectors, it must preserve the structure of the Hilbert spaces. Pointwise convergence obviously preserves the linear space structure, but preservation of the inner product is another matter. Consider now two continuous square-integrable functions $\psi, \phi : \mathbb{R} \to \mathbb{C}$. Let $(\Psi_n)_n$ and $(\Phi_n)_n$, respectively, be sequences of approximations pointwise converging to $\psi$ and $\phi$. The inner products are

$$\langle\phi|\psi\rangle = \int_{\mathbb{R}} \overline{\phi}(x)\psi(x)\,\mathrm{d}x \qquad \text{and} \qquad \langle\Phi_n|\Psi_n\rangle = \sum_X \overline{\Phi}_n(X)\Psi_n(X)$$

the sum being indexed by the integers $-n/2 < X \leqslant n/2$. Writing $\psi_n(X) = n^{-1/4}\Psi_n(X)$, and $\phi_n(X) = n^{-1/4}\Phi_n(X)$ similarly, the condition

$$\langle\phi|\psi\rangle = \lim_{n\to\infty} \langle\phi_n|\psi_n\rangle$$

might be desirable. Certainly, the condition does hold if $\psi$ and $\phi$ are suitably nice and if the errors of measurement tend to zero sufficiently fast, indeed, the sum $\langle\phi_n|\psi_n\rangle$ can, in this case, be regarded as a 'Riemann sum'. Alas, the condition, as it stands, makes little sense as a criterion for convergence, because there are two arbitrary sequences involved. But let us say that the sequence $(\psi_n)_n$ *converges* to $\psi$ provided the equation for $\langle\phi|\psi\rangle$ holds whenever there are no errors in the measurements of $\phi$. In other words, writing

$$\mathrm{res}_n(\phi)(X) := n^{-1/4}\phi(X/\sqrt{n})$$

then the sequence $(\psi_n)_n$ converges to $\psi$ if and only if

$$\langle\phi|\psi\rangle = \lim_{n\to\infty} \langle\mathrm{res}_n(\phi)|\psi_n\rangle$$

for all suitably well-behaved $\phi$. We might as well take 'suitably well behaved' to mean that $\phi$ belongs to the Schwarz space $\mathcal{S}(\mathbb{R})$ (the space of rapidly decreasing functions $\mathbb{R} \to \mathbb{C}$).

One attractive feature of the definition of *convergence* is that it can be applied for an arbitrary vector $\psi \in L^2(\mathbb{R})$, irrespective of whether or not the values of $\psi$ at the sample points are defined.

Even in the case where $\psi \in \mathcal{S}(\mathbb{R})$, our criteria for *convergence* and for *pointwise convergence* are logically independent. For instance, if $\Psi(X) = 0$ for $|X| \leqslant n^{2/3}$ and $\Psi_n(X) = \exp(X^2)$ for $|X| > n^{2/3}$, then $\Psi_n$ converges pointwise to 0, but $\langle\mathrm{res}_n(\phi)|\psi_n\rangle \to \infty$ for $\phi(x) = \exp(-x^2)$, hence $\psi_n$ does not converge. (The author thanks a referee for that example.) On the other hand, if $\psi_n(0) = 1$ and $\psi_n(X) = 0$ for all non-zero $X$, then $\psi_n$ converges to 0 but $\Psi_n$ does not converge pointwise. When [8] and [4] were written, the author expected that any 'pointwise' criterion for convergence of vectors would have to include some condition on the speed of convergence of sample-point values. For this reason, the somewhat obscure term *induction* was used in place of *convergence*. However, the following special case of theorem 3.1 shows that pointwise convergence, together with a harmless caveat on the norms $\|\psi_n\|$, does imply convergence.

**Corollary 2.1.** *In the notation above, suppose that* $\psi_\infty \in \mathcal{S}(\mathbb{R})$. *Suppose that the norms* $\|\psi_n\|$ *are bounded, and that* $\Psi_n$ *converges pointwise to* $\psi_\infty$. *Then* $\psi_n$ *converges to* $\psi_\infty$.

In applications to quantum mechanics, the requirement that the norms $\|\psi_n\|$ are bounded is indeed a harmless, in fact, the given state vectors $\psi_n$ are often normalized.

Warning: given $(\psi_n)_n$ and $(\chi_n)_n$ converging to vectors $\psi$ and $\theta$ in $L^2(\mathbb{R})$, the inner product $\langle\psi_n|\theta_n\rangle$ need not converge to $\langle\psi|\theta\rangle$.

## 3. Pointwise convergence implies convergence

Let us review the inductive resolutions of the kind specified in [5, example 2.A] . The 'continuum' Hilbert space is the space $\mathcal{L}_\infty := L^2(\mathbb{R}^r)$, where $r$ is a positive integer. We fix an infinite set $\mathcal{N}$ of positive integers. For each $n \in \mathcal{N}$, let $\mathcal{X}_n$ be a set, let $\sigma_n$ be a function $\mathcal{X}_n \to \mathbb{R}^r$, let $\nu(n)$ be a positive real number, and suppose that, for every bounded convex subset $U$ of $\mathbb{R}^r$, the preimage $\mathcal{X}_n(U) := \sigma_n^{-1}(U)$ is finite, and the sequence $(|\mathcal{X}_n(U)|/\nu(n)^2)_n$ converges to the measure $|U|$ of $U$. (The elements $X_n \in \mathcal{X}_n$ are to be interpreted as indices of sample points $\sigma_n(X_n) \in \mathbb{R}^r$. Our hypothesis ensures that the sample points $\sigma_n(X_n)$ tend towards being uniformly distributed throughout $\mathbb{R}^r$.) The 'discrete' spaces are the spaces $\mathcal{L}_n := L^2(\mathcal{X}_n)$ consisting of the square-summable functions $\mathcal{X}_n \to \mathbb{C}$. Since $\mathbb{R}^r$ is the union of countably many bounded convex subsets, each set $\mathcal{X}_n$ is countable, hence each Hilbert space $L^2(\mathcal{X}_n)$ is separable. The inner product of two vectors $\psi_n, \chi_n \in \mathcal{L}_n$ is

$$\langle \psi_n | \chi_n \rangle = \sum_{X \in \mathcal{X}_n} \overline{\psi}_n(X) \chi_n(X).$$

Let $\mathcal{S}$ denote the Schwarz subspace $\mathcal{S}(\mathbb{R}^r)$ of $\mathcal{L}_\infty$. We define the *restriction map* $\mathrm{res}_n$ to be the linear map $\mathcal{S} \to \mathcal{L}_n$ such that, for $\phi \in \mathcal{S}$, the value of $\mathrm{res}_n(\phi)$ at an element $X \in \mathcal{X}_n$ is

$$\mathrm{res}_n(\phi)(X) = \phi(\sigma_n(X))/\nu(n).$$

Given $\phi, \theta \in \mathcal{S}$ and a convex subset $V$ of $\mathbb{R}^r$ (not necessarily bounded) then, as the limit of a 'Riemann sum',

$$\int_V \overline{\phi}(x)\theta(x)\,\mathrm{d}x = \lim_{n \in \mathcal{N}} \sum_X \overline{\mathrm{res}_n(\phi)(X)} \cdot \mathrm{res}_n(\theta)(X)$$

where the index $X$ of the sum runs over the preimage $\mathcal{X}_n(V) = \sigma_n^{-1}(V)$. In particular,

$$\langle \phi | \theta \rangle = \lim_{n \in \mathcal{N}} \langle \mathrm{res}_n(\phi) | \mathrm{res}_n(\theta) \rangle.$$

In the terminology of [5], the sequence of Hilbert spaces $(\mathcal{L}_n)_n$ and the sequence of linear maps $(\mathrm{res}_n)_n$ together comprise an inductive resolution of $\mathcal{L}_\infty$.

Given a vector $\psi_\infty \in \mathcal{L}_\infty$ and, for sufficiently large $n \in \mathcal{N}$, vectors $\psi_n \in \mathcal{L}_n$, we call $\psi_\infty$ a *limit* of the sequence $(\psi_n)_n$ provided the norms $\|\psi_n\|$ are bounded, and

$$\langle \phi | \psi_\infty \rangle = \lim_{n \in \mathcal{N}} \langle \mathrm{res}_n(\phi) | \psi_n \rangle$$

for all $\phi \in \mathcal{S}$. Existence and uniqueness properties of limits are established in [5]. When $\psi_\infty$ is the limit of $(\psi_n)_n$, we say that $(\psi_n)_n$ *converges* to $\psi_\infty$, and we write $\psi_\infty = \lim_{n \in \mathcal{N}} \psi_n$. To reiterate: the vectors $\psi_n$ need only be given for sufficiently large $n$ (that is, for all except finitely many $n$).

**Theorem 3.1.** *Using the notation above, let $\psi_\infty \in \mathcal{S}$, and for sufficiently large $n \in \mathcal{N}$, let $\psi_n \in \mathcal{L}_n$. Suppose that the norms $\|\psi_n\|$ are bounded, and furthermore, for all $x \in \mathbb{R}^r$, and all sequences $(X_n)_n$ with $X_n \in \mathcal{X}_n$ and $x = \lim_{n \in \mathcal{N}} \sigma_n(X_n)$, we have $\psi_\infty(x) = \lim_{n \in \mathcal{N}} \nu(n)\psi_n(X_n)$. Then $\psi_\infty = \lim_{n \in \mathcal{N}} \psi_n$.*

**Proof.** We are to show that $\langle \phi | \psi_\infty \rangle = \lim_{ninN} \langle \mathrm{res}_n(\phi) | \psi_n \rangle$ for all $\phi \in \mathcal{S}$. We may assume that $\psi_\infty$ and $\phi$ are normalized. Whenever we consider a fixed element $n \in \mathcal{N}$, we shall assume that $n$ is sufficiently large for all our purposes. Let $\epsilon > 0$. Choose a closed ball $B \subseteq \mathbb{R}^r$ centred at the origin and such that

$$\int_{\mathbb{R}^r - B} |\phi(x)|^2\,\mathrm{d}x \langle \epsilon^2 \rangle \int_{\mathbb{R}^r - B} |\psi_\infty(x)|^2\,\mathrm{d}x.$$

Since $n$ is large, $\mathcal{X}_n(B)|/\nu(n)^2 < 2|B|$. We claim that

$$|\psi_\infty(\sigma_n(X)) - \nu(n)\psi_n(X)| < \epsilon$$

for all $X \in \mathcal{X}_n(B)$. Supposing otherwise, then there exists an infinite subset $\mathcal{N}'$ of $\mathcal{N}$ such that, for all $n \in \mathcal{N}'$, there exists some $Y_n \in \mathcal{X}_n(B)$ satisfying

$$|\psi_\infty(\sigma(Y_n)) - \nu(n)\psi_n(Y_n)| \geqslant \epsilon.$$

By the compactness of $B$, there exists an infinite subset $\mathcal{N}''$ of $\mathcal{N}'$ such that the sequence $(\sigma_n(Y_n))_n$ has a limit $x$ in $B$. We can extend the sequence $(Y_n)_{n\in\mathcal{N}'}$ to a sequence $(X_n)_{n\in\mathcal{N}}$ still with limit $x$. This contradicts the hypothesis on the sequence $(\psi_n)_n$. The claim is established.

It follows that

$$\sum_{X\in\mathcal{X}_n(B)} |\mathrm{res}_n(\psi_\infty)(X) - \psi_n(X)|^2 \leqslant \epsilon^2|\mathcal{X}_n(B)|/\nu(n)^2 < 2|B|\epsilon^2.$$

Noting that $\lim_{n\in\mathcal{N}} \|\mathrm{res}_n(\phi)|^2 = \|\phi\|^2 = 1$, we have

$$\left|\sum_{X\in\mathcal{X}_n(B)} \overline{\mathrm{res}_n(\phi)(X)}\big(\mathrm{res}_n(\psi_\infty)(X) - \psi_n(X)\big)\right| < \epsilon\sqrt{2|B|}.$$

by the Cauchy–Schwarz inequality for series. Meanwhile,

$$\lim_{n\in\mathcal{N}}\left|\sum_{X\in\mathcal{X}_n(\mathbb{R}^r-B)} \overline{\mathrm{res}_n(\phi)(X)}.\mathrm{res}_n(\psi_\infty)(X)\right| = \left|\int_{\mathbb{R}^r-B} \overline{\phi}(x)\psi_\infty(x)\,\mathrm{d}x\right| < \epsilon^2$$

by the Cauchy–Schwarz inequality for integrals. Similarly,

$$\lim_{n\in\mathcal{N}}\left|\sum_{X\in\mathcal{X}_n(\mathbb{R}^r-B)} \overline{\mathrm{res}_n(\phi)(X)}.\psi_n(X)\right| < \epsilon\|\psi_n\|.$$

The latest three inequalities yield

$$|\langle\mathrm{res}_n(\phi)|\mathrm{res}_n(\psi_\infty) - \psi_n\rangle| < \epsilon\left(1 + \sqrt{2|B|} + \|\psi_n\|\right).$$

Since the norms $\|\psi_n\|$ are bounded,

$$\langle\phi|\psi_\infty\rangle = \lim_{n\in\mathcal{N}}\langle\mathrm{res}_n(\phi)|\mathrm{res}_n(\psi_\infty)\rangle = \lim_{n\in\mathcal{N}}\langle\mathrm{res}_n(\phi)|\psi_n\rangle.$$

□

The inductive resolution in the particular case where $\mathcal{L}_\infty = L^2(\mathbb{R})$ and $\mathcal{X}_n = \mathbb{Z} \cap (-n/2, n/2]$ and $\nu(n) = (n/2\pi)^{1/4}$ was considered in [8] and [4]. Under the further assumption that $n_2/n_1$ is a square for all $n_1 \leqslant n_2 \in \mathcal{N}$, it was shown in [4, theorem 2.5] that, for each $j \in \mathbb{N}$, the $j$th Harper function $\sigma_{n,j} \in \mathcal{L}_n$ converges to the $j$th Hermite–Gaussian $h_j \in \mathcal{L}_\infty$. (The significance of this result is that, as a consequence, the Harper function FRFT on $\mathcal{L}_n$ converges to the usual FRFT on $\mathcal{L}_\infty$.) No explicit formula for $\sigma_{n,j}$ is known, nor any recurrence relation with variable $n$ and fixed $j$, so it is to be expected that proofs of the convergence $h_j = \lim_{n\in\mathcal{N}} \sigma_{n,j}$ must be indirect. Nevertheless, a large part of the argument in [4] is concerned with establishing lemmas on rates of convergence. Theorem 3.1 tells us that rate of convergence is not important. It seems likely that theorem 3.1 could provide a simpler argument, and one that dispenses with any special hypothesis on the infinite set of positive integers $\mathcal{N}$.

## 4. The normal approximation theorem

The earliest limit distribution theorem, and arguably the most important still, is the normal approximation theorem, proved by de Moivre [17] using formulae developed by Stirling. It asserts that the normal distribution is the continuum limit of the (symmetrically weighted) binomial distribution. In this section, we show how this paradigm of a correspondence between a 'continuum' scenario and a 'discrete' scenario can be expressed in our sense of convergence of vectors.

A normally distributed random variable $x \in \mathbb{R}$ has associated probability density function $P$ such that

$$P(x) = \exp(-x^2)/\sqrt{\pi}.$$

Now consider a positive integer $n$, write $n = 2\ell + 1$, and let

$$\mathcal{X}_n = \{-\ell, 1 - \ell, \ldots, \ell - 1, \ell\} = \{Z - \ell : Z \in \mathbb{Z} \cap [0, n - 1]\}.$$

A binomially distributed random variable $X \in \mathcal{X}_m$ has associated probability weight function $P_n$ such that

$$P_n(X) = \frac{1}{2^{2\ell}} \binom{2\ell}{l + X}.$$

The normal approximation theorem, as in Révész [21, theorem 2.8] for instance, relates $P$ and $P_n$ thus: given elements $X_n \in \mathcal{X}_n$ such that $X_n = \mathrm{o}(\ell^{2/3})$ as $n$ increases, then

$$P_n(X_n) = \exp(-(1 + \mathrm{o}(1))X_n^2/\ell)/\sqrt{\pi\ell}.$$

Consider a quantum system with state space $\mathcal{L}_\infty = L^2(\mathbb{R})$. Let us assume that the state of the system is expressed by a normalized continuous function $\psi : \mathbb{R} \to \mathbb{C}$. By the *home variable*, we mean the observable associated with the operator $\hat{x}$ such that $\hat{x}\psi(x) = x\psi(x)$. The probability density for observing the home variable to be in the locality of some given value $x$ is $|\psi(x)|^2$. Suppose now that the home variable, as a random variable, is normally distributed. Then $|\psi(x)|^2 = P(x)$. The solutions to this equation are unique only up to an phase factor (continuous in $x$). Applying some foresight, let us impose the condition that the values of $\psi$ are real and non-negative. The $\psi$ must be equal to the *Gaussian function* $h_0$, which is defined by

$$h_0(x) = \pi^{-1/4} \exp(-x^2/2).$$

Now consider a quantum system with state space $\mathcal{L}_n = L^2(\mathcal{X}_n)$. Since $\mathcal{X}_n$ is finite, $\mathcal{L}_n$ is the finite-dimensional inner product space consisting of all the functions $\mathcal{X}_n \to \mathbb{C}$. Let $\psi_n \in \mathcal{L}_n$ be a normalized vector expressing the state of the system. Suppose that the home variable is binomially distributed. Then $|\psi_n(X)|^2 = P_n(X)$. Applying foresight again, we impose the condition that the values of $\psi_n$ are real and non-negative. Then $\psi_n$ must be equal to the *degree zero Kravchuk function* $h_{0,n}$, which is defined by

$$h_{0,n}(m) = \frac{1}{2^\ell}\sqrt{\binom{2\ell}{l + m}}.$$

Taking $\mathcal{N}$ to be the set of all positive integers, and putting $\nu(n) := \ell^{1/4}$, we have specified a particular inductive resolution $(\mathcal{L}_n)_n$ and $(\mathrm{res}_n)_n$ of $\mathcal{L}_\infty$. The normal approximation theorem implies that

$$h_0(x) = \lim_n \ell^{-1/4} h_{0,n}(X_n)$$

when $x = \lim_n X_n/\sqrt{\ell}$. Noting that each $\|h_{0,n}\| = 1$, we deduce, from theorem 3.1:

**Proposition 4.1 (Normal approximation theorem).** *We have* $h_0 = \lim_n h_{0,n}$.

Despite initial appearance, this Hilbert space version is, mathematically and physically, a quite 'natural' rendition of the normal approximation theorem; the degree zero Kravchuk function is a 'natural' discrete analogue and approximation to the Gaussian function. This opinion is supported by the more general result, theorem 5.1 below, but the clincher is the interpretation of the Kravchuk functions in Atakishiyev–Wolf [3] and Atakishiyev *et al* [1]. See also [6] and [7]. We also mention that Hakioğlu–Tependelenlioğlu [12] have shown how binomial wavepackets (degree zero Kravchuk state functions) can spread just as Gaussian wavepackets do.

## 5. Convergence of the Kravchuk functions to the Hermite–Gaussians

Recall that, for a natural number $s$, the *Hermite polynomial* $H_s$ of *degree* $s$ may be defined as the polynomial function $\mathbb{R} \to \mathbb{C}$ such that $H_0(x) = 1$ and $H_1(x) = 2x$ and

$$H_{s+1}(x) - 2xH_s(x) + 2sH_{s-1}(x) = 0.$$

The *Hermite–Gaussian function* $h_s$ of *degree* $s$ is defined to be the rapidly decreasing function $\mathbb{R} \to \mathbb{C}$ such that

$$h_s(x) = H_s(x)h_0(x)\sqrt{s!2^s} = H_s(x)\exp(-x^2/2)/\sqrt{s!2^s\sqrt{\pi}}.$$

It is well known that the Hermite–Gaussians comprise a complete orthonormal set in $\mathcal{L}_\infty$, and that they are the energy eigenstates for a simple harmonic oscillator. They are also the eigenvectors of the continuum fractional Fourier transform.

For a given positive integer $n = 2\ell + 1$, let $s$ be a natural number confined to the range $0 \leqslant s \leqslant 2\ell$. The (symmetrically weighted) Kravchuk polynomial $K_{s,d}$ of *degree* $s$ for *dimension* $n$ may be defined to be the polynomial function $\mathbb{C} \to \mathbb{C}$ such that

$$K_{s,n}(z) = \lim_{c \to -2\ell} F(-z, -s, c, 2)$$

where $F$ is the hypergeometric function. Explicitly,

$$K_{s,n}(z) = \sum_{k=0}^{s} (-2)^k \frac{z(z-1)\cdots(z-k+1)}{k!} \binom{s}{k} \binom{2\ell}{j}^{-1}.$$

The (symmetrically weighted) *Kravchuk function* $h_{s,d}$ of *degree* $s$ for *dimension* $n$ may be defined to be the real-valued function such that

$$h_{s,n}(w) = \frac{(-1)^s}{2^\ell} K_{s,n}(\ell + w) \sqrt{\binom{2\ell}{s} \frac{(2\ell)!}{\Gamma(\ell + w + 1)\Gamma(\ell - w + 1)}}$$

where $-\ell - 1 < w < \ell + 1$. (Shortly, it will become evident that the notation here does not conflict with our earlier definition of the degree zero Kravchuk function $h_{0,n}$.) For an account of the theory of Kravchuk polynomials and Kravchuk functions, we refer to Nikiforov–Uvarov [18] and Vilenkin–Klimyk [25]. A summary of some of their properties may be found in Atakishiyev–Wolf [3].

We prefer to understand the Kravchuk polynomials $K_{s,n}$ as having values defined only at integers $Z$ in the range $0 \leqslant Z \leqslant 2\ell$. (As such, they are, of course, no longer polynomial functions.) Thence

$$K_{s,n}(Z) = \sum_{k=0}^{\min(s,Z)} (-2)^k \binom{Z}{k} \binom{s}{k} \binom{2\ell}{k}^{-1}.$$

Alternatively, the $K_{s,n}$ can be defined by

$$K_{\ell+j,n}(\ell+m) = \binom{2\ell}{\ell+j}^{-1} \sum_{k=\min(0,m+j)}^{\max(\ell+m,\ell+j)} (-1)^k \binom{\ell+m}{k} \binom{\ell-m}{\ell+j-k}$$

where $m, j \in \mathcal{X}_n$. The equivalence of these two explicit formulae is not difficult to derive using equations in Vilenkin–Klimyk [25, section 6.3.1] or Wawrzyńczyk [26, section 8.2] ; for the time being, we leave the derivation as an exercise, but we shall give further details of the manipulations in [7]. By [25, equation 6.8.1.12], the Kravchuk polynomials are also determined by the condition that $K_{0,n}(Z) = 1$ and $K_{1,n}(Z) = 2(Z-\ell)/(2\ell-1)$ and

$$(2\ell - s)K_{s+1,n}(Z) + 2(Z-\ell)K_{s,n}(Z) + sK_{s-1,n}(Z) = 0.$$

Also, we prefer to regard the Kravchuk functions $h_{s,n}$ as functions $\mathcal{X}_n \to \mathbb{C}$. Our defining formula for the Kravchuk functions is now

$$h_{s,n}(m) = \frac{(-1)^s}{2^\ell}\sqrt{\binom{2\ell}{s}\binom{2\ell}{m}}K_{s,n}(\ell+m) = (-1)^s\sqrt{\binom{2\ell}{s}}K_{s,n}(\ell+m)h_{0,n}(m)$$

for $m \in \mathcal{X}_n$. (It is now evident that our two definitions of $h_{0,n}$ coincide.) By [25, equation 6.8.1.9], the Kravchuk function comprise an orthonormal basis $\{h_{s,n} : 0 \leqslant s \leqslant 2\ell\}$ of $\mathcal{L}_n$.

Actually, for the purposes of this paper and its two sequels, the functions $K_{s,n}$ and $h_{s,n}$—restricted to finite domains as above—ought to defined in terms of the Wigner $d$-numbers, without any mention of their continuum extensions. After all, $K_{s,n}$ and $h_{s,n}$ arise in our 'discrete' scenario, and as such, they are entirely 'discrete' entities. Their characterization in terms of $d$-numbers may be found in Nikiforov–Uvarov [18, equation 12.65]. In [7], we shall reintroduce $K_{s,n}$ and $h_{s,n}$ from this representation theoretic perspective, and thence we shall derive their recurrence relations, orthonormality properties, and other fundamental properties. (Let us point out that, when deriving properties of the continuum Kravchuk polynomials and functions, appeals to general properties of the hypergeometric function $F$ demand delicate limiting arguments, since $F$ has singularities when its third argument is a non-positive integer.)

It is well known that the Kravchuk functions (suitably scaled) pointwise converge to the Hermite–Gaussians; the result is noted in Atakishiev–Suslov [2, section 1.2], Atakishiyev–Wolf [3, equation A9], Koekoek–Swarttouw [13, equation 2.21], and Nikiforov–Uvarov [18, equation 12.60] . Nikiforov and Uvarov also indicate a method of proof. Since the argument is brief, let us present it in some detail.

The above recurrence relation for the Kravchuk polynomials can be rewritten as

$$\sqrt{(2\ell-s)(s+1)}h_{s+1,n}(X) - 2mh_{s,n}(X) + \sqrt{s(2\ell-s+1)}h_{s-1,n}(X) = 0.$$

Writing $x = \lim_{n\in\mathcal{N}} X_n$ with each $X_n \in \mathcal{X}_n$ and $|x\sqrt{\ell} - X_n| \leqslant 1$, then

$$\sqrt{2(s+1)}h_{s+1,n}(X_n) - 2xh_{s,n}(X_n) + \sqrt{2s}h_{s-1,n}(X_n) = \mathrm{O}(1/\sqrt{\ell})$$

where, for fixed $b > 0$ and variable $x \in [-b, b]$, the expression $O(1/\sqrt{\ell})$ depends on $b$ but not on $x$ and not on the sequence $(X_n)_n$. Meanwhile, from the above recurrence relation for the Hermite polynomials

$$\sqrt{2(s+1)}h_{s+1}(x) - 2xh_s(x) + \sqrt{2s}h_{s-1}(x) = 0.$$

By the normal approximation theorem, followed by an inductive argument wherein the latest two recurrence relations are compared,

$$h_s(x) = \ell^{1/4}h_{s,n}(X_n) + \mathrm{O}(\ell^{-1/4}).$$

Perforce, we recover the pointwise convergence

$$h_s(x) = \lim_{n\in\mathcal{N}} \ell^{1/4}h_{s,n}(X_n).$$

**Theorem 5.1.** *For every natural number s, we have* $h_s = \lim_{n\in\mathcal{N}} h_{s,n}$.

**Proof.** Each $\|h_{s,n}\| = 1$, so the assertion follows from theorem 4.1 together with the pointwise convergence already established. □

In [6], consolidating results in Atakishiyev–Wolf [3], we shall use theorem 5.1 to show that the Kravchuk function FRFT converges to the usual continuum FRFT. In [7], pursuing ideas in Atakishiyev *et al* [1], theorem 5.1 will be needed to elucidate the way in which the three canonical generators of $su(2)$ are related to energy, momentum and position.

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