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Recursion operator and dispersionless rational Lax representation

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Abstract

We consider equations arising from dispersionless rational Lax representations. A general method to construct recursion operators for such equations is given. Several examples are given, including a degenerate bi-Hamiltonian system with a recursion operator. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently a new method of constructing a recursion operator from Lax representation was introduced in [1]. This construction depends on Lax representation of a given system of PDEs. Let

$$L_t = [A, L], \tag{1}$$

be Lax representation of an integrable nonlinear system of PDEs. Then a hierarchy of symmetries can be given by

$$L_{t_n} = [A_n, L], \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $t_0 = t$, $A_0 = A$ and A_n, $n = 0, 1, 2, \ldots,$ are Gel'fand–Dikkii operators given in terms of L. The recursion relation between symmetries can be written as

$$L_{t_{n+1}} = LL_{t_n} + [R_n, L], \quad n = 0, 1, 2, \ldots, \tag{3}$$

where R_n is an operator such that $\operatorname{ord} R_n = \operatorname{ord} L$.

This symmetry relation allows us to find R_n, hence $L_{t_{n+1}}$, in terms of L and L_{t_n}.

In [1,2] this method was applied to construct recursion operators for Lax equations with different classes of scalar and shift operators, corresponding to field and lattice systems, respectively. In [3] the method was applied to

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dispersionless Lax equations on a Poisson algebra of Laurent series

$$\Lambda = \left\{ \sum_{-\infty}^{+\infty} u_i p^i \colon\ u_i\text{—smooth functions} \right\}, \tag{4}$$

with a polynomial Lax function. The present work is a continuation of [3]. Here we consider a dispersionless Lax equation on the Poisson algebra Λ with a rational Lax function. Such equations one can find in context of topological field theories (see [4,5]).

We have a Lax function

$$L = \frac{\Delta_1}{\Delta_2}, \tag{5}$$

where Δ_1, Δ_2 are polynomials of degree N and M, respectively, and $N > M$. The dispersionless Lax equation is

$$\frac{\partial L}{\partial t_n} = \left\{ \left(L^{\frac{1}{N-M}+n} \right)_{\geqslant 0}, L \right\}, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where the Poisson bracket is given by

$$\{f, g\} = p\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \right).$$

Eq. (6) is of hydrodynamic type. There are several methods for construction of a recursion operator for some equations of hydrodynamic type (see [6–8]). Also a recursion operator can be found with the help of two compatible Hamiltonian formulations of a given equation. For Hamiltonian formulations of equations of hydrodynamic type see Refs. [9,10] and for Hamiltonian formulations of the equations admitting a dispersionless Lax representation see Refs. [11–15].

We construct a recursion operator for a hierarchy of symmetries (6), using a dispersionless Lax representation. First we study the symmetry relation (3) for the rational Lax function. Then we give some examples of calculation of a recursion operator. In particular, we find a recursion operator $\mathcal{R}$ for Eq. (6) with the Lax function

$$L = p + S + \frac{P}{p+Q}, \tag{7}$$

which leads to the system [11]

$$S_t = P_x, \qquad P_t = PS_x - QP_x - PQ_x, \qquad Q_t = QS_x - QQ_x. \tag{8}$$

The recursion operator is given by

$$\mathcal{R} = \begin{pmatrix} S & 1 & PQ^{-1} + P_x D_x^{-1} \cdot Q \\ 2P & S - Q & -2P + (PS_x - (PQ)_x) D_x^{-1} \cdot Q \\ Q & 1 & PQ^{-1} + S - Q + (QS_x - QQ_x) D_x^{-1} \cdot Q \end{pmatrix}. \tag{9}$$

In [11] bi-Hamiltonian representation of this equation was constructed with Hamiltonian operators

$$\mathcal{D}_1 = \begin{pmatrix} 0 & P & Q \\ P & -2PQ & -Q^2 \\ Q & -Q^2 & 0 \end{pmatrix} D_x + \begin{pmatrix} 0 & P_x & Q_x \\ 0 & -(PQ)_x & -QQ_x \\ 0 & -QQ_x & 0 \end{pmatrix}, \tag{10}$$

and

$$\mathcal{D}_2 = \begin{pmatrix} 2P & P(S-3Q) & Q(S-Q) \\ P(S-3Q) & P\big(2P - 2SQ + 4Q^2\big) & Q\big(2P - SQ + Q^2\big) \\ Q(S-Q) & Q\big(2P - SQ + Q^2\big) & 2Q^2 \end{pmatrix} D_x$$

$$+\begin{pmatrix} P_x & SP_x - 2(PQ)_x & SQ_x - QQ_x \\ PS_x - (QP)_x & \left(-SPQ + P^2 + 2PQ^2\right)_x & Q_x\left(2P + Q^2 - SQ\right) \\ QS_x - QQ_x & Q(2P_x + 2QQ_x - S_x - SQQ_x) & 2QQ_x \end{pmatrix}. \tag{11}$$

These Hamiltonian operators are degenerate, so, one cannot use them to find a recursion operator. But it turns out that they are related to the recursion operator $\mathcal{R}$. One can easily check that the following equality holds

$$\mathcal{R}\mathcal{D}_1 = \mathcal{D}_2. \tag{12}$$

We observe that the degeneracy in the bi-Hamiltonian operators is due to the following fact. Let $p' = p + F$ then the Lax function becomes

$$L = p' + G + \frac{P}{p'}. \tag{13}$$

This means that we have two independent variables P and G, where $G = S - F$. The equation corresponding to the Lax function (13) has been studied in [3].

To remove degeneracy one can take the Lax function as

$$L = p + S + \frac{P}{p} + \sum_{i=1}^{m} \frac{Q_i}{p + F_i}. \tag{14}$$

As an example we shall consider the Eq. (6) with the Lax function

$$L = p + S + \frac{P}{p} + \frac{Q}{p + F}. \tag{15}$$

2. Symmetry relation for rational dispersionless Lax representation

Following [1] we consider the hierarchy of symmetries for the dispersionless Lax equation (6) with the Lax function (5)

$$\frac{\partial L}{\partial t_n} = \left\{\left(L^{\frac{1}{N-M}+n}\right)_{\geqslant 0}, L\right\}, \quad n = 0, 1, 2, \ldots, \tag{16}$$

Lemma 1. *For any $n = 0, 1, 2, \ldots,$*

$$\frac{\partial L}{\partial t_n} = L\frac{\partial L}{\partial t_{n-1}} + \{R_n, L\}. \tag{17}$$

Function R_n has a form

$$R_n = A + \frac{B}{\Delta_2}, \tag{18}$$

where A is a polynomial of degree $(N - M)$ and B is a polynomial of degree $(M - 1)$.

Proof. We have

$$\left(L^{\frac{1}{N-M}+n}\right)_{\geqslant 0} = \left[L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{\geqslant 0} + L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{<0}\right]_{\geqslant 0}.$$

So,

$$\left(L^{\frac{1}{N-M}+n}\right)_{\geqslant 0} = L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{\geqslant 0} + \left(L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{<0}\right)_{\geqslant 0} - \left(L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{\geqslant 0}\right)_{<0}.$$

If we take

$$R_n = \left(L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{<0}\right)_{\geqslant 0} - \left(L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{\geqslant 0}\right)_{<0}, \tag{19}$$

then

$$\left(L^{\frac{1}{N-M}+n}\right)_{\geqslant 0} = L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{\geqslant 0} + R_n.$$

Hence,

$$\frac{\partial L}{\partial t_n} = \left\{\left(L^{\frac{1}{N-M}+n}\right)_{\geqslant 0}, L\right\} = \left\{L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{\geqslant 0} + R_n, L\right\} = L\frac{\partial L}{\partial t_n} + \{R_n, L\},$$

and (17) is satisfied. The remainder R_n has form (18). Indeed in (19) we set

$$A = \left(L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{<0}\right)_{\geqslant 0},$$

and

$$B = \Delta_2 \cdot \left(L\left(L^{\frac{1}{N-M}+(n-1)}\right)_{\geqslant 0}\right)_{<0}.$$

Then A is a polynomial of degree $(N-M-1)$ and B is a polynomial of degree $(M-1)$. □

Now we can apply the Lemma 1 to find recursion operators.

3. Examples

Example 2. Let us consider the Eq. (8) given in introduction.

Lemma 3. *A recursion operator for* (8) *is given by* (9).

Proof. Using (18) for R_n, we have $R_n = A + \frac{B}{p+Q}$. So, the symmetry relation (17) is

$$\begin{aligned}
&\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n}\cdot\frac{1}{p+Q} + \frac{\partial Q}{\partial t_n}\cdot\frac{P}{(p+Q)^2} \\
&\quad = \left(p + S + \frac{P}{p+Q}\right)\left(\frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}}\cdot\frac{1}{p+Q} + \frac{\partial Q}{\partial t_{n-1}}\cdot\frac{P}{(p+Q)^2}\right) \\
&\qquad + p\left(A_x + \frac{B_x}{p+Q} + \frac{-BQ_x}{(p+Q)^2}\right)\left(1 + \frac{-P}{(p+Q)^2}\right) - \frac{pB}{(p+Q)^2}\left(S_x + \frac{P_x}{p+Q} + \frac{-PQ_x}{(p+Q)^2}\right).
\end{aligned}$$

To have the equality the coefficients of p and $(p+Q)^{-3}$ must be zero. It gives the recursion relations to find A and B. Then the coefficients of p^0, $(p+Q)^{-1}$, $(p+Q)^{-2}$ give expressions for $\frac{\partial S}{\partial t_n}$, $\frac{\partial P}{\partial t_n}$, $\frac{\partial Q}{\partial t_n}$. □

Example 4. The dispersionless Lax equation (6) with the Lax function (15), for $n = 1$, gives the following system

$$S_t = P_x + Q_x, \qquad P_t = PS_x, \qquad Q_t = QS_x - FQ_x - QF_x, \qquad F_t = FS_x - FF_x. \tag{20}$$

Lemma 5. *A recursion operator for* (20) *is given by*

$$\begin{pmatrix} S & 2+P_x D_x^{-1}\cdot P^{-1} & 1 & QF^{-1}+Q_x D_x^{-1}\cdot F^{-1} \\ 2P & \begin{array}{l} S+QF^{-1}+PS_x D_x^{-1}\cdot P^{-1} \\ \quad +PF^{-1}\big(Q_x-QF^{-1}F_x\big)D_x^{-1}\cdot P^{-1} \end{array} & PF^{-1} & \begin{array}{l} -2PQF^{-2} \\ \quad -PF^{-1}\big(Q_x-QF^{-1}F_x\big)D_x^{-1}\cdot F^{-1} \end{array} \\ 2Q & \begin{array}{l} -QF^{-1} \\ \quad -PF^{-1}(Q_x-QF^{-1}F_x)D_x^{-1}\cdot P^{-1} \end{array} & S-F-PF^{-1} & \begin{array}{l} -2PQF^{-2}-2Q \\ \quad +PF^{-1}\big(Q_x-QF^{-1}F_x\big)D_x^{-1}\cdot F^{-1} \\ \quad +(QS_x-QF_x-FQ_x)D_x^{-1}\cdot F^{-1} \end{array} \\ F & 1+\big(P_x-PF^{-1}F_x\big)D_x^{-1}\cdot P^{-1} & -1 & \begin{array}{l} PF^{-1}-F+(FS_x-FF_x)D_x^{-1}\cdot F^{-1} \\ \quad -\big(P_x-PF^{-1}F_x\big)D_x^{-1}\cdot F^{-1} \end{array} \end{pmatrix}. \tag{21}$$

Proof. Using (18) for R_n, we have $R_n = C + \frac{A}{p} + \frac{B}{p+F}$. So, the symmetry relation (17) is

$$\begin{aligned} &\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n}\cdot\frac{1}{p} + \frac{\partial Q}{\partial t_n}\cdot\frac{1}{(p+F)} + \frac{\partial F}{\partial t_n}\cdot\frac{-Q}{(p+F)^2} \\ &\quad = \left(p+S+\frac{P}{p}+\frac{Q}{p+F}\right)\left(\frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}}\cdot\frac{1}{p} + \frac{\partial Q}{\partial t_{n-1}}\cdot\frac{1}{(p+F)} + \frac{\partial F}{\partial t_{n-1}}\cdot\frac{-Q}{(p+F)^2}\right) \\ &\qquad + p\left(\frac{-B}{p^2}+\frac{-C}{(p+F)^2}\right)\left(S_x+\frac{P_x}{p}+\frac{Q_x}{(p+F)}+\frac{-QF_x}{(p+F)^2}\right) \\ &\qquad - p\left(A_x+\frac{B_x}{p}+\frac{C_x}{(p+F)+\frac{-CF_x}{(p+F)^2}}\right)\left(1+\frac{P}{p}+\frac{-Q}{(p+F)^2}\right). \end{aligned}$$

Therefore, the coefficients of p, p^{-2} and $(p+F)^{-3}$ must be zero, it gives recursion relations to find A, B and C. Then the coefficients of p^0, p^{-1}, $(p+F)^{-1}$ and $(p+F)^{-2}$, give expressions for $\frac{\partial S}{\partial t_n}$, $\frac{\partial P}{\partial t_n}$, $\frac{\partial Q}{\partial t_n}$ and $\frac{\partial F}{\partial t_n}$. □

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References

[1] M. Gürses, A. Karasu, V.V. Sokolov, J. Math. Phys. 40 (1999) 6473.
[2] M. Blaszak, Rep. Math. Phys. 48 (1-2) (2001) 27.
[3] M. Gürses, K. Zheltukhin, J. Math. Phys. 42 (2001) 1309.
[4] B.A. Dubrovin, Geometry of 2D Topological Field Theories, Lecture Notes in Mathematics, Vol. 1620, Springer, Berlin, 1993, pp. 120–348.
[5] S. Aoyama, Y. Kodama, Commun. Math. Phys. 182 (1996) 185.
[6] M.B. Sheftel, Generalized hydrodynamic-type systems, in: N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3, CRC Press, New York, 1996, pp. 169–189.
[7] V.M. Teshukov, LIIAN 106 (1989) 25.
[8] A.P. Fordy, B. Gürel, Theoret. Math. Phys. (1999).
[9] B.A. Dubrovin, S.P. Novikov, Sov. Math. Dokl. 27 (1983) 665.

[10] E.V. Ferepantov, Hydrodynamic-type systems, in: N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 1, CRC Press, New York, 1994, pp. 303–331.
[11] I.A.B. Strachan, J. Math. Phys. 40 (1999) 5058.
[12] D.B. Fairlie, I.A.B. Strachan, Inverse Problems 12 (1998) 885.
[13] J.C. Brunelli, M. Gürses, K. Zheltukhin, Rev. Math. Phys. 13 (4) (2001) 529.
[14] J.C. Brunelli, A. Das, Phys. Lett. A 235 (1997) 597.
[15] L.-C. Li, Commun. Math. Phys. 203 (1999) 573.