[6] Liu, Y. K., 1976, ''Discussion on Measurement of Angular Acceleration of a Rigid Body Using Linear Accelerometers,'' ASME J. Appl. Mech., **43**, pp. 977–978; Author's closure, pp. 378.
[7] Mital, N. K., and King, A. I., 1979, ''Computation of Rigid-Body Rotation in Three-Dimensional Space From Body-Fixed Linear Acceleration Measurements,'' ASME J. Appl. Mech., **46**, pp. 925–930.
[8] Morris, J. R., 1973, ''Accelerometry-A technique for the measurement of human body movements,'' J. Biomech., **6**, pp. 729–736.
[9] Nebot, E., Durrant-White, H., 1999, ''Initial Calibration and Alignament of Low-Cost Inertial Navigation Units for Land Vehicle Applications,'' J. Rob. Syst., **16**, pp. 81–92.
[10] Padgaonkar, A. J., Krieger, K. W., and King, A. I., 1975, ''Measurement of Angular Acceleration of a Rigid Body Using Linear Accelerometers,'' ASME J. Appl. Mech., **42**, pp. 552–558.

Quadratic Optimality of the Zero-Phase Repetitive Control

Hakan Köroğlu
e-mail: koroglu@ee.bilkent.edu.tr

Ömer Morgül

Bilkent University, Department of Electrical & Electronics Engineering, Bilkent, 06533, Ankara, Turkey

We consider the quadratically optimal repetitive control problem in discrete-time and show that the existing zero-phase repetitive controller is quadratically optimal for stable plants.
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I Introduction

In the beginning of the 80s, the study on the control strategies for the tracking/rejection of (unknown) periodic signals (with known period) evolved as a new discipline, which then began to be referred to as ''repetitive control.'' The design of discrete-time repetitive controllers was considered by Tomizuka et al. in [1] and a prototype repetitive controller was developed using the zero phase error tracking controller (ZPETC) of [2]. This was then modified in [3,4] for the improvement of the stochastic behavior and the stability robustness. Though studied in [3], the optimality of the modified zero-phase repetitive control is not fully elaborated. In this note, we first study the quadratically optimal repetitive control problem in Section 2 and then show in Section 3 that the modified zero-phase repetitive controller is quadratically optimal for stable plants.

II Quadratically Optimal Repetitive Control

In this section, we derive a condition for the quadratic optimality of discrete-time repetitive control systems. We consider the linear time-invariant feedback system of Fig. 1 with a periodic $x=r-d$ signal of the form

$$x(t)=\sum_{i=0}^{n-1} X_{\mathrm{m}}(\omega_i)\cos(\omega_i t+\theta_i), \tag{1}$$

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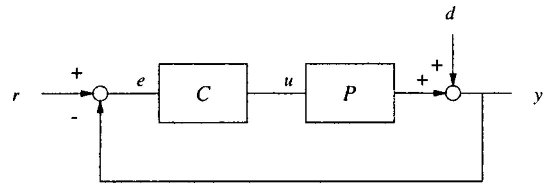

Fig. 1 Unity feedback control system

where n is the period, $X_{\mathrm{m}}(\omega_i)$ are the magnitudes of the signal at $\omega_i=2\pi i/n$, and θ_i are the phases. As is well known, periodic signals of period n contain n frequencies given by $\omega_i=2\pi i/n$; $i=0,\ldots,n-1$ and can be expressed as in (1). Next, we consider the quadratic cost defined by

$$J=\sum_{i=0}^{n-1} |E_{\mathrm{m}}(\omega_i)|^2+\lambda_i|U_{\mathrm{m}}(\omega_i)|^2, \tag{2}$$

where $E_{\mathrm{m}}(\omega_i)$ and $U_{\mathrm{m}}(\omega_i)$ denote the magnitudes of, respectively, the error (e) and the control input (u) signals at ω_i, and λ_is denote non-negative constants. By the minimization of the quadratic cost of (2), the tracking error, as well as the control input, are kept small when the reference (r)/disturbance (d) signals are periodic with period n. In fact, this cost has a time-domain equivalent, which is also quadratic (see [5]). The values of λ_is determine the level of penalization on the power of the control input at the frequencies present in the considered periodic signals. With $\lambda_i=0$, the tracking error is minimized (in a quadratic sense) by the minimization of this cost. In the sequel, we will call the systems minimizing this cost as quadratically optimal.

In the following theorem, we give the condition on $L=PC$ for the minimization of J. For notational simplicity, we use $H(\omega)$ to denote $H(e^{j\omega})$.

Theorem 1. Consider the control system of Fig. 1 with the $x=r-d$ signal of (1) and assume that the system is stable. The feedback system is quadratically optimal with λ_i if and only if

$$L(\omega_i)=\lambda_i^{-1}|P(\omega_i)|^2; \quad \forall i, i=0,\ldots,n-1. \tag{3}$$

The optimal cost is given by

$$J^{\mathrm{opt}}=\sum_{i=0}^{n-1} (1+\lambda_i^{-1}|P(\omega_i)|^2)^{-1}|X_{\mathrm{m}}(\omega_i)|^2. \tag{4}$$

Proof: It follows from the feedback system relations that $|U_{\mathrm{m}}(\omega_i)|=|C(\omega_i)||S(\omega_i)||X_{\mathrm{m}}(\omega_i)|$ and $|E_{\mathrm{m}}(\omega_i)|=|S(\omega_i)||X_{\mathrm{m}}(\omega_i)|$, where $S=1/(1+L)$. Thus we can express J as

$$J=\sum_{i=0}^{n-1} (1+\lambda_i|C(\omega_i)|^2)|S(\omega_i)|^2|X_{\mathrm{m}}(\omega_i)|^2.$$

With T defined as $T=L/(1+L)$, we have $|S|^2=|1-T|^2$ and $|C|^2|S|^2=|P|^{-2}|T|^2$. Expanding $|1-T|^2$, we can write

$$J=\sum_{i=0}^{n-1} [|G(\omega_i)|^2|T(\omega_i)|^2-2\Re\{T(\omega_i)\}+1]|X_{\mathrm{m}}(\omega_i)|^2,$$

where $|G(\omega_i)|^2$ is defined as $|G(\omega_i)|^2=1+\lambda_i|P(\omega_i)|^{-2}$. Taking the term in the square brackets in $|G(\omega_i)|^2$ parentheses and completing the squares, we can reorganize this expression as

$$J=\sum_{i=0}^{n-1} |G(\omega_i)|^2|T(\omega_i)-|G(\omega_i)|^{-2}|^2|X_{\mathrm{m}}(\omega_i)|^2+J^{\mathrm{opt}}.$$

If $T(\omega_i)=|G(\omega_i)|^{-2}$ is satisfied for all i, then J will be minimum, as J^{opt} is a term that is independent of $C(\omega_i)$. Using the inverse relation $L=T/(1-T)$, this condition can be transformed to $L(\omega_i)=(|G(\omega_i)|^2-1)^{-1}$, which is equivalent to (3). If the controller is not of restricted complexity, it is always possible to find a stable closed-loop that satisfies (3), and this proves the only if part.

Remark 1. If $X_{\text{m}}(\omega_i)=0$ for some i, it is not necessary to have the conditions of (3) satisfied at those i.

III Quadratic Optimality of the Zero-Phase Repetitive Control

Discrete-time modified repetitive control structures are formed by the inclusion of the modified delayed positive feedback unit in the feedback loop. This unit has the transfer function given by

$$C_{\text{R}}(z)=\frac{F(z)z^{-n}}{1-F(z)z^{-n}}, \tag{5}$$

where n is the period of the considered signals and F is a stable low-pass filter which satisfies $|F(\omega)|\leqslant 1$. In order to preserve the internal stability, this unit should be accompanied with an appropriate controller. The prototype repetitive controllers developed by Tomizuka et al. in [1,3,4] use the ZPETC of [2] as the accompanying part. ZPETC was developed originally for feedforward tracking purposes. If the transfer function of a stable and causal plant is denoted as $P(z)=z^{-\delta_P}N_P(z^{-1})/D_P(z^{-1})$, with N_P and D_P being coprime numerator/denominator polynomials having nonzero leading coefficients and δ_P being the plant delay, the ZPETC is given by

$$C_{\text{ZP}}(z)=\frac{z^{\delta_P}D_P(z^{-1})N_P^-(z)}{\|N_P^-\|_\infty^2 N_P^+(z^{-1})}. \tag{6}$$

Here, N_P^+ and N_P^- denote, respectively, the stable (i.e., cancelable) and the unstable (i.e., noncancelable) parts of N_P, and $\|N_P^-\|_\infty$ is defined as $\|N_P^-\|_\infty=\sup_\omega|N_P^-(e^{-j\omega})|$. Thus, the modified repetitive controller is formed as

$$C_{\text{ZPR}}(z)=kC_{\text{R}}(z)C_{\text{ZP}}(z), \tag{7}$$

where k denotes a scalar which is usually referred to as the repetitive control gain. The control system of Fig. 1 is stable with $C=C_{\text{ZPR}}$ for a stable plant P, if $k\in(0,2)$ and $|F(\omega)|\leqslant 1$. For the controller to be implementable, we should, moreover, have $n\geqslant\delta_P+\deg N_P^- -\delta_F$, where δ_F is the filter delay (i.e. $F(z)=z^{-\delta_F}N_F(z^{-1})/D_F(z^{-1})$), which is allowed to be negative. The perfect repetitive controller structure of [1], which supplies zero tracking error, can be recovered with $F=1$.

The modified form of the repetitive controller (i.e., the repetitive controller with $F\neq 1$) offers improved stability robustness [4] and stochastic behavior [3] at the cost of degraded tracking/rejection performance especially over the high-pass band. We show below that it also minimizes the quadratic cost of (2) if F is a filter of zero-phase nature (as proposed in [4,6]).

Theorem 2. Let the plant of the control system of Fig. 1 have a causal and stable transfer function P given by $P(z)=z^{-\delta_P}N_P(z^{-1})/D_P(z^{-1})$. The feedback system will be quadratically optimal with $C=C_{\text{ZPR}}$ if F is chosen as $F(z)=M(z^{-1})M(z)$, where M is a polynomial with a degree less than or equal to $n-\delta_P-\deg N_P^-$. The λ_is of the optimized cost are given by

$$\lambda_i=\frac{\|N_P^-\|_\infty^2(|M(\omega_i)|^{-2}-1)|N_P^+(\omega_i)|^2}{k|D_P(\omega_i)|^2}. \tag{8}$$

The optimum value of the cost with these λ_is is

$$J_{\text{ZPR}}^{\text{opt}}=\sum_{i=0}^{n-1}\left(1+\frac{k|N_P^-(\omega_i)|^2}{\|N_P^-\|_\infty^2(|M(\omega_i)|^{-2}-1)}\right)^{-1}|X_{\text{m}}(\omega_i)|^2. \tag{9}$$

Proof: The loop gain of the feedback system with P and $C=C_{\text{ZPR}}$ can be found as $L_{\text{ZPR}}(z)=k\|N_P^-\|_\infty^{-2}N_P^-(z)N_P^-(z^{-1})C_{\text{R}}(z)$. Since $e^{-j\omega_i n}=1$ for $\omega_i=2\pi i/n$, we have $L_{\text{ZPR}}(\omega_i)=k\|N_P^-\|_\infty^{-2}|N_P^-(\omega_i)|^2F(\omega_i)(1-F(\omega_i))^{-1}$. With $F(z)=M(z^{-1})M(z)$, we have $F(\omega_i)=|M(\omega_i)|^2$ and thus $L_{\text{ZPR}}(\omega_i)$s are all zero-phase. Hence by Theorem 1, the feedback system with C_{ZPR} is quadratically optimal with $\lambda_i=L_{\text{ZPR}}^{-1}(\omega_i)|P(\omega_i)|^2$, which can be found as in (8). Inserting (8) in (4), we can find the optimum cost for this case as in (9).

Remark 2. It is possible to verify the well-known facts about the tracking/rejection performance of the perfect and the modified repetitive control systems via the help of (8) and (9). For the perfect repetitive control case, we have $F=M=1$ and thus $\lambda_i=0$. Hence, in this case we have no penalization on the power of the control input and the cost to be minimized is simply the variance of the tracking error. With $\lambda_i=0$, we have $J_{\text{ZPR}}^{\text{opt}}=0$, which means that perfect discrete-time repetitive control systems supply zero tracking error. In the modified repetitive control case, F (and thus M) is-usually-of low-pass nature and hence λ_is are small for is which are close to 0 or n, and large for λ_is which are close to $n/2$. This corresponds to more penalization on the power of the control input over the high-pass band and thus degraded tracking/rejection performance at the high frequency region.

IV Concluding Remarks

We considered the quadratically optimal repetitive control problem and have shown that the modified zero-phase repetitive controllers are quadratically optimal for stable plants. As pointed out in [3], the zero-phase repetitive controller can be constructed for an unstable plant by first stabilizing this plant via an inner loop, and then using the transfer function of the stabilized loop. In this case, the overall system will not necessarily be quadratically optimal (with the cost defined using the original control input). This is because the stabilizing controller will affect the control input to the plant. Yet, a quadratically optimal repetitive controller structure can be developed for general (i.e., stable as well as unstable) plants by using the ideas and techniques presented in [5]. This will be a generalization of the zero-phase repetitive control for general plants and is left for a future work.

References

[1] Tomizuka, M., Tsao, T. C., and Chew, K. K., 1989, ''Analysis and synthesis of discrete-time repetitive controllers,'' ASME J. Dyn. Syst., Meas., Control, **111**, No. 3, pp. 353–358.

[2] Tomizuka, M., 1987, ''Zero phase error tracking algorithm for digital control,'' ASME J. Dyn. Syst., Meas., Control, **109**, No. 1, pp. 65–68.

[3] Chew, K. K., and Tomizuka, M., 1990, ''Steady-state and stochastic performance of a modified discrete-time prototype repetitive controller,'' ASME J. Dyn. Syst., Meas., Control, **112**, No. 1, pp. 35–41.

[4] Tsao, T. C., and Tomizuka, M., 1994, ''Robust adaptive and repetitive digital tracking control and application to hydraulic servo for noncircular machining,'' ASME J. Dyn. Syst., Meas., Control, **116**, No. 1, pp. 24–32.

[5] Köroğlu, H., and Morgül, Ö., 1999, ''Discrete-time LQ optimal repetitive control,'' *Proceedings of American Control Conference*, San Diego, CA, June, pp. 3287–3291.

[6] Tomizuka, M., 1993, ''On the design of digital tracking controllers,'' ASME J. Dyn. Syst., Meas., Control, **115**, No. 2B, pp. 412–418.