Entanglement and the SU(2) phase states in atomic systems

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We show that a system of $2n$ identical two-level atoms interacting with n cavity photons manifests entanglement and that the set of entangled states coincides with the so-called SU(2) phase states. In particular, violation of classical realism in terms of the Greenberger-Horne-Zeilinger and Clauser-Horne-Shimoni-Holt conditions is proved. We discuss a property of entanglement expressed in terms of local measurements. We also show that generation of entangled states in the atom-photon systems under consideration strongly depends on the choice of initial conditions and that the parasitic influence of cavity detuning can be compensated through the use of Kerr medium.

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I. INTRODUCTION

It has been recognized that entanglement phenomenon touches on the conceptual problems of reality and locality in quantum physics as well as the more technological aspects of quantum communications, cryptography, and computing. In particular, the methods of quantum key distribution in communication channels secured from eavesdropping are based on the use of entangled states [1–5] (for recent review, see Refs. [6,7]). In turn, the realization of quantum computer [8] is dependent on the ability to form entangled states of initially uncorrelated single-particle states [9].

In recent years, many successful experiments have been performed to verify the violation of Bell's inequalities and Greenberger-Horne-Zeilinger (GHZ) equality [10,11] and to develop the methods of engineered entanglement for quantum cryptography and quantum key distribution. In particular, the recent advances in the field of cavity QED and techniques of atom manipulation, trapping, and cooling enable a number of experiments that investigate the entanglement in the atomic systems (see Refs. [11–17] and references therein).

It has been shown recently [18] that a pure entangled state of two atoms can be obtained in an optical resonator through the exchange by a single photon. The main idea in Ref. [18] is that a single excitation of the system is either carried by a photon or shared between the atoms. If a photon can leak out from the resonator, the absence of photon counts in the process of continuous monitoring of the cavity decay can be associated with the presence of the pure entangled atomic state. The importance of this scheme is caused by the fact that its realization seems to be easily available with present experimental technique.

The main objective of this paper is to show that the entangled states in the "atoms-plus-photons" systems of the type discussed in Ref. [18] can be represented by the so-called SU(2) phase states corresponding to the SU(2) algebra of the odd "spin"

$$j=\frac{1}{2}\left[\binom{2n}{n}-1\right], \tag{1}$$

where $2n$ is the even number of atoms and $n=1,2,\ldots$ is the number of cavity photons. In particular, the system considered in Ref. [18] corresponds to the phase states of "spin" $j=1/2$. The SU(2) phase states were introduced in Ref. [19] for an arbitrary spin and then generalized in Refs. [20,21] to the case of the SU(2) subalgebra in the Weyl-Heisenberg algebra of photon operators (for recent review, see Ref. [22]). From the mathematical point of view, this is the system of

$$N=2j+1$$

qubits defined in the Hilbert space

$$\mathcal{H}_N=(\mathbf{C}^2)^{\otimes N}$$

with the componentwise action of $\mathrm{SU}(2)^N$. In particular, we show that these states violate the classical realism and discuss their realization.

On the other hand, we will discuss a condition of entanglement that has been proposed recently [23]. Let us note in this connection that, in the usual treatment of entanglement, the entangled states of a two-component (in general, multicomponent) system are considered as the nonseparable states with respect to the subsystems (e.g., see Ref. [24]). For example, if the individual components of a two-component system are described by the states $|\xi_i\rangle$ and $|\chi_i\rangle$, respectively, the state

$$|\psi_{ent}\rangle=\sum_i b_i|\xi_i\rangle\otimes|\chi_i\rangle,$$

$$\langle\xi_i|\xi_k\rangle=\langle\chi_i|\chi_k\rangle=\delta_{ik}, \quad \sum_i |b_i|^2=1,$$

is entangled if $b_i\neq 0$ for at least two distinct values of the subscript i. From the mathematical point of view, the entanglement is caused by the combination of the superposition principle in quantum mechanics with the tensor product structure of the space of state of the two-component or multicomponent system [25].

Very often, the existence of entanglement is verified in terms of violation of Bell's inequalities and their generalizations [26–31]. Another way is based on the use of GHZ theorem [10]. A possibility to introduce more general inequalities is also discussed [32].

It should be noted that the use of Bell's inequalities and their numerous generalizations demonstrate nothing but the nonexistence of hidden variables. Moreover, it is possible to say that the unique, general, and mathematically correct definition of entanglement still does not exist (e.g., see Ref. [32]).

An interesting approach has been proposed recently [32]. Considering the state shared between Alice and Bob as a quantum communication channel, the authors of Ref. [32] concluded that the information in the case of entanglement is carried mostly by the correlations between the ends of the channel. These correlations manifest themselves by means of the local measurements on the sides of the channel [23].

Following Ref. [23], consider a composite system defined in the Hilbert space

$$\mathcal{H} = \otimes_l \mathcal{H}^{(l)}, \quad l \geqslant 2.$$

Let G be the group of dynamical symmetry of a subsystem in the composite system. Then the Hermitian operators g associated with representation of G in $\mathcal{H}^{(l)}$ define the set of local measurement on the corresponding side of the channel provided by a state $|\psi\rangle \in \mathcal{H}$. For example, in the case of $\mathcal{H}^{(l)} = \mathbf{C}^2$, corresponding to the Einstein-Podolsky and Rosen (EPR) spin-$\frac{1}{2}$ system, $G = SU(2)$ and the set of local measurements can be specified by the infinitesimal generators of the SL(2) group

$$\{g\} = \{\sigma_k^{(l)}\}, \quad k = 1,2,3,$$

which is the complexification of the SU(2) group.

It was shown in Ref. [23] that the maximum correlation between the ends of the channel corresponds to the states such that

$$\forall g, \quad \langle g \rangle = 0.$$

This statement can be illustrated by the atoms-plus-photons systems under consideration. Consider first the set of two identical two-level atoms. Let $|e_l\rangle$ and $|g_l\rangle$ denote the excited and the ground atomic states of the l^{th} atom, respectively. Then, the entangled, maximum excited atomic states in the system "2 atoms plus 1 photon" considered in Ref. [18] are

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|e_1 g_2\rangle \pm |g_1 e_2\rangle). \tag{2}$$

Then, the local measurement g can be described by the Pauli matrices

$$\sigma_1^{(l)} = |e_l\rangle\langle g_l| + |g_l\rangle\langle e_l|,$$

$$\sigma_2^{(l)} = -i|e_l\rangle\langle g_l| + i|g_l\rangle\langle e_l|,$$

$$\sigma_3^{(l)} = |e_l\rangle\langle e_l| - |g_l\rangle\langle g_l|, \tag{3}$$

i.e., by the infinitesimal generators of the algebra SL(2). It is now a straightforward matter to check that

$$\forall i,l \quad \langle \psi_\pm | \sigma_i^{(l)} | \psi_\pm \rangle = 0, \tag{4}$$

where averaging is taken over the states (4). Another example is provided by the GHZ states [10]

$$|\psi_\pm^{(GHZ)}\rangle = \frac{1}{\sqrt{2}}(|e_1 e_2 e_3\rangle \pm |g_1 g_2 g_3\rangle), \tag{5}$$

corresponding to the maximum atomic excitation in the $3+3$ system. It is easily seen that the averaging of the local operators (3) over Eq. (5) gives the same result as Eq. (4).

This property (4) can be used to define the entangled states.

We will show that the SU(2) phase states of spin j defined by Eq. (1) in a $(2n+n)$-type atom-photon system obey the nonseparability conditions, have the property (4), and manifest the violation of classical realism expressed in terms of the GHZ [10] and CHSH (Clauser-Horne-Shimoni-Holt) [33] conditions.

The paper is organized as follows. In Sec. II, we consider the representation of the SU(2) phase states. As a particular example, we examine the system of two identical two-level atoms, interacting with a single cavity photon and show that the maximum entangled atomic states of the Ref. [18] belong to the class of the SU(2) phase states of spin $j=1/2$. Let us stress that hereafter the maximum entanglement is defined in the usual way by the maximum of reduced entropy (e.g., see Refs. [23,25,27,32]). Then, we generalize this result on the case of $2n+n$ system. As a nontrivial example, we consider in Sec. III the system of four identical two-level atoms interacting with the two cavity photons. In this case, the set of entangled, maximum excited atomic states is provided by the six orthogonal SU(2) phase states of spin $j=5/2$. For these states, we prove violation of classical realism through the use of GHZ and CHSH conditions. In Sec. IV, we discuss how the entangled atomic states can be achieved in the process of steady-state evolution. In particular, we show that the maximum entanglement can be achieved if the initial state of the system contains the photons and does not contain the atomic excitations. We also show that the presence of the cavity detuning hampers the creation of pure entangled states and that the parasitic influence of detuning can be compensated through the use of the Kerr medium inside the cavity. Finally, in Sec. V, we briefly discuss the obtained results.

II. REPRESENTATION OF THE SU(2) PHASE STATES

An arbitrary spin j can be described by the generators J_+, J_-, J_z of the SU(2) algebra such that

$$[J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm,$$

$$J^2 = J_z^2 + \tfrac{1}{2}(J_+J_- + J_-J_+) = j(j+1)\times\mathbf{1}, \tag{6}$$

where $\mathbf{1}$ is the unit operator in the $(2j+1)$-dimensional Hilbert space. Since

$$J_{\pm}=J_x \pm iJ_y ,$$

it is possible to say that the generators J_+ ,J_- ,J_z in Eq. (6) correspond to the Cartesian representation of the SU(2) algebra. Following Ref. [19], one can introduce the representation in spherical coordinates via the polar decomposition of Eq. (6) of the form

$$J_+=J_r\boldsymbol{\epsilon}, \quad J_r=J_r^+ , \quad \boldsymbol{\epsilon}\boldsymbol{\epsilon}^+=\mathbf{1}, \tag{7}$$

where the Hermitian operator J_r corresponds to the radial contribution, white $\boldsymbol{\epsilon}$ gives the exponential of the azimuthal phase operator. It is a straightforward matter to show that $\boldsymbol{\epsilon}$ can be represented by the following $(2j+1)\times(2j+1)$ matrix:

$$\boldsymbol{\epsilon}=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\\ e^{i\psi} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{8}$$

in the $(2j+1)$-dimensional Hilbert space. Here ψ is an arbitrary real parameter (reference phase). The eigenstates of the operator (8)

$$\boldsymbol{\epsilon}|\phi_n^{(j)}\rangle=e^{i\phi_n^{(j)}}|\phi_n^{(j)}\rangle, \quad n=1, \dots ,(2j+1), \tag{9}$$

form the basis of the so-called phase states

$$|\psi_n^{(j)}\rangle=\frac{1}{\sqrt{2j+1}}\sum_{k=0}^{2j} e^{ik\phi_n^{(j)}}|\psi_k\rangle \tag{10}$$

dual with respect to the basis of individual states $|\psi_k\rangle$ of the Hilbert space.

As a physical example of some considerable interest, consider now the system of the two identical two-level atom interacting with the single cavity photon (see Ref. [18]). If the cavity photon is absorbed by either atom, the atomic subsystem can be observed in the following states

$$|\psi_1\rangle=|e_1g_2\rangle, \quad |\psi_2\rangle=|g_1e_2\rangle, \tag{11}$$

where $|e_1g_2\rangle=|e_1\rangle\otimes|g_2\rangle$ and $|e\rangle$ and $|g\rangle$ denote the excited and ground atomic states, respectively. The subscript marks the atom. Using the atomic basis (11), we can construct the following representation of the SU(2) algebra:

$$J_+=|e_1g_2\rangle\langle g_1e_2|, \quad J_-=|g_1e_2\rangle\langle e_1g_2|,$$

$$J_3=\tfrac{1}{2}\,(|e_1g_2\rangle\langle e_1g_2|-|g_1e_2\rangle\langle g_1e_2|). \tag{12}$$

This representation formally corresponds to Eq. (6) at the spin $j=1/2$. Then, the corresponding exponential of the phase operator (8) takes the form

$$\boldsymbol{\epsilon}=|e_1g_2\rangle\langle g_1e_2|+e^{i\psi}|g_1e_2\rangle\langle e_1g_2|. \tag{13}$$

In turn, the phase states (9) and (10) are

$$|\phi_{\pm}\rangle=\frac{1}{\sqrt{2}}(|e_1g_2\rangle+e^{i\phi_{\pm}}|g_1e_2\rangle), \tag{14}$$

$$\phi_{\pm}=\psi/2+(1\mp 1)\pi/2.$$

It is easily seen that the phase states (14) form the set of entangled atomic states in the two-atom system under consideration. Definitely, these states obey the nonseparability condition. It is also seen that Eq. (14) coincides with the maximally entangled states (2) of Ref. [18] when the reference phase $\psi=0$.

Consider now a general $2n+n$ system at $n\geqslant 1$. Then, the maximum excited atomic states

$$|\psi_i\rangle=|\{e\}_n ,\{g\}_n\rangle, \tag{15}$$

can be used to construct a representation of the SU(2) algebra (6) of spin j defined in Eq. (1). Here $i=1,2, \dots ,N$ and

$$N=2j+1=\binom{2n}{n}$$

is the total number of such a states. In the basis (15), we can construct the polar decomposition of the SU(2) algebra of spin (1) and the corresponding exponential of the phase operator (8) and the phase states (10). Let us rename the states (15) as follows:

$$|\psi_k\rangle\rightarrow|\psi_{k'}\rangle, \quad k'\equiv k-1=0, \dots ,N-1.$$

Then, the SU(2) phase states (10) take the form

$$|\phi_k\rangle=\frac{1}{\sqrt{N}}\sum_{k'=0}^{N-1} e^{ik'\phi_k}|\psi_{k'}\rangle, \tag{16}$$

where

$$\phi_k=(\psi+2k\pi)/N.$$

These states (16) form a basis dual with respect to Eq. (15) and spanning the Hilbert space of the maximum excited atomic states in the $2n+n$ system under consideration. By construction, the phase states (16) are nonseparable with respect to contributions of individual atoms and thus entangled [24]. Let us stress that the choice of the phase factors in Eq. (16) is irrelevant to entanglement, which holds for arbitrary phase factors. This choice is caused by the aspiration for getting the dual with respect to basis (15) of entangled states.

It is easily seen that the states (16) obey the condition (4). In fact, the action of the flip operators $\sigma_{1,2}^{(l)}$ in Eq. (3) on the states (16) leads to the change of the number of either excited or deexcited atoms:

$$\sigma_{1,2}^{(l)}|\psi_k\rangle\rightarrow\begin{cases}|\{e\}_{n-1},\{g\}_{n+1}\rangle, l\in\{g\}\\ |\{e\}_{n+1},\{g\}_{n-1}\rangle, l\in\{e\}\end{cases}$$

and therefore $\langle\sigma_{1,2}^{(l)}\rangle=0$ in the case of averaging over the states (16). Since each state (15) contains equal number of excited and deexcited atoms, the action of the parity operator in Eq. (3) on the phase states (16) should lead to the state that differs from Eq. (16) by the multiplication of a certain n terms by the factor of -1. Hence

$$\langle \sigma_3^{(l)} \rangle = \frac{1}{N} \left(\sum_{i=1}^{N/2} 1 - \sum_{i=N/2+1}^{N} 1 \right) = 0.$$

By construction, N is always an even number.

Thus, the SU(2) phase states (16), corresponding to the maximum excited atomic states in the $2n+n$ system, are entangled because they are nonseparable and, at the same time, obey the condition (4) for the local measurements. In the following section, we show that the states (16) manifest violation of classical realism as well.

Before we begin to discuss this subject, let us note that the SU(2) phase states of the atomic system under consideration with integer spin do not provide the entanglement. Consider as an example the system of three identical two-level atoms, interacting with a single cavity photon. There are the three excited atomic states

$$|e_1 g_2 g_3\rangle, \quad |g_1 e_2 g_3\rangle, \quad |g_1 g_2 e_3\rangle \tag{17}$$

and the three dual phase states of the type as shown in Eq. (16)

$$|\psi_k\rangle = \frac{1}{\sqrt{3}} (|e_1 g_2 g_3\rangle + e^{i\phi_k} |g_1 e_2 g_3\rangle + e^{2i\phi_k} |g_1 g_2 e_3\rangle). \tag{18}$$

It is clear that the states (18) are the phase states of spin $j=1$. Here

$$\phi_k = (\psi + 2k\pi)/3, \quad k = 0,1,2.$$

It is easily seen that the phase states (18) cannot be factorized with respect to atoms. At the same time, the average of the parity operator $\sigma_3^{(l)}$ in Eq. (3) over the states (18) is

$$\forall k,l \quad \langle \psi_k | \sigma_3^{(l)} | \psi_k \rangle = -\frac{1}{3},$$

although the averages of the flip operators are

$$\forall k,l \quad \langle \psi_k | \sigma_{1,2}^{(l)} | \psi_k \rangle = 0.$$

Thus, the nonseparable states (18) do not obey the condition (4). At the same time, these states do not manifest the maximum entanglement as well. Let us stress that the nonseparability is not a sufficient condition of maximum entanglement [24]. For example, from the measurement of the state of the first atom we can only learn that either the atoms 2 and 3 are both in the ground state with reliability or they are in the two-atom entangled state of the type discussed in Ref. [18]. Similar result can be obtained for the system of three atoms interacting with two cavity photons. The only maximum entangled state of the system of three atoms is provided by the superposition of GHZ states (5).

III. THE 4+2 SYSTEM

To show that the phase states (16) of a $2n+n$ system violate the classical realism, consider the system of four identical two-level atoms interacting with two cavity photons. The maximum excited atomic states at $n=2$ are

$$|e_1 e_2 g_3 g_4\rangle, \quad |e_1 g_2 e_3 g_4\rangle, \quad |e_1 g_2 g_3 e_4\rangle,$$
$$|g_1 e_2 e_3 g_4\rangle, \quad |g_1 e_2 g_3 e_4\rangle, \quad |g_1 g_2 e_3 e_4\rangle. \tag{19}$$

These orthonormal states form the six-dimensional basis of the Hilbert space in which the representation of the generators (6) has the form

$$\begin{aligned} J_+ = {} & \sqrt{5} |e_1 e_2 g_3 g_4\rangle\langle e_1 g_2 e_3 g_4| + \sqrt{8} |e_1 g_2 e_3 g_4\rangle\langle e_1 g_2 g_3 e_4| \\ & + 3 |e_1 g_2 g_3 e_4\rangle\langle g_1 e_2 e_3 g_4| + \sqrt{8} |g_1 e_2 e_3 g_4\rangle\langle g_1 e_2 g_3 e_4| \\ & + \sqrt{5} |g_1 e_2 g_3 e_4\rangle\langle g_1 g_2 e_3 e_4|, \end{aligned}$$

$$\begin{aligned} J_3 = {} & \tfrac{5}{2} |e_1 e_2 g_3 g_4\rangle\langle e_1 e_2 g_3 g_4| + \tfrac{3}{2} |e_1 g_2 e_3 g_4\rangle\langle e_1 g_2 e_3 g_4| \\ & + \tfrac{1}{2} |e_1 g_2 g_3 e_4\rangle\langle e_1 g_2 g_3 e_4| - \tfrac{1}{2} |g_1 e_2 e_3 g_4\rangle\langle g_1 e_2 e_3 g_4| \\ & - \tfrac{3}{2} |g_1 e_2 g_3 e_4\rangle\langle g_1 e_2 g_3 e_4| - \tfrac{5}{2} |g_1 g_2 e_3 e_4\rangle\langle g_1 g_2 e_3 e_4|. \end{aligned}$$

By construction, they describe the spin $j=5/2$ system. In turn, the exponential of the phase operator (8) takes the form

$$\begin{aligned} \epsilon = {} & |e_1 e_2 g_3 g_4\rangle\langle e_1 g_2 e_3 g_4| + |e_1 g_2 e_3 g_4\rangle\langle e_1 g_2 g_3 e_4| \\ & + |e_1 g_2 g_3 e_4\rangle\langle g_1 e_2 e_3 g_4| + |g_1 e_2 e_3 g_4\rangle\langle g_1 e_2 g_3 e_4| \\ & + |g_1 e_2 g_3 e_4\rangle\langle g_1 g_2 e_3 e_4| + e^{i\psi} |g_1 g_2 e_3 e_4\rangle\langle e_1 e_2 g_3 g_4|. \end{aligned}$$

Then, the six phase states (9) have the form (16) with $N=6$ and

$$\phi_k = \frac{\psi}{6} + \frac{k\pi}{3}, \quad k = 0,1,\ldots,5. \tag{20}$$

As well as for Eq. (16), these states are nonseparable and hence entangled and obey the condition (4) for local variables.

To show that these phase states violate the classical realism, let us first represent the states (16) at $N=6$ in the following way

$$|\phi_k\rangle = \frac{1}{\sqrt{3}} (|\chi_{1k}\rangle + e^{i\phi_k} |\chi_{2k}\rangle + e^{2i\phi_k} |\chi_{3k}\rangle), \tag{21}$$

where

$$|\chi_{1k}\rangle=\frac{1}{\sqrt{2}}(|e_1e_2g_3g_4\rangle+e^{5i\phi_k}|g_1g_2e_3e_4\rangle),$$

$$|\chi_{2k}\rangle=\frac{1}{\sqrt{2}}(|g_1e_2e_3g_4\rangle+e^{3i\phi_k}|e_1g_2g_3e_4\rangle),$$

$$|\chi_{3k}\rangle=\frac{1}{\sqrt{2}}(|g_1e_2g_3e_4\rangle+e^{i\phi_k}|e_1g_2e_3g_4\rangle). \qquad (22)$$

It is easily seen that each set of six states $|\chi_{pk}\rangle$ with $p=1,2,3$ and $k=0,\ldots,5$ consists of the nonseparable and hence entangled states. Consider, for example, the states $|\chi_{1k}\rangle$ in Eq. (22). Because of the definition of the phase angle ϕ_k at $N=6$, they consist of the three sets of the pairwise orthogonal states

$$\{|\chi_{10}\rangle,|\chi_{13}\rangle\},\quad \{|\chi_{11}\rangle,|\chi_{14}\rangle\},\quad \{|\chi_{12}\rangle,|\chi_{15}\rangle\}.$$

It is also seen that the second and third sets here are obtained from the first set by the successive rotations of the reference frame.

Now the violation of classical realism can be proved through the use of the GHZ theorem [10]. Consider first the state $|\chi_{10}\rangle$ in Eq. (22). It is easy to verify that this state obey the following conditions:

$$\forall i,l \quad \otimes_{l=1}^{4}\sigma_i^{(l)}|\chi_{10}\rangle=|\chi_{10}\rangle \qquad (23)$$

and

$$\sigma_1^{(1)}\sigma_1^{(2)}\sigma_2^{(3)}\sigma_2^{(4)}|\chi_{10}\rangle=-|\chi_{10}\rangle,$$

$$\sigma_2^{(1)}\sigma_2^{(2)}\sigma_1^{(3)}\sigma_1^{(4)}|\chi_{10}\rangle=-|\chi_{10}\rangle,$$

$$\sigma_1^{(1)}\sigma_2^{(2)}\sigma_1^{(3)}\sigma_2^{(4)}|\chi_{10}\rangle=|\chi_{10}\rangle,$$

$$\sigma_1^{(1)}\sigma_2^{(2)}\sigma_2^{(3)}\sigma_1^{(4)}|\chi_{10}\rangle=|\chi_{10}\rangle,$$

$$\sigma_2^{(1)}\sigma_1^{(2)}\sigma_2^{(3)}\sigma_1^{(4)}|\chi_{10}\rangle=|\chi_{10}\rangle,$$

$$\sigma_2^{(1)}\sigma_1^{(2)}\sigma_1^{(3)}\sigma_2^{(4)}|\chi_{10}\rangle=|\chi_{10}\rangle. \qquad (24)$$

It is possible to say that these equalities (23) and (24) express a kind of EPR "action at distance" in the maximum excited states of the system of four atoms interacting with two photons. In other words, the correlations represented by Eqs. (23) and (24) permit us to determine in a unique way the state of the fourth atom via measurement of the states of other three atoms.

The operator equalities (23) and (24) can be used to obtain the relations similar to those in the GHZ theorem. Following Ref. [10], we have to assign the classical quantities $m_i^{(l)}$ to the local operators. Here

$$m_1^{(l)},m_2^{(l)}=\pm 1.$$

Then, it follows from Eq. (23) that

$$\prod_{l=1}^{4} m_1^{(l)}=1. \qquad (25)$$

At the same time, it follows from Eq. (24) that

$$[\sigma_1^{(1)}\sigma_1^{(2)}\sigma_2^{(3)}\sigma_2^{(4)}][\sigma_1^{(1)}\sigma_2^{(2)}\sigma_1^{(3)}\sigma_2^{(4)}]\times[\sigma_1^{(1)}\sigma_2^{(2)}\sigma_2^{(3)}\sigma_1^{(4)}]|\chi_{10}\rangle=-|\chi_{10}\rangle.$$

Employing the classical variables instead of the local operators allows this to be cast into the form

$$(m_1^{(1)})^3m_1^{(2)}(m_2^{(2)})^2m_1^{(3)}(m_2^{(3)})^2m_1^{(4)}(m_2^{(4)})^2=-1.$$

Since $(m_1^{(l)})^2=(m_2^{(l)})^2=1$, we get an equivalent equality

$$m_1^{(1)}m_1^{(2)}m_1^{(3)}m_1^{(4)}=-1,$$

which contradicts Eq. (25). Hence, the state $|\chi_{10}\rangle$ in Eq. (22) obey the GHZ theorem. Similar result can be obtained for all other states in Eq. (22) and hence, for the phase states (21).

Our consideration so far have applied to the local measurements touching on a single atom. We now note that the phase states (21) allow another kind of entanglement in the case of pairwise measurement. Consider again the state $|\chi_{10}\rangle$ in Eq. (22) and assume that the measurements a and b corresponds to a pair of atoms:

$$a=\cos\theta_a|e_1e_2\rangle\langle e_1e_2|+\sin\theta_a(|e_1e_2\rangle\langle g_1g_2|+|g_1g_2\rangle\times\langle e_1e_2|)-\cos\theta_a|g_1g_2\rangle\langle g_1g_2|,$$

$$b=\cos\theta_b|e_3e_4\rangle\langle e_3e_4|+\sin\theta_b(|e_3e_4\rangle\langle g_3g_4|+|g_3g_4\rangle\times\langle e_3e_4|)-\cos\theta_b|g_3g_4\rangle\langle g_3g_4|. \qquad (26)$$

Assume now that we make the two measurements a and a' with the angles $\theta_1=\pi$ and $\theta_a'=\pi/2$ and the two more measurements b and b' with the angles $\theta_b'=-\theta_b$, respectively. Then, the averaging over the state $|\chi_{10}\rangle$ gives

$$\langle ab\rangle=\langle ab'\rangle=\cos\theta_b,\quad \langle a'b\rangle=\sin\theta_b=-\langle a'b'\rangle.$$

Employing the CHSH inequality [33]

$$|\langle ab\rangle+\langle a'b\rangle+\langle a'b'\rangle-\langle ab'\rangle|\leqslant 2 \qquad (27)$$

then gives

$$|\cos\theta_b-\sin\theta_b|\leqslant 1.$$

Violation of this inequality and hence, of the classical realism occurs at small negative θ_b, when we can put

$$|\cos\theta_b-\sin\theta_b|\sim 1+|\theta_b|>1.$$

Similar consideration can be done for all states in Eq. (22) through the use of proper pairwise measurements. At the same time, the phase states (21) do not manifest entanglement with respect to the pairwise measurements.

The phase states (16) for the $6+3,8+4,\ldots$ systems, corresponding to the spin (1) equal to $19/2,69/2,\ldots$, respectively, can be considered as above.

IV. INITIAL CONDITIONS AND ATOMIC ENTANGLEMENT

It is clear that the evolution of the $2n+n$ system strongly depends on the choice of initial conditions. To trace the proper choice leading to the atomic entanglement, let us ignore the relaxation processes. Then, the steady-state evolution of the $2n+n$ system under consideration is governed by the Hamiltonian

$$H=\Delta a^{+}a+\omega_0\mathcal{N}+\gamma\sum_l (R_l^{+}a+a^{+}R_l). \tag{28}$$

Here Δ is the cavity detuning, ω_0 is the atomic transition frequency, γ is the atom-field coupling constant, and operators a and a^{+} describe the cavity photons,

$$\mathcal{N}=a^{+}a+\sum_l |e_l\rangle\langle e_l|\otimes_{l'\neq l}\mathbf{1}^{(l)},$$

and the atomic operators are defined as follows:

$$R_l^{+}=|e_l\rangle\langle g_l|\otimes_{l'\neq l}\mathbf{1}^{(l')}.$$

Here $\mathbf{1}^{(l)}$ denotes the unit operator in the two-dimensional Hilbert space of the l^{th} atom. It is seen that $[\mathcal{N},H]=0$. It is also seen that the atomic operators are similar, in a certain sense, to the local operators (3). In fact

$$R_l^{\pm}=\frac{\sigma_1^{(l)}\pm i\sigma_2^{(l)}}{2}.$$

Consider first the case of two atoms and single cavity photon when $l=1,2$ and the Hamiltonian (28) coincides with that of Ref. [18]. For simplicity, we use here the same coupling constant γ for both atoms. Our consideration can easily be generalized on the case of coupling constant depending on the atomic position. Let us note that, in the case of only two atoms, the Hamiltonian (28) can be represented as follows

$$H\rightarrow H_\phi=\Delta a^{+}a+\omega_0\mathcal{N}_\phi+\gamma\sqrt{2}(\mathcal{R}^{+}a+a^{+}\mathcal{R}), \tag{29}$$

where

$$\mathcal{N}_\phi=a^{+}a+\sum_{k=\pm 1}|\phi_k\rangle\langle\phi_k|$$

and

$$\mathcal{R}^{+}=|\phi_{+}\rangle\langle g_1g_2|.$$

Here $|\phi_\pm\rangle$ denote the phase states (14).

Using the Hamiltonian (29) as the generator of evolution, for the time-dependent wave function we get

$$\begin{aligned}|\Psi(t)\rangle&=e^{-iH_\phi t}|\Psi(0)\rangle\\&=[C_{-}(t)|\phi_{-}\rangle+C_{+}(t)|\phi_{+}\rangle]\otimes|0\rangle_{ph}\\&\quad+C(t)|g_1g_2\rangle\otimes|1\rangle_{ph},\end{aligned} \tag{30}$$

where $|\cdots\rangle_{ph}$ denotes the states of the cavity field. The coefficients $C_\pm(t)$ and $C(t)$ in Eq. (30) are completely determined by the initial conditions and the normalization condition.

It is easily seen that the state $|\phi_{-}\rangle\otimes|0\rangle_{ph}$ is the eigenstate of the Hamiltonian (29). Hence, at

$$C_{-}(0)=1,\quad C_{+}(0)=C(0)=0,$$

the atomic phase state $|\phi_{-}\rangle$ in Eq. (14) provides the stationary, maximum entangled atomic state in the system under consideration [18]. At the same time, it is not very clear how to prepare such a state.

Therefore we consider a more realistic initial state provided by excitation of either atom, while the cavity field is in the vacuum state. To realize such a state, we can assume, for example, that one of the atoms (initially deexcited) is trapped in the cavity, while the second atom (initially excited) slowly passes through the cavity like in the experiments discussed in Refs. [14,15]. assume for definiteness that

$$|\Psi(0)\rangle=|e_1g_2\rangle\otimes|0\rangle_{ph}. \tag{31}$$

Then, the coefficients of the wave function (30) take the form

$$C_{-}(t)=\frac{1}{\sqrt{2}}e^{-i\omega_0 t},$$

$$C_{+}(t)=\frac{1}{\sqrt{2}}\left[\cos(\Omega t)+\frac{i\Delta}{2\Omega}\sin(\Omega t)\right]e^{-i(\omega_0+\Delta/2)t},$$

$$C(t)=-\frac{i\gamma}{\Omega}e^{-i(\omega_0+\Delta/2)t}\sin(\Omega t),$$

where $\Omega=[2\gamma^2+(\Delta/2)^2]^{1/2}$. At first site, the probabilities

$$P_\pm(t)=|\langle 0|_{ph}\otimes\langle\phi_{pm}|\Psi(t)\rangle|^2=|C_\pm(t)|^2$$

to observe the states (14) corresponding to the maximum atomic entanglement, are

$$P_{-}(t)=\tfrac{1}{2},$$

$$P_{+}(t)=\frac{\Delta^2}{8\Omega^2}+\frac{\gamma^2}{\Omega^2}\cos^2(\Omega t)\leqslant\frac{1}{2},$$

respectively. At the same time, the absence of photon counts, which is considered in Ref. [18] as a sign of the atomic entanglement, corresponds here to the case when both probabilities $P_\pm(t_k)=1/2$ at a certain time t_k. In other words, the mutually orthogonal entangled states (14) have the same probability to be observed at $t=t_k$. This means that there is no atomic entanglement at all but we definitely know which atom is in the excited state.

Consider one more realistic initial state when both atoms are trapped in the cavity in deexcited state, while the cavity field contains a photon:

$$|\Psi(0)\rangle = |g_1 g_2\rangle \otimes |1\rangle_{ph}. \tag{32}$$

Then, for all times we get $C_-(t)=0$ and

$$C_+(t) = -\frac{i\gamma\sqrt{2}}{\Omega} e^{-i(\omega_0+\Delta/2)t}\sin(\Omega t),$$

$$C(t) = \left[\cos(\Omega t) - \frac{i\Delta}{2\Omega}\sin(\Omega t)\right] e^{-i(\omega_0+\Delta/2)t}.$$

Hence, under this initial condition, the entangled state $|\phi_-\rangle$ cannot be achieved at all, while the second entangled state $|\phi_+\rangle$ in Eq. (14) can be achieved. It is seen that, in the case of initial state (32), the probability to detect the photon is

$$P_{ph}(t) = |C(t)|^2 = \cos^2(\Omega t) + \frac{\Delta^2}{4\Omega^2}\sin^2(\Omega t).$$

This expression takes the minimum value

$$\min P_{ph} = P_{ph}(t_m) = \frac{\Delta^2}{4\Omega^2}$$

at $t=t_m=\pi(2m+1)/2\Omega, \quad m=0,1,\ldots$. At the same time t_m, the probability to have the entangled atomic state $|\phi_+\rangle$ takes the maximum value

$$P_+(t_m) = |C_+(t_m)|^2 = \frac{2\gamma^2}{2\gamma^2 + (\Delta/2)^2}.$$

It is seen that the pure atomic entanglement with $P_+(t_m) = 1$ is realized at $t=t_m$ only in the absence of the cavity detuning when $\Delta \to 0$.

The parasitic influence of the cavity detuning can be compensated through the use of Kerr medium filling the cavity. In this case, the Hamiltonian (28) should be supplemented by the term [34]

$$H_\kappa = \kappa(a^+a)^2,$$

which leads to the following renormalization of the Rabi frequency:

$$\Omega \to \Omega_\kappa = \sqrt{2\gamma^2 + (\Delta+\kappa)^2/4}.$$

Then, the proper choice of the Kerr parameter $\kappa = -\Delta$ should lead to the pure entangled atomic state $|\phi_+\rangle$ at a certain times.

Consider now the case of four atoms and two photons. In contrast to the previous case, neither phase state in Eq. (21) is an eigenstate of the Hamiltonian (28). Then, the choice of the initial state either as a state with two excited atoms or as a state with one excited atom plus cavity photon does not lead to a pure atomic entanglement. As in the case of two atoms, the pure atomic entanglement can be reached under the choice of the state with the absence of the atomic excitations in the initial state. The influence of the cavity detuning can be compensated by the presence of Kerr medium as well as in the case of two atoms.

V. CONCLUSION

Let us briefly discuss the obtained results. For the system of two identical two-level atoms interacting with a single photon as proposed in Ref. [18] it is shown that the maximum entangled atomic states are represented by the SU(2) phase states of spin 1/2. Moreover, it is shown that the SU(2) phase states of the half-integer spin j (1) form a certain class of maximum entangled atomic states in the system of $2n$ atoms interacting with n photons. In particular, the violation of classical realism is shown.

It should be noted in this connection that the above considered SU(2) phase states do not represent a unique way to construct the maximum entangled states in the multiatom systems and that some other symmetries, for example the SU($\mathcal{N}$) can be considered as well. Moreover, in some cases the SU(2) phase states cannot be used to determine the maximum entangled states at all. Consider for example the case of two identical two-level atoms interacting with two photons, when the atomic subsystem can be specified by the four states

$$|e_1e_2\rangle, \quad |e_1g_2\rangle, \quad |g_1e_2\rangle, \quad |g_1g_2\rangle.$$

By performing a similar analysis to that described in Sec. II, it is easy to construct the corresponding set of the SU(2) phase states

$$|\phi_k\rangle = \tfrac{1}{2}\left(|e_1e_2\rangle + e^{i\phi_k}|e_1g_2\rangle + e^{2i\phi_k}|g_1g_2\rangle + e^{3i\phi_k}|g_1e_2\rangle\right),$$

$$\phi_k = \frac{\psi}{4} + \frac{k\pi}{2}, \quad k=0,1,2,3,$$

which do not manifest the maximum entanglement. At the same time, the general criterion (4) permits us to determine infinitely many maximum entangled states in this case [23]. An example is provided by the following set of orthonormal maximum entangled states:

$$|\psi_1\rangle = \tfrac{1}{2}\left(|e_1e_2\rangle + |g_1g_2\rangle + i|e_1g_2\rangle + i|g_1e_2\rangle\right),$$

$$|\psi_2\rangle = \tfrac{1}{2}\left(|e_1e_2\rangle - |g_1g_2\rangle - i|e_1g_2\rangle + i|g_1e_2\rangle\right),$$

$$|\psi_3\rangle = \tfrac{1}{2}\left(i|e_1e_2\rangle + i|g_1g_2\rangle + |e_1g_2\rangle + |g_1e_2\rangle\right),$$

$$|\psi_4\rangle = \tfrac{1}{2}\left(-i|e_1e_2\rangle + i|g_1g_2\rangle + |e_1g_2\rangle - |g_1e_2\rangle\right).$$

In fact, the Eq. (4) gives a general condition [23], while the SU(2) phase states can manifest the maximum entanglement only under a certain condition [special choice of the effective spin (1)].

Nevertheless, the SU(2) phase states considered in Secs. II and III represent an important example of the atomic entangled states. First of all, they can be easily realized in the atomic systems in a cavity. In fact, these states have a simple physical meaning. In addition to Eq. (9), the SU(2) phase states can be defined to be the eigenstates of the Hermitian ‘‘cosine’’ operator [19,22]

$$C = \tfrac{1}{2}(\epsilon + \epsilon^+),$$

where ϵ is defined by Eq. (8). This operator C can be considered as a "Hamiltonian," describing the correlations between the different atoms. For example, in the case of the two atoms interacting with the single photon, the operator C takes the form

$$C=\sigma_{+}^{(1)}\sigma_{-}^{(2)}+\sigma_{-}^{(2)}\sigma_{+}^{(1)}, \qquad (33)$$

where

$$\sigma_{\pm}^{(l)}=\frac{\sigma_{1}^{(l)}\pm i\sigma_{2}^{(l)}}{2}.$$

The operator structure of Eq. (33) coincides with that of the so-called model of plane rotator, which is a particular case of the Heisenberg model of ferromagnetism widely used in statistical physics [35] and in quantum information theory [36].

Let us also stress that the SU(2) phase states similar to those considered in Secs. II and III have been discussed recently in the context of quantum coding [37].

It is also known that the SU(2) phase states have direct connection with the quantum description of polarization of spherical photons emitted by the multipole transitions in atoms and molecules [21,22,38]. Therefore, the polarization entanglement of photons can be examined in direct analogy to the above discussed atomic entanglement [39]. At the same time, the consideration of spherical photons requires the use of more quantum degrees of freedom. Consider as an example the cascade decay of a two-level atom specified by the transition [40]

$$|J=2,m=0\rangle\rightarrow|J'=0,m'=0\rangle.$$

Here J,J' and m,m' denote the angular momentum and projection of the angular momentum of the excited and the ground atomic states, respectively. This transition gives rise to an entangled photon twins [40]. Each photon carries spin 1, but because of the conservation of the angular momentum in the process of radiation, the sum of projections of the angular momenta of the two photons should be equal to zero. Denoting the state of a photon with given m by $|m\rangle$, we get the three possible states of the photon subsystem:

$$|+1\rangle\otimes|-1\rangle, \quad |0\rangle\otimes|0\rangle, \quad |-1\rangle\otimes|+1\rangle.$$

These three "individual" states can be used to construct the dual basis of the SU(2) phase states [21]

$$|\phi_k\rangle=\frac{1}{\sqrt{3}}(|+1\rangle\otimes|-1\rangle+e^{i\phi_k}|0\rangle\otimes|0\rangle+e^{2i\phi_k}|-1\rangle\otimes|+1\rangle,$$

$$\phi_k=\frac{\psi+2k\pi}{3}, \quad k=0,1,2, \qquad (34)$$

similar to Eq. (18). It can be easily seen that these states manifest the maximum entanglement.

Similar entangled states have been discussed in the context of the so-called biphoton excitations [41] (photon pairs in symmetric Fock states). They can also be used in quantum cryptography [42].

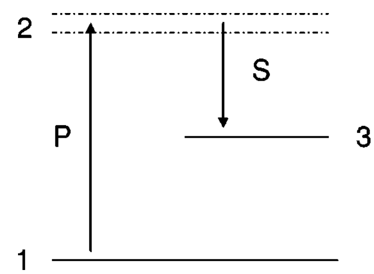

FIG. 1. Atomic Raman-type interaction with pump (P) and Stokes (S) photons.

Let us stress that the general condition of the type as in Eq. (4) is also valid in the case of states (34). However, the definition of local measurement should be changed in this case. Because of the number of degrees of freedom per photon is equal to 3, the Hermitian operators associated with the SU(3) group should be considered instead of the infinitesimal generators of the SL(2) group. For example, the set of Stokes operators of Ref. [21], corresponding to the representation of the SU(3) subalgebra in the Weyl-Heisenberg algebra of spherical photons, can be used to define the complete set of local measurements in this case.

It is shown in Sec. IV that the realization of a pure atomic entanglement in the $(2n+n)$-type atom-plus-photon systems strongly depends on the choice of initial state. That is the entangled states can be reached in the process of steady-state evolution only if all $2n$ atoms are initially in the deexcited states, while the cavity contains just n photons. This condition has an intuitively clear explanation: the excitations of different atoms have the same probability and therefore each photon in the $2n+n$ system is shared with a couple of atoms.

It is also shown in Sec. IV that the presence of cavity detuning hampers the creation of a pure entangled atomic state. This negative effect can be compensated through the use of Kerr medium in the cavity.

We now note that the practical realization of a long-lived, maximum entanglement in a quantum-mechanical system strongly depends on the interaction between this system and environment. The point is that the state of a closed quantum-mechanical system changes periodically, providing the maximum entanglement as an instant event only at a certain times (see Sec. IV). Such a periodicity is caused by a finite number of degrees of freedom in the system. To destruct such a periodicity, it is necessary to connect the system to a "heat bath," which would tune in the system to a required state. In Ref. [18], it has been proposed to support the atomic entanglement by the cavity losses. In this case, the absence of the photon counting outside the cavity can be associated with the existence of the entangled atomic state in the cavity.

Let us stress that an advantage of the use of the SU(2) phase states as the maximum entangled atomic states consists in the simple preparation of the initial states discussed in Sec. IV.

In view of realization of atomic entanglement with the present experimental technique, it seems to be more convenient if the existence of entangled state in a cavity would manifest itself via a signal photon rather than the absence of photon leakage from the cavity. In this case, there should be at least two modes such that one of them (the cavity mode) provides the correlation between the atoms, while the second can freely leave the resonator to signalize the existence of the entanglement. Such a process can be realized through the use of Raman process in atoms shown in Fig. 1 (e.g., see Ref. [43]). Here the dipole transitions are allowed between the levels 1 and 2 and 2 and 3, while forbidden between 1 and 3 because of the parity conservation. In the simplest case, we should assume that the two identical atoms of this type are located in a cavity, which has a very high quality with respect to the pumping mode ω_P, while the Stokes photons with frequency ω_{Sk} can leak away freely.

Assume that the atoms are initially in the ground state 1, the Stokes field is in the vacuum state, and the pump field consists of a single photon. The evolution of the system can lead to the absorption of the cavity photon by either atom with further emission of the Stokes photon that leaves the cavity. After that the atoms are in entangled state, corresponding to the excitation of the atomic level 3 shared between the atoms. Since the inverse process cannot be realized without assistance of the Stokes photon, such a state represents a durable atomic entangled state.

It is clear that the above consideration of the atomic entanglement in the multiatom system can be generalized with ease in the case of Raman process in atoms. In other words, the SU(2) phase states similar to Eq. (16) form the class of the maximum entangled atomic states in the case of Raman-type processes in the three-level atoms as well. An evident advantage of the use of the Raman process is the long-lived maximum entanglement in atomic subsystem.

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[1] C. H. Bennet and G. Brassard, in *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India,* edited by R. M. Vasagam (IEEE New York, 1984).
[2] A. K. Ekert, Phys. Rev. Lett. **68**, 661 (1991).
[3] C. H. Bennet, G. Brassard, and A. K. Ekert, Sci. Am. (Int. Ed.) **276**, 50 (1992).
[4] A. K. Ekert, J. G. Rarity, P. G. Tapster, and G. M. Palma, Phys. Rev. Lett. **69**, 1293 (1992).
[5] C. H. Bennet, F. Bessette, G. Brassard, L. Salivan, and J. Smolin, J. Cryptology **5**, 3 (1992).
[6] *The Physics of Quantum Information*, edited by D. Bouwmeester, A. K. Ekert, and A. Zeilinger (Springer-Verlag, Berlin, 2000).
[7] *Quantum Communications, Computing, and Measurements*, edited by P. Tombesi and O. Hirota (Kluwer Academic/Plenum Publishers, New York, 2001).
[8] P. W. Shor, in *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science*, edited by S. Goldwasser (IEEE Computer Society Press, Los Alamos, CA, 1994).
[9] J. G. Rarity and P. R. Tapster, Phys. Rev. A **59**, R35 (1999).
[10] D. M. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, The Netherlands, 1989).
[11] K. Wódkiewicz, Liwei Wang, and J. H. Eberly, Phys. Rev. A **47**, 3280 (1993).
[12] C. J. Hood, W. Lange, H. Mabichi, and H. J. Kimble, Phys. Rev. Lett. **10**, 4710 (1995).
[13] C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).
[14] M. Weidinger, B. T. H. Varcoe, R. Heerland, and H. Walther, Phys. Rev. Lett. **82**, 3795 (1999); S. Haroche, in *Latin-American School of Physics XXXIELAF*, edited by Shahen Hacyan, Rocio Jauregui, and Ramon Lopez-Pena, AIP Conf. Proc. **464** (AIP, Woodbury, NY, 1995), p. 45.
[15] G. Rempe, Ann. Phys. (Leipzig) **9**, 843 (2000).
[16] *Advances in Atomic, Molecular, and Optical Physics*, edited by B. Bederson and H. Walther (Academic Press, New York, 2000), Vol. 42.
[17] B. Julsgaard, A. Kozhekin, and E. Polzik, Nature (London) **413**, 400 (2001).
[18] M. B. Plenio, S. F. Huelga, A. Beige, and P. L. Knight, Phys. Rev. A **59**, 2468 (1999).
[19] A. Vourdas, Phys. Rev. A **41**, 1653 (1990).
[20] A. S. Shumovsky and Ö. E. Müstecaplioğlu, J. Mod. Opt. **45**, 619 (1998).
[21] A. S. Shumovsky, J. Phys. A **32**, 6589 (1999).
[22] A.S. Shumovsky, in *Modern Nonlinear Optics*, 2nd ed., Advances in Chemical Physics, Vol. 119 edited by M. W. Evans (Wiley, New York, 2001), Pt. 1.
[23] A. A. Klyachko and A. S. Shumovsky, e-print quant-ph/0203099.
[24] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[25] M. B. Plenio and V. Verdal, Contemp. Phys. **39**, 431 (1998).
[26] D. M. Greenberger, M. Horne, A. Simony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[27] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[28] A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 460 (1993).
[29] R. Horodecki, P. Horodecki, and M. Horodecki Phys. Lett. A **200**, 340 (1995).
[30] A. Peres, Found. Phys. **29**, 589 (1999).
[31] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 010102(R) (2001).
[32] Č. Brukner, M. Žukovski, and A. Zeilinger, e-print quant-ph/0106119v1.
[33] J. F. Clauser, M. A. Horne, A. Shimoni, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[34] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, New York, 1997).
[35] E.g., see: N. N. Bogolubov Jr., B. I. Sadovnikov, and A. S. Shumovsky, *Mathematical Methods of Statistical Mechanics of Model Systems* (CRC Press, Boca Ration, FL, 1994).

[36] J. I. Cirac, Nature (London) **409**, 63 (2001).
[37] A. Vourdas, Phys. Rev. A **65**, 042321 (2002).
[38] A. S. Shumovsky and Ö. E. Müstecaplioğlu, Phys. Rev. Lett. **80**, 1202 (1998).
[39] See: A. S. Shumovsky, in *Quantum Communications, Computing, and Measurements* (Ref. [7]).
[40] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995).
[41] A. V. Burlakov, M. V. Chekhova, O. A. Karabutova, D. N. Klyshko, and S. P. Kulik, Phys. Rev. A **60**, R4209 (1999).
[42] H. Bechmann-Pasquinucci and A. Peres, Phys. Rev. Lett. **85**, 3313 (2000).
[43] C. K. Law and J. H. Eberly, Phys. Rev. A **47**, 3195 (1993); R. R. Puri, C. K. Law, and J. H. Eberly, *ibid.* **50**, 4212 (1994); C. Cabrillo, J. I. Cirac, P. Garsia-Fernandez, and P. Zoller, *ibid.* **59**, 1025 (1999); M. Hennrich, T. Legero, K. Khun, and G. Rempe, Phys. Rev. Lett. **85**, 4872 (2000); A. Beige, W. J. Munro, and P. L. Knight, Phys. Rev. A **62**, 052102 (2000).